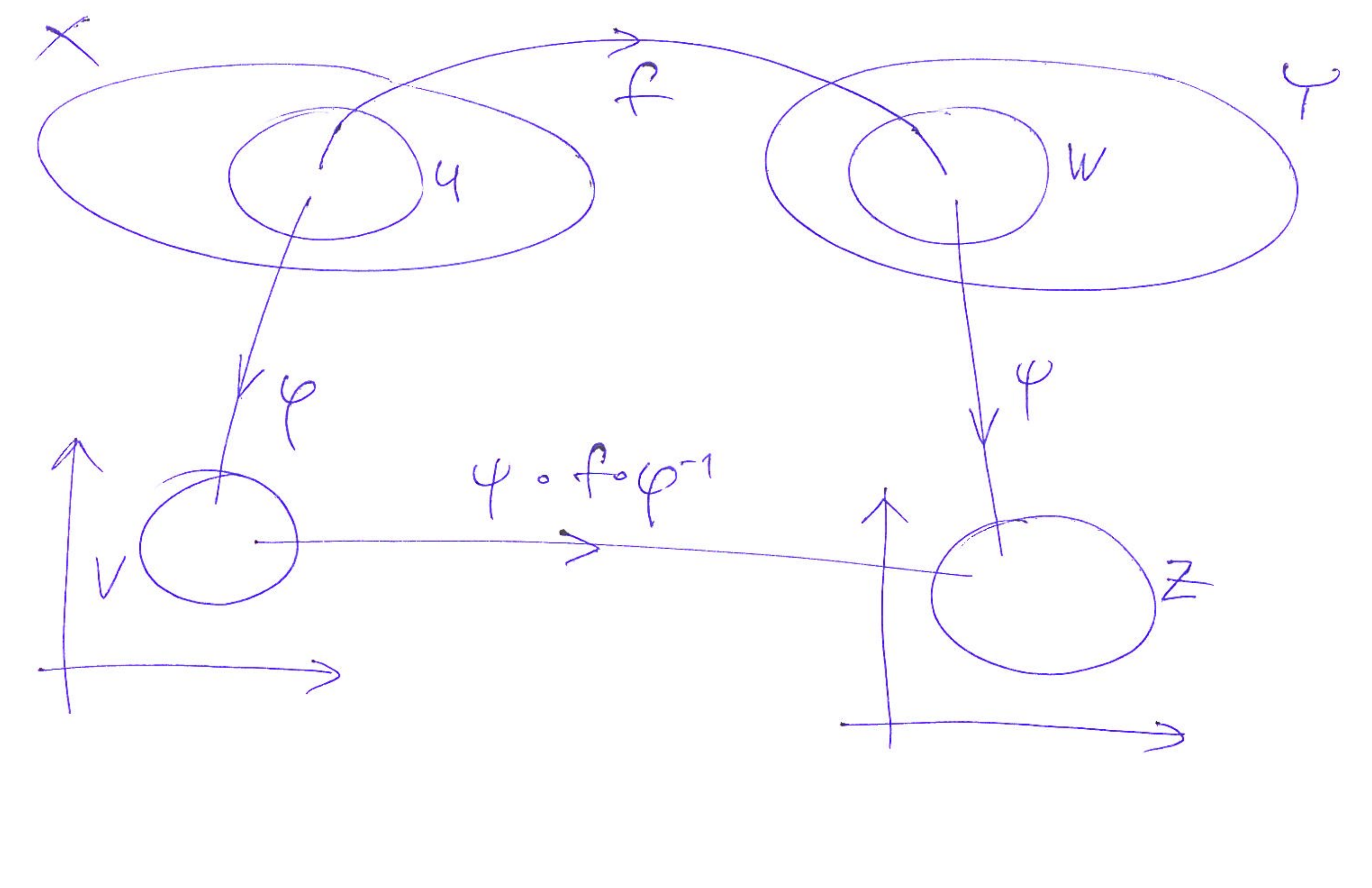
Remark: We call an affine curve (AC) regular if $\left(\frac{\partial P}{\partial z} \Big| \frac{\partial P}{\partial w}\right) \neq (0,0)$ on X. Then we show that such a RS X is not compact. On the other hand, Riemann showed that every compact RS is algebraic, that is, we can obtain it by compactifying an algebraic curve.
"adding some points at ∞"

—— x ——

HOLOMORPHIC MAPS



DEF. Let X be a RS with an atlas $\mathcal{A}$,
and Y — " — — " — $\mathcal{B}$.
Then a continuous map $f: X \to Y$ is called *holomorphic* if f is holomorphic in local co-ordinates, that is, for each chart $(U, V, \varphi) \in \mathcal{A}$ and $(W, Z, \psi) \in \mathcal{B}$, the map $\psi \circ f \circ \varphi^{-1}$ is holomorphic on its domain of definition.

