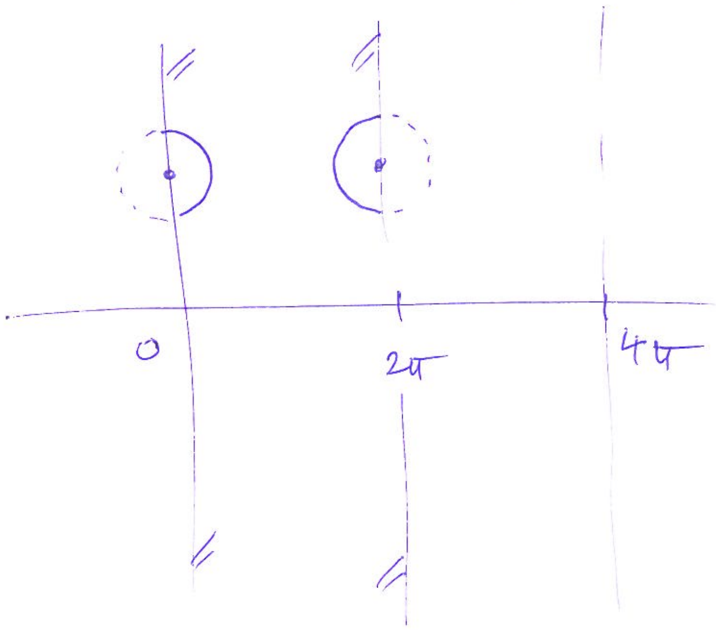
QU01

Cylinder

For $z_1, z_2 \in \mathbb{C}$ we write $z_1 \sim z_2$ if $z_2 = z_1 + 2\pi n$ for some $n \in \mathbb{Z}$. Denote by $\mathbb{C}/_{2\pi\mathbb{Z}}$ the set of equivalence classes w.r.t. $\sim$ and by $\pi: \mathbb{C} \longrightarrow \mathbb{C}/_{2\pi\mathbb{Z}}$ the canonical projection.



We endow $\mathbb{C}/2\pi\mathbb{Z}$ with the quotient topology, that is, $U \subset \mathbb{C}/2\pi\mathbb{Z}$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{C}$.

We can view $\mathbb{C}/2\pi\mathbb{Z}$ as the strip

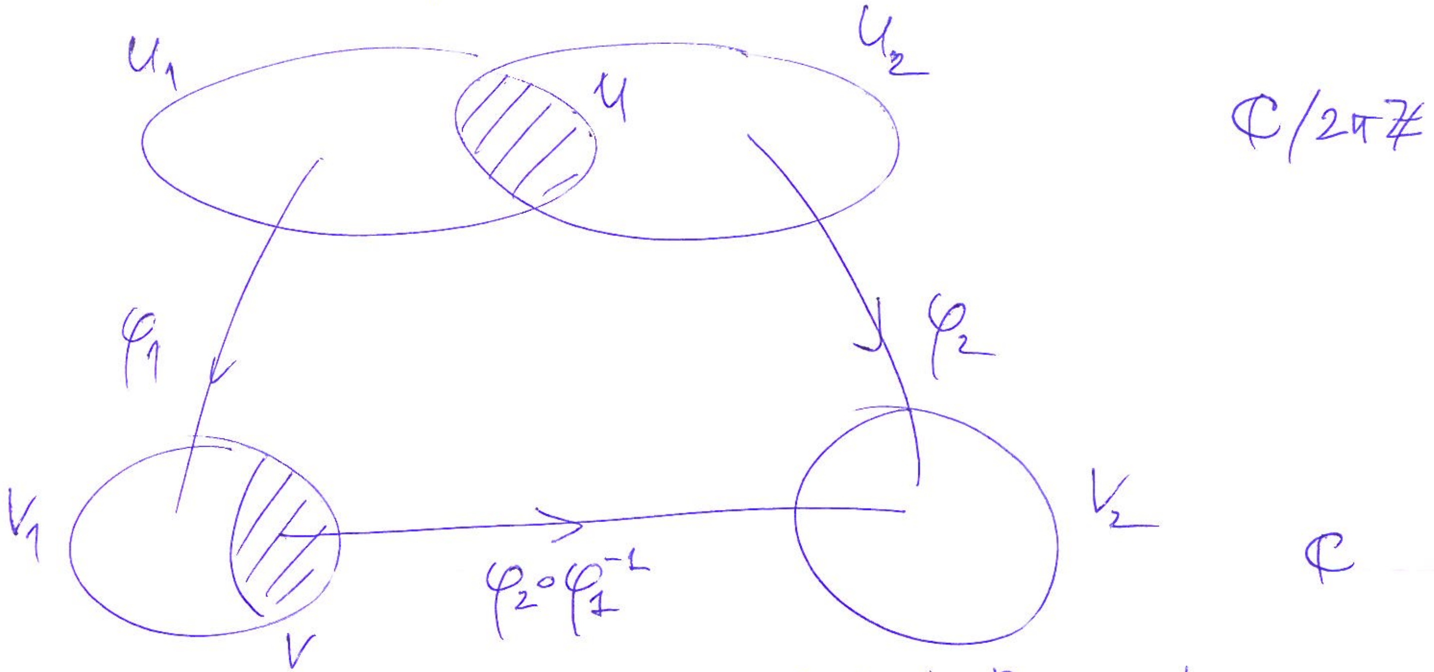
$$\{z \in \mathbb{C} \mid 0 \leq \operatorname{Re} z \leq 2\pi\}$$

