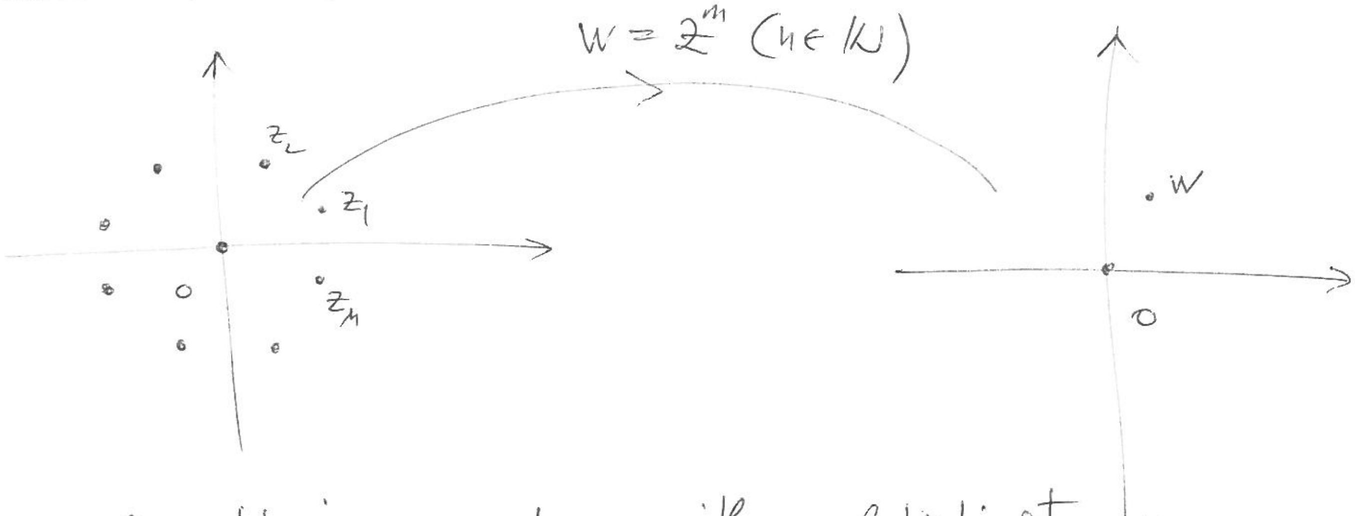
Example



- $w = z^n$ attains 0 at 0 with multiplicity n
  $w \neq 0$ at n different points $z_1, \dots, z_n \neq 0$
  with multiplicity 1
- We show that any holomorphic map locally looks like $w = z^n$ for some $n \in \mathbb{N}$.

