Consequences of the theorem on the local form

Let $f: X \to Y$ be a non-constant holomorphic map between RSs and X be connected, then

(a) f is an open map, i.e., for each open $G \subset X$, $f(G)$ is open in Y.

(b) For each $y \in Y$, the set $f^{-1}(\{y\}) = \{x \in X \mid f(x) = y\}$ has no accumulation points in X.

(c) The set $\{x \in X \mid n_f(x) > 1\}$ has no accumulation points in X.

(d) TEST FOR A LOCAL INVERSE: FSAE (= the following statements are equivalent to each another):

(i) $m_f(x) = 1$

(ii) there is a ngh $\mathcal{U}$ of x such that $f|_\mathcal{U}$ is one-to-one

(iii) there are open ngh $\mathcal{U}$ of x and V of y such that $f|_\mathcal{U}: \mathcal{U} \xrightarrow{\text{onto}} V$ is a conformal map.

$\boxed{\text{THEOREM}}$ Let $f: X \to Y$ be a non-constant holomorphic map and let X be compact and connected. Then $f(X)$ is a compact component of Y. In particular, a non-constant holomorphic map between compact connected RSs is surjective.

Pf: Since f is continuous $f(X)$ is compact (closed) and connected. Since f is open $f(X)$ is open in Y. Then $f(X)$ is a component (i.e., a maximal connected subset) of Y. Indeed, let $M \supsetneq f(X)$. Then

$$M = \underbrace{(M \cap f(X))}_{\neq \emptyset} \cup \underbrace{(M \cap (Y \setminus f(X)))}_{\neq \emptyset}.$$

open in M

$\boxed{\text{Corollary}}$ (Fundamental theorem of algebra) Each complex polynomial P of the degree at least 1 has a root.

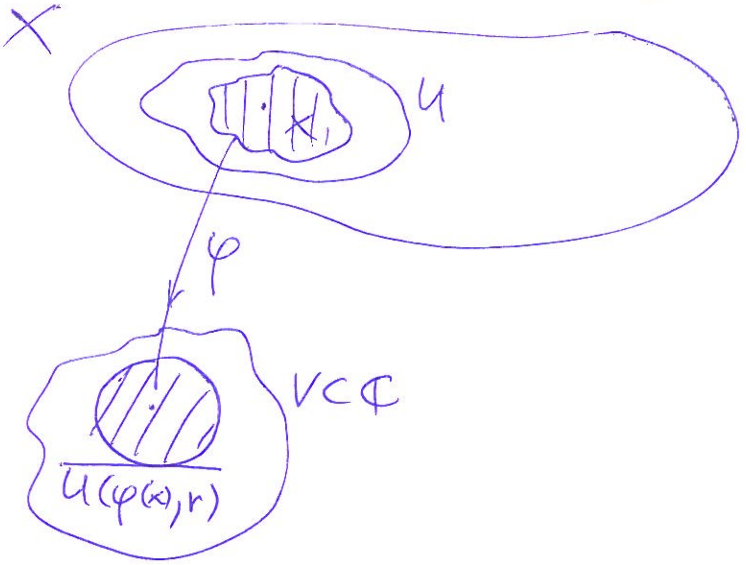
⌈ Indeed, $P: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a non-constant holomorphic map and, by theorem, P is onto. ⌋

Remark: We know more. In fact, each polynomial of the degree d has just d roots, if multiplicities are counted. In what follows, we generalize this too.

Locally compact spaces and proper mappings

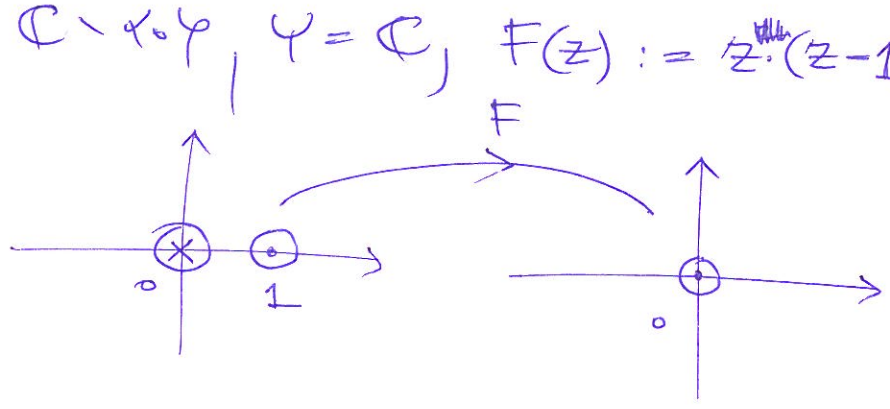
DEF. A topological space X is called locally compact if X is Hausdorff and each $x \in X$ admits a compact ngh.