but we identify the corresponding points on the opposite edges. In other words, by gluing together the opposite edges of the strip we get a cylinder in $\mathbb{R}^3$.

Exercise Find a homeomorphism $\mathbb{C}/2\pi\mathbb{Z}$ onto the cylinder $S^1 \times \mathbb{R}$ where S^1 is the unit circle in $\mathbb{R}^2 \simeq \mathbb{C}$.

Conformal structure

Let $V \subset \mathbb{C}$ be open such that no two different points in V are equivalent under $\sim$. Then $\pi|_V$ is a homeomorphism of V onto an open set U in $\mathbb{C}/2\pi\mathbb{Z}$ and $(U, V, (\pi|_V)^{-1})$ is a local map on $\mathbb{C}/2\pi\mathbb{Z}$. Actually, such local maps endow $\mathbb{C}/2\pi\mathbb{Z}$ with a conformal structure. Indeed, let (U_1, V_1, φ_1) and (U_2, V_2, φ_2) be two such local maps with $U := U_1 \cap U_2 \neq \emptyset$.



We prove, say, that $\varphi_2 \circ \varphi_1^{-1}$ is holomorphic on $V := \varphi_1(U)$. WLOG, assume that U is connected (otherwise, take any component of U).

Then, for each $z \in V$,

$$\varphi_2 \circ \varphi_1^{-1}(z) = z + 2\pi n(z)$$

where $n : V \to \mathbb{Z}$. Since V is a domain and n is continuous n is constant on V.

Functions on $\mathbb{C}/2\pi\mathbb{Z}$

Let f be a function on $\mathbb{C}/2\pi\mathbb{Z}$. Then $F := f \circ \pi$ is a function on $\mathbb{C}$ which is <u>2π-periodic</u>, that is,

$$F(z + 2\pi) = F(z), \quad z \in \mathbb{C}.$$

Obviously, the correspondence $f \mapsto F$ between functions on $\mathbb{C}/2\pi\mathbb{Z}$ and 2π-periodic fun's on $\mathbb{C}$ is one-to-one. Moreover, a function f on $\mathbb{C}/2\pi\mathbb{Z}$ is continuous, holomorphic or meromorphic iff so is the corresponding F.

Exercise

QV04

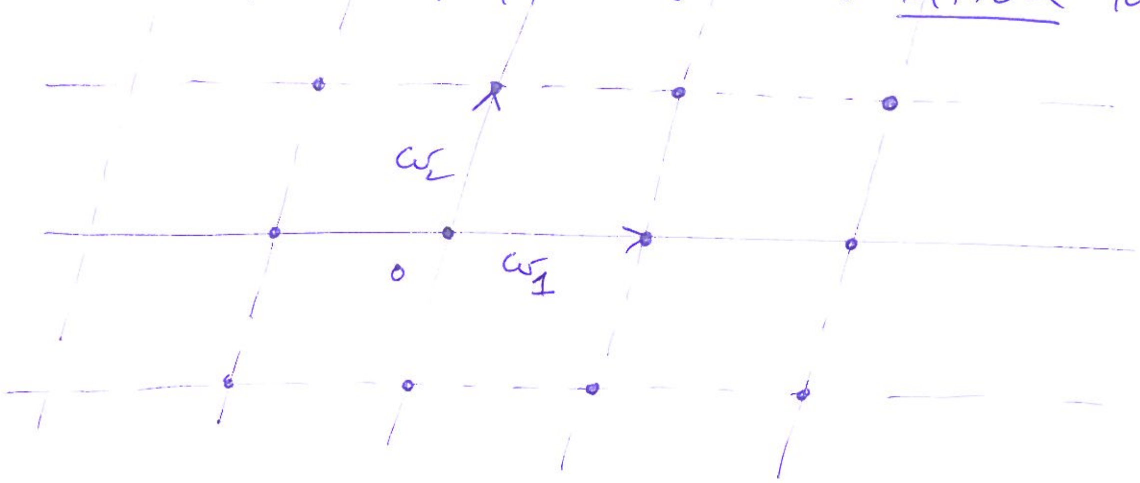
1. We can define the cylinder $\mathbb{C}/\mathbb{Z}\omega$ for any $0 \neq \omega \in \mathbb{C}$ (instead of 2π).