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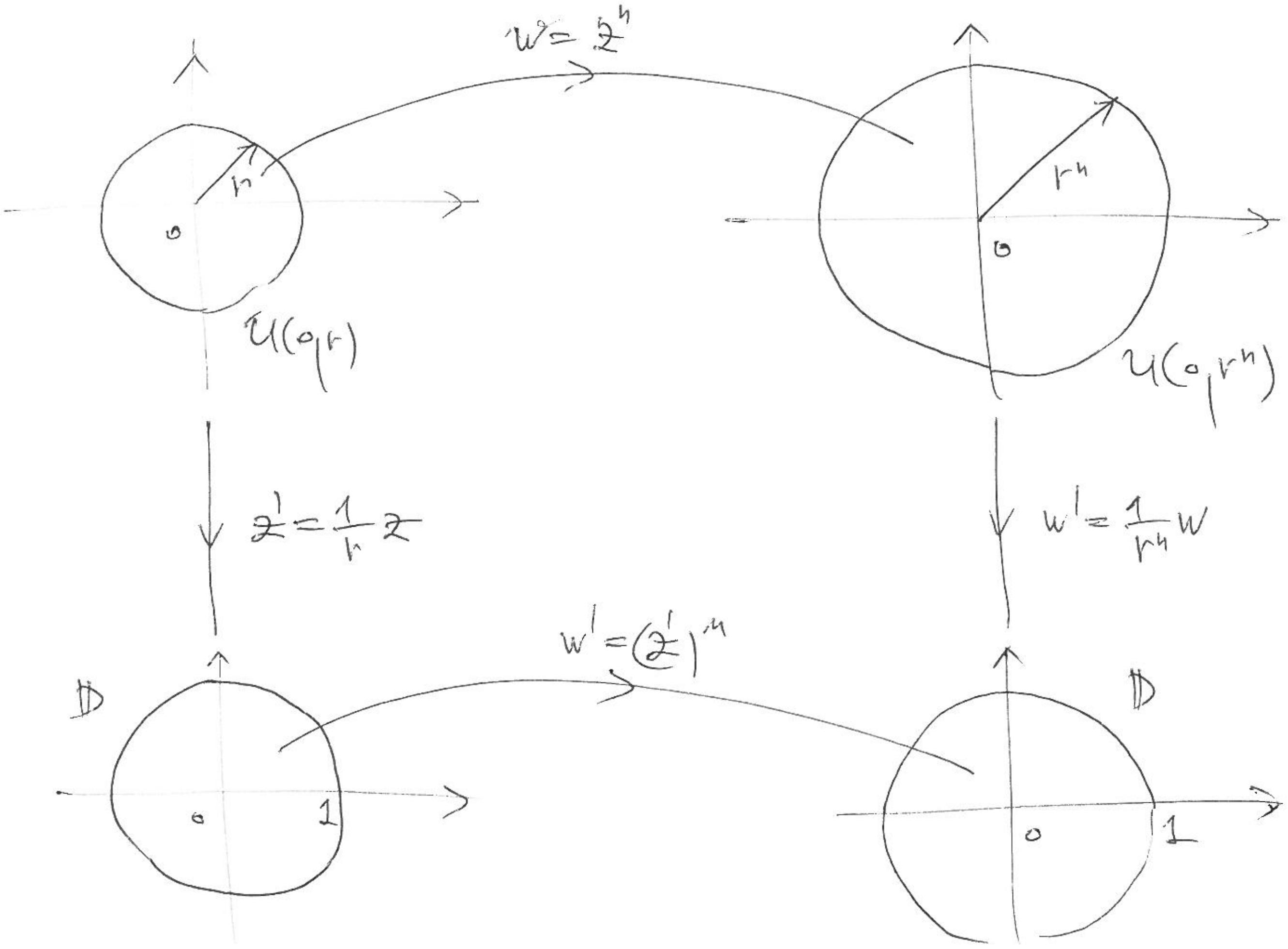
Denote $U(z_0, r) := \{ z \in \mathbb{C} \mid |z - z_0| < r \}$ and $\mathbb{D} := U(0,1)$.



