DEFINITION OF RS AND EXAMPLES

RS = 1-dim complex manifold

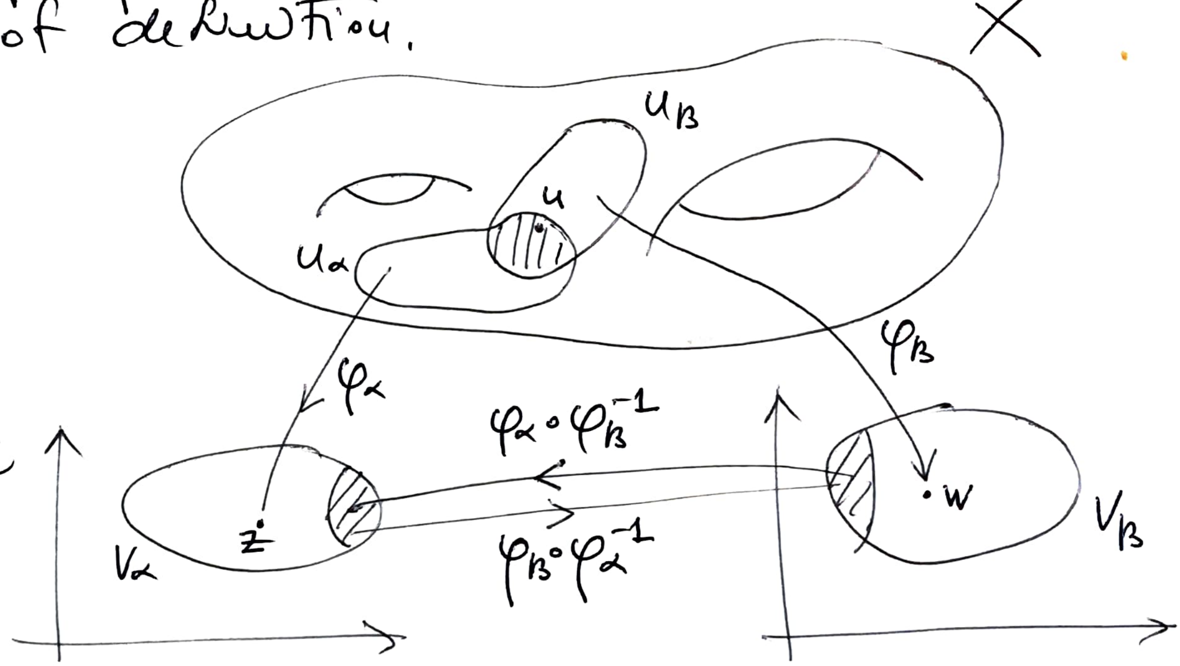
<u>DEF</u> A <u>Riemann surface</u> is given by the following data:

(1.) A Hausdorff topological space X (with a countable basis of open sets);

(2.) a collection of open sets $U_\alpha \subset X$, $\alpha \in I$ such that $X = \bigcup_{\alpha \in I} U_\alpha$;

(3.) For each $\alpha \in I$, a homeomorphism $\varphi_\alpha : U_\alpha \to V_\alpha$ is given where V_α is an open set in $\mathbb{C}$, with the property that for all $\alpha, \beta \in I$ the composite map

(*) $\varphi_\alpha \circ \varphi_\beta^{-1}$ is <u>holomorphic</u> on its domain of definition.



Terminology

(i) $(U_\alpha, V_\alpha, \varphi_\alpha)$ ... a chart or local co-ordinates on X

(ii) $\{(U_\alpha, V_\alpha, \varphi_\alpha) \mid \alpha \in I\}$ ... an atlas of charts on X

(iii) If $U_\alpha \cap U_\beta \neq \emptyset$, the transition function

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \underbrace{\varphi_\beta(U_\alpha \cap U_\beta)}_{\text{open in } \mathbb{C}} \longrightarrow \underbrace{\varphi_\alpha(U_\alpha \cap U_\beta)}_{\text{open in } \mathbb{C}}$$

is a homeomorphism, and it is a conformal map. Indeed, $\varphi_\alpha \circ \varphi_\beta^{-1}$ is holomorphic and so is its inverse $\varphi_\beta \circ \varphi_\alpha^{-1}$.

If $U \subset \mathbb{C}$ is open, we call a map $f : U \to \mathbb{C}$ conformal provided that f is one-to-one and holomorphic. Recall that f is conformal iff f and f^{-1} are both holomorphic.

REMARK: 1.) A RS X is locally homeomorphic with $\mathbb{C}$ and it is covered by the given atlas of charts with holomorphic transition functions.

2.) Let $u \in U_\alpha \cap U_\beta \subset X$. Then $z = \varphi_\alpha(u)$ and $w = \varphi_\beta(u)$ are different local co-ordinates of u and $z = \varphi_\alpha \circ \varphi_\beta^{-1}(w)$, $w = \varphi_\beta \circ \varphi_\alpha^{-1}(z)$ are their conformal transformations.