Remark: (i) $\psi \circ f \circ \varphi^{-1} : \underbrace{\varphi(U \cap f^{-1}(W))}_{\text{open in } \mathbb{C} \text{ or } \emptyset} \longrightarrow Z$

(ii) Let $\bar{\mathcal{A}} \supset \mathcal{A}$ and $\bar{\mathcal{B}} \supset \mathcal{B}$ be conformal structures on X and Y, respectively. Then $f: X \to Y$ is holomorphic w.r.t. $\mathcal{A}, \mathcal{B}$ iff
— " — — " — $\bar{\mathcal{A}}, \bar{\mathcal{B}}$.

(iii) In an analogous way, we define smooth maps between smooth surfaces etc.

DEF. We say that RS X and Y are *conformally equivalent* (write $X \simeq Y$) if there is a $\underbrace{\text{one-to-one holomorphic}}_{\text{conformal}}$ map $f: X \xrightarrow{\text{onto}} Y$.

Remark: A map $f: X \xrightarrow{\text{onto}} Y$ is conformal iff so is its inverse $f^{-1}: Y \xrightarrow{\text{onto}} X$.
In particular, the relation $\simeq$ is an equivalence.

$\boxed{\text{Example}}$ Let S^2 be the unit sphere in $\mathbb{R}^3$, i.e.,

$$S^2 = \{[x_1, x_2, x_3] \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}.$$

Then $S^2 \simeq \$:= \mathbb{C} \cup \{\infty\}$ because the stereographic projection $\phi: S^2 \xrightarrow{\text{onto}} \$$ is a conformal map.

DEF. Let X be a RS. Then a holomorphic map $f: X \to \mathbb{C}$ is called a holomorphic function on X. Notation: $\mathrm{Hol}(X)$

$\boxed{\text{Example}}$ $\mathrm{Hol}(\$) = \{\text{constant functions}\}$
[HINT: USE LIOUVILLE'S THEOREM.]

Remark: We show that if X is a compact connected RS, then $\mathrm{Hol}(X) = \{\text{constant fun's}\}$

$\boxed{\text{Example}}$ Let $X \subset \mathbb{C}$ be a domain. Then a function f on X is called meromorphic if, at each $z_0 \in X$, either f is holomorphic or f has a pole. Putting $f \equiv \infty$ on the set P_f of poles of f, we have that $f: X \to \hat{\mathbb{C}}$ is continuous. Moreover, $f: X \to \hat{\mathbb{C}}$ is a holomorphic map in the sense of RS iff either f is meromorphic on X, or $f \equiv \infty$ on X.

[Indeed, f has a pole at z_0 of degree p iff $1/f$ has a zero at z_0 of degree p.]

DEF. Let X be a RS. Then a holomorphic map $f: X \to \hat{\mathbb{C}}$ is called a meromorphic function on X if ~~it is not~~ $f \not\equiv \infty$ on any component of X. Notation: $M(X)$

$\boxed{\text{Example}}$ $M(\hat{\mathbb{C}}) = \{\text{rational functions}\}$

Unique presentation by principal parts

Let $f \in M(\mathbb{S})$. Then

$$(*) \quad f(z) = \sum_{i=1}^{m} \underbrace{\sum_{j=1}^{m_i} \frac{a_{ij}}{(z-p_i)^j}}_{\substack{\text{the principal part of} \\ \text{the Laurent series} \\ \text{of } f \text{ near a pole } p_i}} + p(z)$$

where p is a polynomial, p_i are the finite poles of f, $a_{i m_i} \neq 0$ and m_i is the degree of the pole p_i.

Remark: This is an expression of a rational function f into simple fractions.

Unique presentation by zeros and poles

Let $0 \not\equiv f \in M(\mathbb{S})$. Then

$$(\Delta) \quad f(z) = c \cdot \frac{\prod_{i=1}^{m} (z - z_i)}{\prod_{j=1}^{m} (z - p_j)}$$

where $c \neq 0$, z_i are the finite zeroes of f and p_j are the finite poles of f (repeated as necessary).

Remark: We have analogues of unique presentations (*) and (Δ) for arbitrary compact RS.

$\boxed{\text{Observation}}$ Every meromorphic function $f \not\equiv 0$ on $\mathbb{S}$ has just as many zeroes as poles, if multiplicities are counted.

Proof: If $n \geq m$, then ∞ is a pole of the degree $n-m$. If $n < m$, then ∞ is a zeroes of the degree $m-n$. ▨