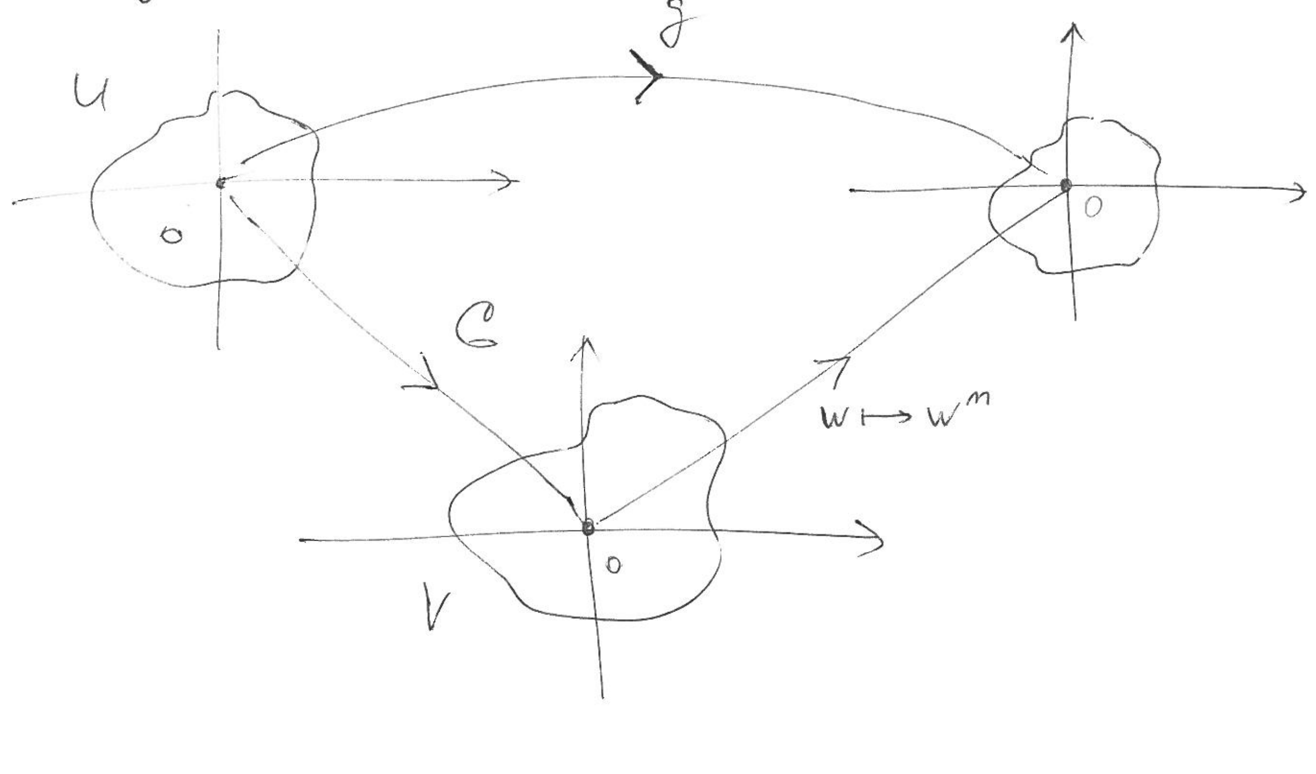
The local form of a holomorphic map

LEMMA: Let g be a holomorphic function on a ngh. of $0 \in \mathbb{C}$, $g(0)=0$ and g is not constant on any ngh. of 0. Then there is a unique $n \in \mathbb{N}$ such that

1. $g(0)=0=g'(0)=\dots=g^{(n-1)}(0)$, $g^{(n)}(0)\neq 0$;
2. there are open ngh. U, V of 0 and a conformal map $\mathcal{C}: U \xrightarrow{\text{onto}} V$ such that

$$g = \mathcal{C}^n \text{ on } \mathcal{U}.$$

The number n is the multiplicity of the zero 0 of g.



RS18

Pf: Consider the Taylor series expansion of g about 0 and let n be the ~~lowest~~ order of the first non-zero term (such must exist!):

$$g(z) = \underset{\neq 0}{a_n} z^n + a_{n+1} z^{n+1} + \cdots$$

Since $a_k = g^{(k)}(0)/k!$ we get (1).

Moreover, we have

$$g(z) = a_n z^n h(z)$$

where $h(z) = 1 + b_1 z + b_2 z^2 + \cdots$ with $b_i = \dfrac{a_{n+i}}{a_n}$.