Riemann surface theory studies properties independent of the choice of co-ordinate chart.

3.) For any atlas $\mathcal{A}$ on X, there is a unique maximal atlas $\tilde{\mathcal{A}}$ on X such that $\tilde{\mathcal{A}} \supset \mathcal{A}$. We call $\tilde{\mathcal{A}}$ a conformal structure on X.
If atlases $\mathcal{A}$ and $\mathcal{B}$ on X give the same conformal structure (i.e., $\mathcal{A} \cup \mathcal{B}$ is an atlas on X), then we consider the corresponding RS's as the same.

4.) If we take the same definition but in (*) replace the word 'holomorphic' by smooth (with positive Jacobian) we get the definition of an (oriented) smooth surface.

Let $U \subset \mathbb{C}$ be open and $f: U \to \mathbb{C}$.
Since $\mathbb{C} \simeq \mathbb{R}^2$, $z = x+iy = (x,y)$ we can view f as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$. Here f is smooth if f is C^∞ in the sense of two real variables.

$\boxed{\text{Claim}}$ If the complex derivative $f'(z_0)$ exists, then Jacobian $J_f(z_0) = |f'(z_0)|^2$.

Proof: If $f = f_1 + i f_2$, then

$$J_f = \det \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix}.$$

At z_0, using CR conditions, we get

$$J_f = \left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial x}\right)^2 = \left|\frac{\partial f}{\partial x}\right|^2 = |f'|^2. \quad \boxtimes$$

By Claim, all Riemann surfaces are oriented smooth surfaces.

On the other hand, every oriented smooth surface X in $\mathbb{R}^3$ has a unique compatible conformal structure.

5.) If we take the same definition but replace $\mathbb{C}$ with $\mathbb{R}^n$ and the word 'holomorphic' in (*) with smooth we get the definition of an n-dimensional smooth manifold.

If we replace $\mathbb{C}$ with $\mathbb{C}^n$ and, in (*), we consider maps holomorphic in the sense of several complex variables, we get the definition of an n-dimensional complex manifold.

In particular, RS is a 1-dim complex manifold.

Examples

$\boxed{1.}$ Let X be an open set in $\mathbb{C}$.

Atlas: (X, Id).

In particular, $X = \mathbb{C}$ or the unit disc

$$\mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \}.$$

Let Y be an open subset of a RS X.

If $\mathcal{A}$ is an atlas on X, then

$$\mathcal{A}|_Y := \{ (U \cap Y, \varphi(U \cap Y), \varphi|_{U \cap Y}) \mid (U, V, \varphi) \in \mathcal{A} \}$$

is an atlas on Y and $(Y, \mathcal{A}|_Y)$ is a RS.

$\boxed{2.}$ Riemann sphere $\mathbb{S} = \mathbb{C} \cup \{\infty\}$

Atlas: $(\mathbb{C}, \varphi_1)$ where $\varphi_1(z) := z$;

$(\mathbb{S} \setminus \{0\}, \varphi_2)$ where $\varphi_2(z) := \frac{1}{z}, \ z \in \mathbb{C} \setminus \{0\},$

$:= 0, \ z = \infty.$

Then $\varphi_1 \circ \varphi_2^{-1}(z) = \frac{1}{z} = \varphi_2 \circ \varphi_1^{-1}(z), \ z \in \mathbb{C} \setminus \{0\}.$

Algebraic curves

- studied in Algebraic Geometry

Let $P(z,w)$ be a polynomial in two complex variables, that is,

$$P(z,w) = \sum_{n,m \in \mathbb{N}_0} a_{nm} z^n w^m$$

where $a_{nm} \neq 0$ for a finitely many $(n,m) \in \mathbb{N}_0^2$.

Put

(affine curve)

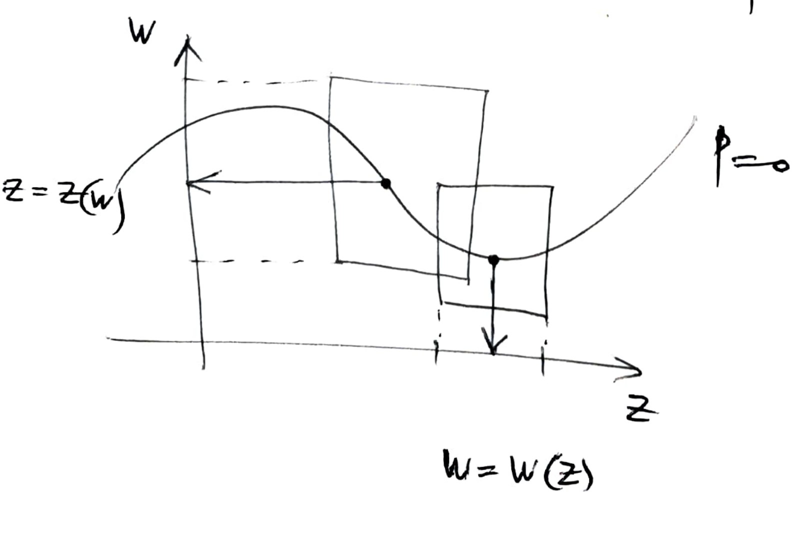
(AC) $\quad X := \{ (z,w) \in \mathbb{C}^2 \mid P(z,w) = 0 \}$,

the zero set of the polynomial P.