Example Every RS is locally compact.
manifold



DEF. A continuous map $f: X \to Y$ between locally compact spaces X, Y is called proper if, for each compact $K \subset Y$, $f^{-1}(K)$ is compact.

Example (i) $X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C}$, $F(z) := z \cdot (z-1)$ is not proper



(ii) Let $f: X \to Y$ be holomorphic and let X be compact. Then f is proper. RS 27

⌈ Let K be a compact in Y. Since Y is Hausdorff, K is closed, and $f^{-1}(K)$ is closed in X by the continuity of f. Since X is a compact $f^{-1}(K)$ is a compact too. ⌋

$\boxed{\text{LEMMA 1}}$ Let $f: X \to Y$ be a proper map. Then $f(A)$ is closed in Y for each closed $A \subset X$.

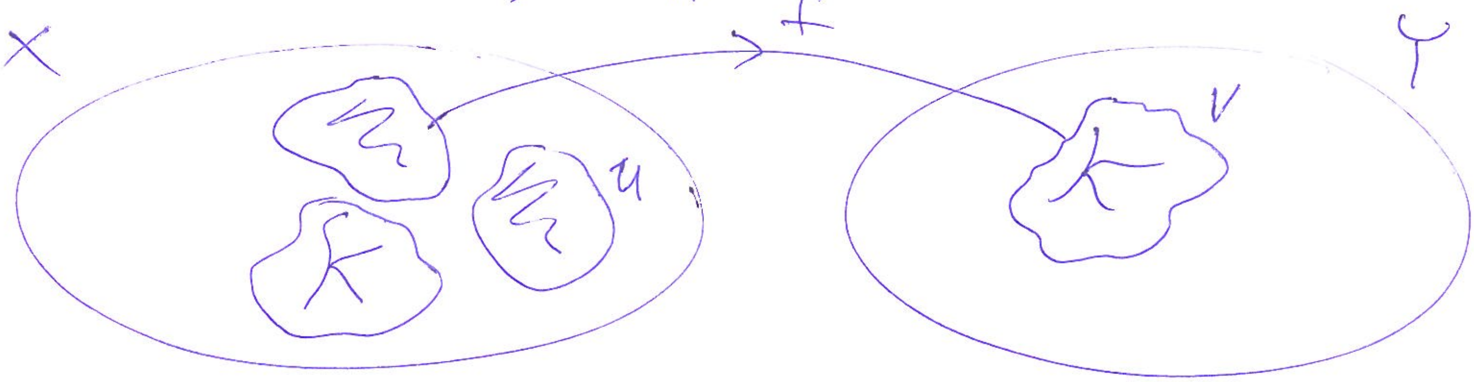
Pf: (i) $A \subset X$ is closed iff, for each compact K in X, $K \cap A$ is compact. Indeed, the implication $\Rightarrow$ is obvious. As for $\Leftarrow$, assume that $A \subset X$ is not closed. Choose $x \in \overline{A} \setminus A$ and a compact nbh. K of x. Then $x \in \overline{A \cap K} \setminus (A \cap K)$, and $K \cap A$ is not compact.

(ii) Let A be a closed set in X and K be a compact in Y. Then

$$K \cap f(A) = f(\underbrace{A \cap \underbrace{f^{-1}(K)}_{\text{cpt}}}_{\text{cont.}}) \text{ is compact.}$$ ▨

$\boxed{\text{LEMMA 2}}$ Let $f : X \to Y$ be <u>proper</u>.

Let K be a compact in Y and $U \subset X$ be open such that $f^{-1}(K) \subset U$.



There is an open set V in Y such that $K \subset V$ and $f^{-1}(V) \subset U$.

<u>Pf</u>: By Lemma 1, since $X \setminus U$ is closed and f is proper $f(X \setminus U)$ is closed in Y. Put

$$V := Y \setminus f(X \setminus U). \quad \blacksquare$$

THEOREM Let $f: X \to Y$ be a non-constant <u>proper</u> holomorphic map between <u>connected</u> R.S. X and Y. Then f is surjective and, for each $y \in Y$, $f^{-1}(y)$ is a finite set.

Put $$d(y) := \sum_{x \in f^{-1}(y)} m_f(x), \quad y \in Y.$$

Then the number $d(y)$ does not depend of $y \in Y$. We write $\deg f := d(y)$, $y \in Y$, and call $\deg(f)$ <u>the degree</u> of f.

<u>Remark</u>: (i) For a <u>constant</u> $f: X \to Y$, we define $\deg f := 0$.

(ii) $d(y)$ is the number of solutions $x \in X$ the equation $y = f(x)$, if multiplicities are counted.