There is a ngh $\tilde{U}$ of 0 such that

$$h(\tilde{U}) \subset \mathbb{C} \setminus (-\infty, 0].$$

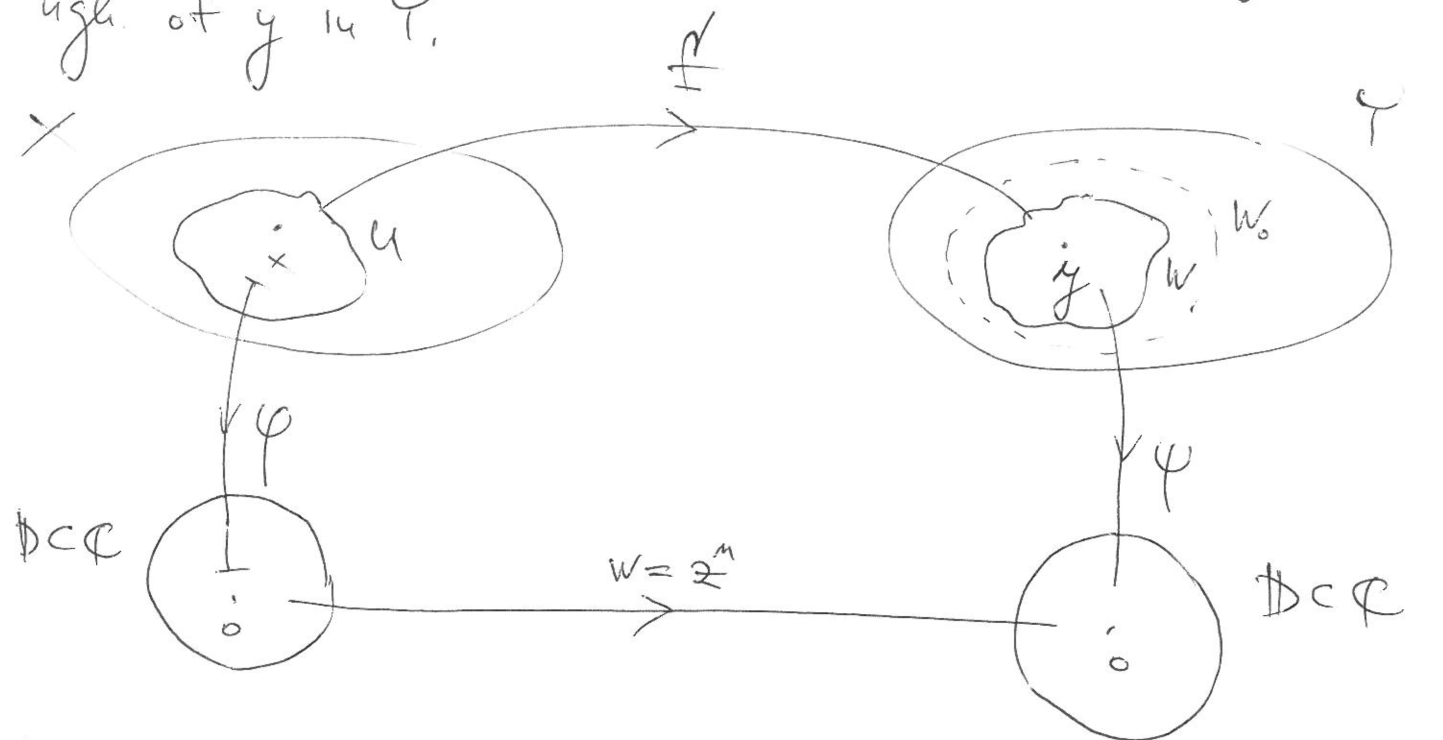
Then $H(z) := \sqrt[n]{h(z)}$, $z \in \tilde{U}$ is holomorphic and, putting $G(z) := \sqrt[n]{a_n}\, z H(z)$, $z \in \tilde{U}$, we get $g(z) = (G(z))^n$, $z \in \tilde{U}$.

Here $\sqrt[n]{w} := \exp\left(\frac{1}{n} \log w\right)$, $w \in \mathbb{C} \setminus \{0\}$ is the principal value of the n-th root of w.

Since $G'(0) = \lim_{z \to 0} \dfrac{G(z)}{z} = \sqrt[n]{a_n} \underbrace{\lim_{z\to 0} H(z)}_{= H(0) = 1} = \sqrt[n]{a_n} \neq 0$

there are ngh $U \subset \tilde{U}$ and V of 0 such that $G : U \xrightarrow{\text{onto}} V$ is a conformal map. ▨

$\boxed{\text{Theorem}}$ Let X, Y be RS, $f: X \to Y$ be holomorphic, $x \in X$ and $y = f(x)$. Assume that f is not constant on any ngh of x. Let W_0 be a given ngh of y in Y.