$\boxed{\text{FACT}}$ If $\left(\frac{\partial P}{\partial z}, \frac{\partial P}{\partial w} \right) \neq (0,0)$ on X, then X is a RS with local co-ordinates given by

(i) $(z,w) \longrightarrow w$ on neighborhoods of points at which $\frac{\partial P}{\partial z} \neq 0$;

(ii) $(z,w) \longrightarrow z$ on ngh of points at which $\frac{\partial P}{\partial w} \neq 0$.



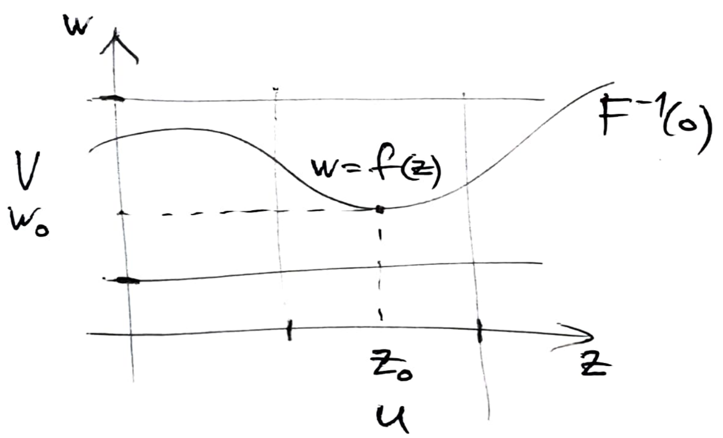
A proof of FACT follows from the following theorem.

$\boxed{\text{Theorem}}$ (on implicit function, complex version)

Let $F : \mathbb{C}^2 \to \mathbb{C}$ be a holomorphic function in a ngh. of $(z_0, w_0) \in \mathbb{C}^2$, i.e., $F = F(z, w)$ ~~and~~ has (continuous) $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial w}$ there. Let $F(z_0, w_0) = 0$.

(a) Assume $\frac{\partial F}{\partial w}(z_0, w_0) \neq 0$. Then $F^{-1}(0)$ in some ngh. of (z_0, w_0) is the graph of a holomorphic function $w = w(z)$. To be more precise, there are a ngh ~~U~~ $U \subset \mathbb{C}$ of z_0, a ngh $V \subset \mathbb{C}$ of w_0 and a holomorphic function $f : U \to V$ such that

$$F^{-1}(0) \cap (U \times V) = \operatorname{graph} f \overset{\text{def.}}{:=} \{(z, f(z)) \mid z \in U\}.$$

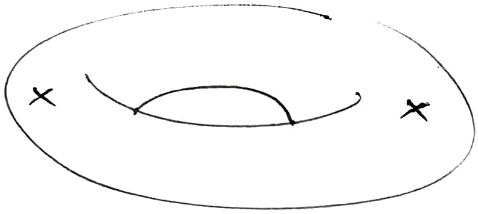


(b) Assume $\frac{\partial F}{\partial z}(z_0, w_0) \neq 0$. Then $F^{-1}(0)$ in some ngh of (z_0, w_0) is the graph of a holomorphic function $z = z(w)$.

(c) If $\frac{\partial F}{\partial z}(z_0, w_0) \neq 0$ and $\frac{\partial F}{\partial w}(z_0, w_0) \neq 0$, then the functions $z = z(w)$ and $w = w(z)$ of (a) and (b) are inverse of each other.

<u>Proof</u>: We can view F also as a function from $\mathbb{R}^4$ to $\mathbb{R}^2$. Since F is C^1 we can use a 'real' version of Theorem on an implicit function. To verify that corresponding Jacobians are non-zero and that $w = w(z)$, $z = z(w)$ are holomorphic, we use the CR conditions. $\boxed{\text{Exercise}}$ ▨

$\boxed{\text{Example}}$ Consider $X := \{ (z, w) \in \mathbb{C}^2 \mid w^2 = p(z) \}$ where $p(z) = (z^2 - 1)(z^2 - 4)$. We know that

$X \simeq$ 

Then X is a RS because, for

$P(z, w) := w^2 - p(z)$, we have

- $\frac{\partial P}{\partial w} = 2w \neq 0$ iff $w \neq 0$;
- $\frac{\partial P}{\partial z} = -p' \neq 0$ if $w = 0$ and $z = \underbrace{\pm 1, \pm 2}_{\text{simple roots of } p}$