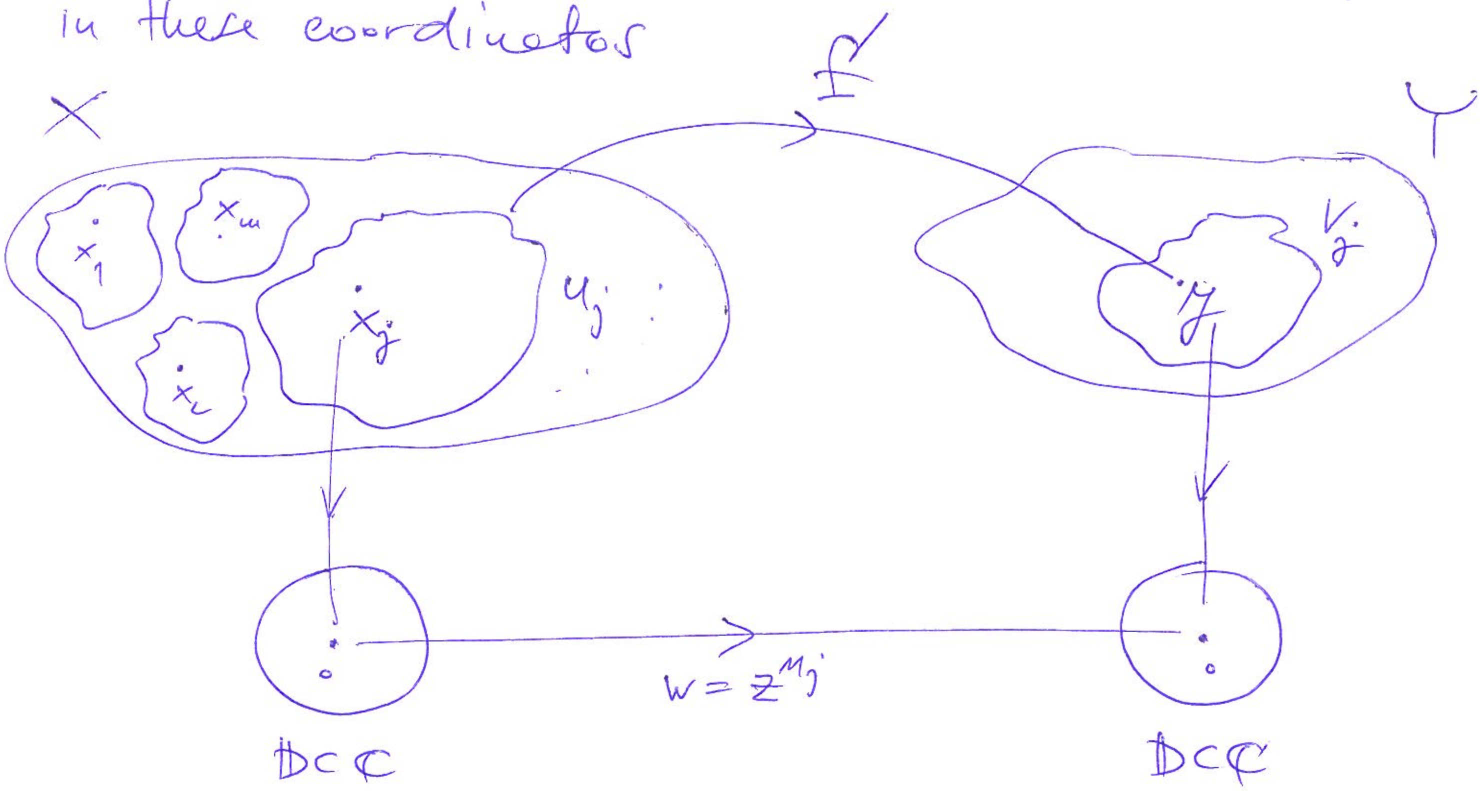
(iii) Actually, $\deg f$ is a 'topological invariant'.

<u>Pf</u>: We have that $f(X) = Y$ because $f(X)$ is open, closed (by LEMMA 1) and connected. Moreover, for each $y \in Y$, $f^{-1}(y)$ is a compact without accumulation points in X.

To prove that d is constant on Y it is sufficient to show that d is locally constant on Y because Y is connected.

— x —

Let $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_m\}$. For each $j = 1, \dots, m$, there are local co-ordinates $(U_j, \mathbb{D}, \varphi_j)$ around x_j and $(V_j, \mathbb{D}, \psi_j)$ around y such that f looks like $w = z^{n_j}$ for some $n_j \in \mathbb{N}$ in these coordinates



WLOG, we can assume that $U_i \cap U_j = \emptyset$ for $i \neq j$.

Put $V := V_1 \cap \dots \cap V_m$.

WLOG, we can assume that $f^{-1}(V) \subset U_1 \cup \dots \cup U_m$ using LEMMA 2. Then $d = n_1 + \dots + n_m$ on V. ▨