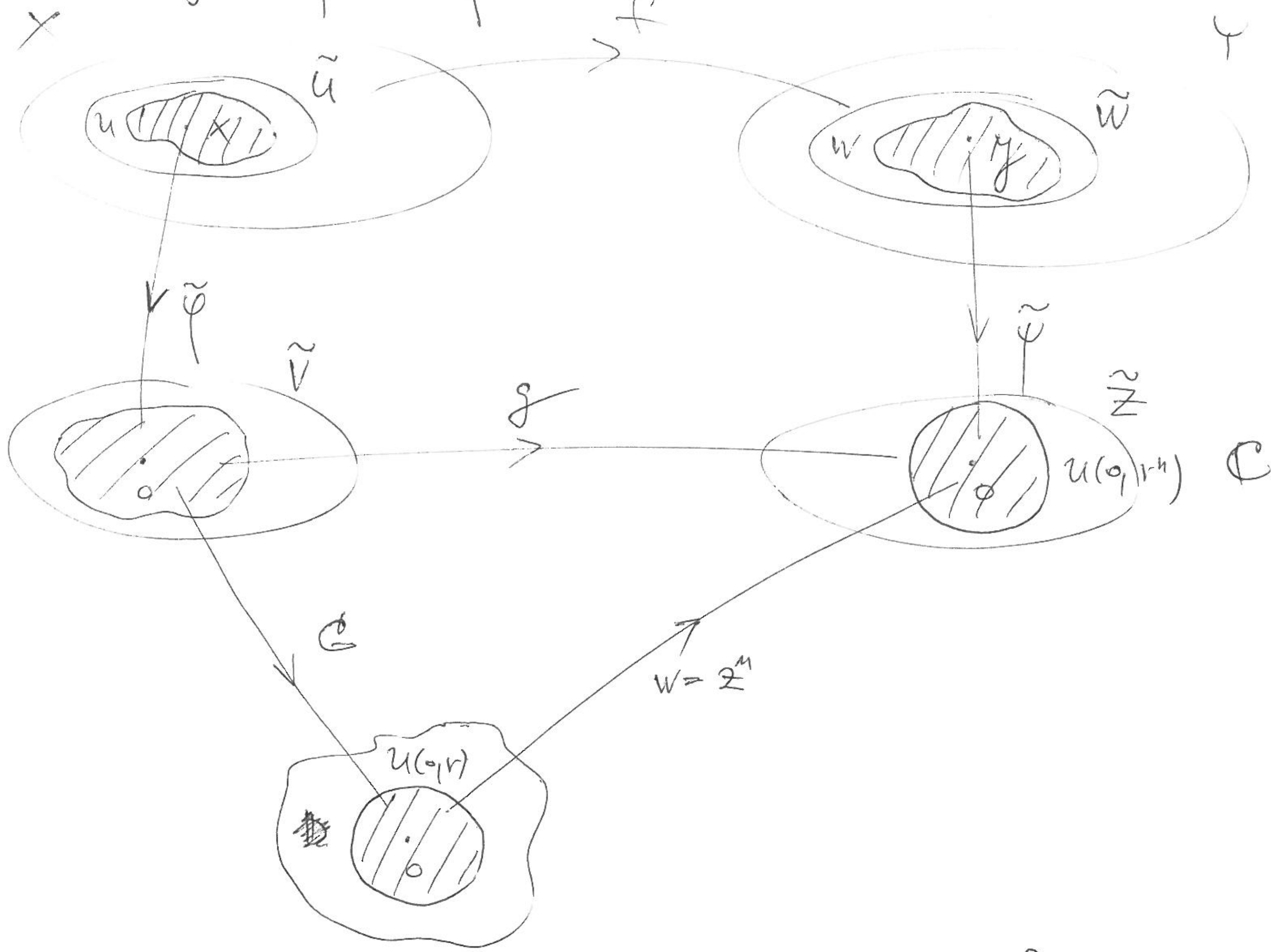
Then there are local co-ordinates $(U, \mathbb{D}, \varphi)$ on X and $(W, \mathbb{D}, \psi)$ on Y such that $x \in U$, $\varphi(x) = 0$, $y \in W$, $\psi(y) = 0$, $W \subset W_0$ and, for some $n \in \mathbb{N}$,

$$\psi \circ f \circ \varphi^{-1}(z) = z^n, \quad z \in \mathbb{D}.$$

<u>Pf</u>: Let $(\tilde{U}, \tilde{V}, \tilde{\varphi})$ be a local chart on X,
and $(\tilde{W}, \tilde{Z}, \tilde{\psi})$ —"— on Y
such that $\tilde{W} \subset W_0$ and $f(\tilde{U}) \subset \tilde{W}$.
$x \in \tilde{U}$, $y \in \tilde{W}$

WLOG (= without loss of generality), we can assume that $\tilde{\varphi}(x) = 0 = \tilde{\psi}(y)$ (otherwise, consider $\tilde{\varphi} - \tilde{\varphi}(x)$ and $\tilde{\psi} - \tilde{\psi}(y)$).

Put $g := \tilde{\psi} \circ f \circ \tilde{\varphi}^{-1}$ on $\tilde{V}$.