2. All cylinders are conformally equivalent. In particular, find a conformal transformation of $\mathbb{C}/\mathbb{Z}\omega$ onto $\mathbb{C} \setminus \{0\}$.

[HINT: $z \in \mathbb{C} \longmapsto \exp(2\pi i z/\omega) \in \mathbb{C} \setminus \{0\}$]

Torus

Let ω_1, ω_2 are linearly independent vectors in the plane, i.e., $0 \neq \omega_1, \omega_2 \in \mathbb{C}$ and $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$.
Let $L := \{ n\omega_1 + m\omega_2 \mid n, m \in \mathbb{Z} \}$ be a lattice in $\mathbb{C}$.



We write $z_1 \sim_L z_2$ for $z_1, z_2 \in \mathbb{C}$ if $z_2 - z_1 \in L$.
Denote by $T_L := \mathbb{C}/L$ the set of equivalence classes with respect to $\sim_L$ and by $\pi_L : \mathbb{C} \longrightarrow \mathbb{C}/L$ the canonical projection.
I shall often omit the subscript L on T_L and π_L. We endow T with the quotient topology, that is, $U \subset T$ is open iff $\pi^{-1}(U)$ is open in $\mathbb{C}$.

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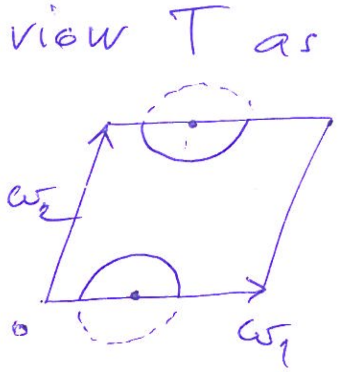
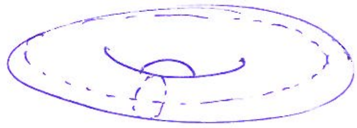
We can view T as the closed parallelogram



but we identify the corresponding points on the opposite edges. In other words, by gluing together the opposite edges of the parallelogram we get a torus in $\mathbb{R}^3$.

Exercise Find explicitly homeomorphisms between T, $S^1 \times S^1$ and T^2. Here $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle in $\mathbb{C}$ and T^2 is a torus in $\mathbb{R}^3$.



Hint: $[\lambda\omega_1 + \mu\omega_2] \in T \longmapsto (e^{2\pi i\lambda}, e^{2\pi i\mu}) \in S^1 \times S^1$
$\lambda, \mu \in \mathbb{R}$

Conformal structure on T

Let $V \subset \mathbb{C}$ be open such that no two different points in V are equivalent under $\sim_L$. Then $\pi|_V$ is a homeomorphism of V onto an open U in T and $(U, V, (\pi|_V)_{-1})$ is a local map on T. Actually, such local maps endow T with a conformal structure. Indeed, we have

Exercise Any transition function ϕ between such local maps on each component of its domain of definition is a translation $\phi(z) = z + n\omega_1 + m\omega_2$ for some $n, m \in \mathbb{Z}$.

Functions on T

Let f be a function on T. Then $F := f \circ \pi$ is a function on $\mathbb{C}$ which is doubly periodic with periods ω_1 and ω_2, that is,

$$F(z + \omega_1) = F(z) = F(z + \omega_2), \quad z \in \mathbb{C}.$$

Obviously, the correspondence $f \longmapsto F$ between functions on T and doubly periodic functions on $\mathbb{C}$ is one-to-one. Moreover, a function f on T is continuous, holomorphic or meromorphic iff so is the corresponding F.

Example Doubly periodic holomorphic functions on $\mathbb{C}$ are constant. In other words,

$$\mathrm{Hol}(T) = \{\text{constant functions}\}.$$

DEF. Elliptic functions are doubly periodic meromorphic functions on $\mathbb{C}$, i.e., the meromorphic functions on T.

$\boxed{\text{Exercise}}$ Of course, all tori ~~tori~~ are homeomorphic. Easily, T_L is conformally equivalent to $T_\tau := T_{\mathbb{Z}+\mathbb{Z}\tau}$ for some $\tau \in \mathbb{H}$ where

$$\mathbb{H} := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}$$

is the upper half-plane. [take $\tau := \frac{\omega_2}{\omega_1}$ or $\frac{\omega_1}{\omega_2}$.]

Moreover, $T_\tau \simeq T_{\tau'}$ iff $\tau' = \varphi(\tau)$ for some linear fractional transformation of the form

$$(*) \quad \varphi(z) = \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$.

The transformations $(*)$ form the so-called modular group $\underbrace{}_{PSL(2,\mathbb{Z})}$. Then $\mathbb{H}/PSL(2,\mathbb{Z})$ is the moduli space for tori ~~tori~~, that is, the space parametrising the conformal classes of tori. Here $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm 1\}$.