Take G for g as in LEMMA. Then, for $r > 0$ small enough, put

$$U := \tilde{\varphi}^{-1} \circ G^{-1}(U(0,r)),$$
$$\varphi := G \circ \tilde{\varphi}|_U$$
$$W := \tilde{\psi}^{-1}(U(0,r^n))$$
$$\psi := \tilde{\psi}|_W$$

To get $r = 1$, apply a simple change of co-ordinates as in Example on page $\boxed{\text{RS } 16.5}$.

∎

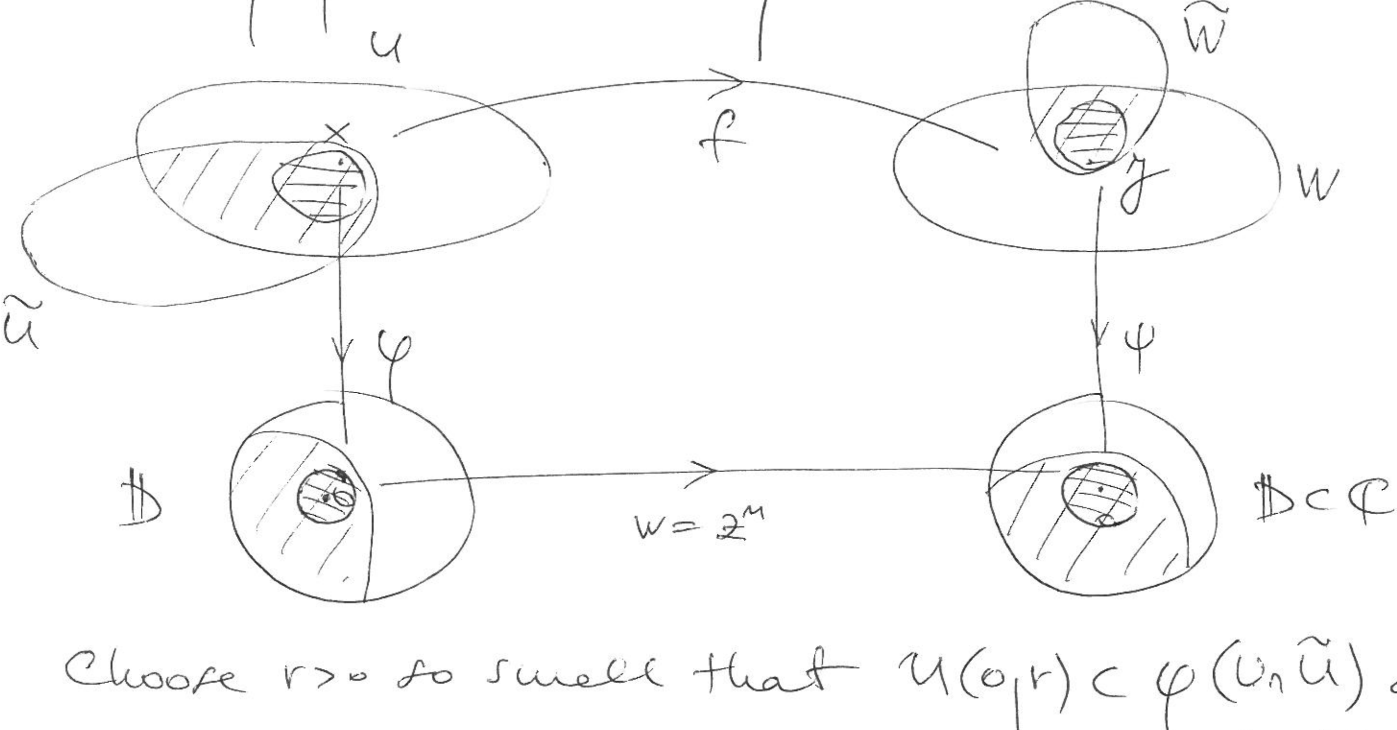
REMARK: The number n from THEOREM does not depend on the choice of local co-ordinates. We write $n_f(x) = n$ and call $n_f(x)$ the valency of f at x.

To prove uniqueness of n, we use the following property: For U, W, n as in THEOREM, we have that, for each $w \in W \setminus \{y\}$, the set

$$f^{-1}(\{w\}) \cap (U \setminus \{x\})$$

has just n elements.

⌈Let $\tilde{U}, \tilde{W}, \tilde{n}$ also satisfy THEOREM. Then



Choose $r > 0$ so small that $\mathcal{U}(0, r) \subset \varphi(U \cap \tilde{U})$ and $\mathcal{U}(0, r^n) \subset \psi(W \cap \tilde{W})$.

For each $w \in W$ such that $0 \neq \psi(w) \in \mathcal{U}(0, r^n)$, we have $f^{-1}(\{w\}) \cap (U \setminus \{x\}) \subset f^{-1}(\{w\}) \cap (\tilde{U} \setminus \{x\})$, hence $n \leq \tilde{n}$. Similarly, $n \geq \tilde{n}$, and $n = \tilde{n}$. ∎⌋

Corollary 1 Let $f: X \to Y$ be a holomorphic map between RS X and Y. If f is constant on some ngh of a point $x_0 \in X$, then f is constant on the (connected) component of X which contains x_0.

Pf: WLOG, assume that X is connected.

Put $M := \{x \in X \mid f \text{ is constant on some ngh. of } x\}$.

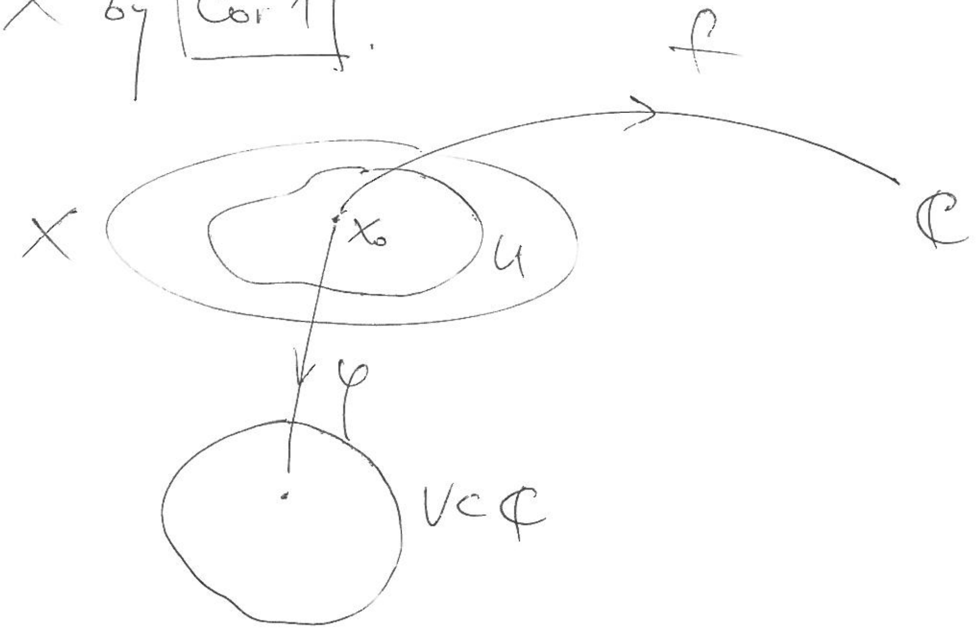
Then $\emptyset \neq M$ is open in X. But M is also closed in M. Indeed, for each $x \in X \setminus M$, the function f on some ngh of x looks like $w = z^m$ for some $m \in \mathbb{N}$ by THEOREM. ▨

Corollary 2 (Maximum modulus principle)

Let X be a connected RS and $f \in \mathcal{H}ol(X)$. If $|f|$ attains a local maximum on X, then f is constant on X.

Pf: Let (U, V, φ) be a chart on X such that, for some $x_0 \in U$, we have $|f(x_0)| \geq |f|$ on U. WLOG, we can assume that $V \subset \mathbb{C}$ is connected (otherwise, take the component V containing $\varphi(x_0)$). Then, by the classical maximum modulus principle, $f \circ \varphi^{-1}$ is constant on V. Hence

f is constant on $\mathcal{U}$, and on the whole of X by $\boxed{\text{Cor 1}}$.



$\boxed{\text{Corollary 3}}$ If X is a compact connected RS, then $Hol(X) = \{\text{constant functions on } X\}$.

Pf: It follows from $\boxed{\text{Cor 2}}$ because $|f|$ attains a maximum on the compact X. $\blacksquare$