Quotients

Let X be a RS and let $\mathrm{Aut}(X)$ be the group of conformal transformations of X onto X. Let Γ be a subgroup of $\mathrm{Aut}(X)$ such that the action of Γ on X is <u>properly discontinuous</u>, that is, for each $x \in X$ there is a ngh. U of x such that the sets gU, $g \in \Gamma$, are pairwise disjoint. (PD)

In particular, the action Γ on X is <u>free</u>, that is, if $g \in \Gamma$, $x \in X$ and $gx = x$, then g is the identity map. In many interesting cases, the free action of Γ implies (PD).

For $x, y \in X$ we write $x \sim_\Gamma y$ if $y = gx$ for some $g \in \Gamma$.

We denote by X/Γ the set of equivalence classes w.r.t. $\sim_\Gamma$. Due to (PD) it is possible to introduce a <u>conformal structure</u> on X/Γ as for cylinders and tori. $\boxed{\text{Exercise}}$

Example

(1) <u>Cylinder</u>: $\mathbb{C}/\mathbb{Z}\omega \simeq \mathbb{C}/\Gamma$ where

$\Gamma := \{ g_n \mid n \in \mathbb{Z} \} \simeq \mathbb{Z}$ with $g_n(z) := z + n\omega$, $z \in \mathbb{C}$

(2) <u>Torus</u>: $\mathbb{C}/\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \simeq \mathbb{C}/\Gamma$ where

$\Gamma := \{ g_{n,m} \mid n, m \in \mathbb{Z} \} \simeq \mathbb{Z} \times \mathbb{Z}$ with

$$g_{n,m}(z) := z + n\omega_1 + m\omega_2$$

For a function f on X/Γ, $F := f \circ \pi$ is the corresponding function on X which is <u>Γ-invariant</u>, i.e., $\forall g \in \Gamma \ \forall x \in X$

$$F(gx) = F(x).$$

This correspondence is again one-to-one and a function f on X/Γ is continuous, holomorphic, meromorphic etc. iff so is the corresponding F on X, here $\pi : X \longrightarrow X/\Gamma$ is the canonical projection.

For example,

$$\underset{\text{meromorphic}}{M(X/\Gamma)} \cong \{ F \in M(X) \mid F \text{ is } \Gamma\text{-invariant} \}.$$

Example (i.e., quotients of $\hat{\mathbb{C}}$)

1. Elliptic RS[*]: $X \simeq \hat{\mathbb{C}}$

We know that $\mathrm{Aut}(\hat{\mathbb{C}})$ is the group of linear fractional maps (also Möbius transformations)

$$z \longmapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$.
Each $g \in \mathrm{Aut}(\hat{\mathbb{C}})$ has a fixed point on $\hat{\mathbb{C}}$,*) the only $\Gamma \subset \mathrm{Aut}(\hat{\mathbb{C}})$ with (PD) is trivial, i.e., $\Gamma = \{id\}$.

2. Parabolic RS (i.e., quotients of $\mathbb{C}$):

$X \simeq \mathbb{C}$, $\mathbb{C} \setminus \{0\}$ (cylinder), $\mathbb{C}/L$ (tori)

We know that $\mathrm{Aut}(\mathbb{C})$ is the group of linear maps

$$z \longmapsto az + b$$

where $a, b \in \mathbb{C}$ and $a \neq 0$.[1] Such a map has no fixed point on $\mathbb{C}$ iff $a = 1$ and $b \neq 0$, that is, it is a non-trivial translation.

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[1] Indeed, $\varphi \in \mathrm{Aut}(\mathbb{C})$ is holomorphic and one-to-one on $\mathbb{C}$. Thus φ has a simple pole at ∞.

*) GAMELIN, Complex Analysis, 2000, Exercise 12 for II.7, p. 69.

(3.) <u>Hyperbolic</u> RS (i.e., quotients of $\mathbb{D} \simeq \mathbb{H}$): $X \simeq \mathbb{H}/\Gamma$ where Γ is a

unit disc | upper half-plane

QUOT

<u>discrete</u> subgroup of $\mathrm{Aut}(\mathbb{H})$ such that its action on $\mathbb{H}$ is <u>free</u>. It is known that Γ is (FD). Here Γ is discrete if there is a ngh of the identity in $\mathrm{Aut}(\mathbb{H})$ consisting only of the identity.

We know that $\mathrm{Aut}(\mathbb{H})$ is the group of maps

$$z \longmapsto \frac{az+b}{cz+d}$$

where $a,b,c,d \in \mathbb{R}$ and $ad-bc=1$.

Thus $\mathrm{Aut}(\mathbb{H}) \simeq \underset{\text{projective special linear group}}{PSL(2,\mathbb{R})} = SL(2,\mathbb{R})\Big/\underset{\text{identity matrix}}{\{\pm 1\}}$

where $SL(2,\mathbb{R}) = \left\{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2} \,\middle|\, \det A = 1\right\}$

$\mathbb{C}$ $\quad$ $\mathbb{C}$

$\mathbb{Z}$ $\quad$ $\mathbb{Z}$

(i) the <u>moduli space for tori</u> $\mathbb{H}/PSL(2,\mathbb{Z})$

(ii) $\mathbb{C} \setminus \{0,1\} \simeq \mathbb{H}/\Gamma$ where

$$\Gamma := \{A \in PSL(2,\mathbb{Z}) \mid a,d \text{ odd and } b,c \text{ even}\}$$

[a proof of LITTLE PICARD THEOREM, see RUDIN, REAL AND COMPLEX ANALYSIS, 3rd ed., 1987, Section 16]

(iii) Fix a prime p and let $\tilde{\Gamma}_p$ be the set of $A \in SL(2,\mathbb{Z})$ such that $A = \pm 1$ modulo p. Put $\Gamma_p := \tilde{\Gamma}_p / \{\pm 1\}$. Then $X_p = \mathbb{H}/\Gamma_p$ are RS. Important in Number Theory! | QU09

modular curves

In the course Complex Analysis 1, we proved

$\boxed{\text{RIEMANN'S THEOREM}}$ If $\emptyset \neq G \subsetneq \mathbb{C}$ is a <u>simply connected</u> domain, then

$$G \simeq \mathbb{D} \simeq \mathbb{H}.$$

$\boxed{\text{UNIFORMIZATION THEOREM}}$ (Riemann ~ 1850, a complete proof after 50 years POINCARÉ, KOEBE)

A connected RS X is <u>simply connected</u> iff $X \simeq \hat{\mathbb{C}}, \mathbb{C}$ or $\mathbb{H}$. Here $\simeq$ means the conformal equivalence and X is <u>simply connected</u> if every continuous closed curve in X is homotopic to a constant curve.

<u>Proof</u>: HARD! SEE e.g. GAMELIN, COMPLEX ANALYSIS, 2000. ▨

$\boxed{\text{Corollary}}$ If X is a connected RS, then $X \simeq \tilde{X}/\Gamma$ where $\tilde{X} = \mathbb{S}, \mathbb{C}$ or $\mathbb{H}$ and Γ is a subgroup of $\mathrm{Aut}(\tilde{X})$ with the property (PD). In the previous examples, we give all the possibilities.

Pf: We explain this in more details later if time permits. ▨

$\boxed{\text{Remark}}$ Every domain (= open connected set) G in $\mathbb{C}$ is an example of connected RS. not compact

It is easy to see that G is a parabolic RS iff $G = \mathbb{C}$ or $\mathbb{C} \setminus \{z_0\}$ for some $z_0 \in \mathbb{C}$. Otherwise, G is a hyperbolic RS. $\boxed{\text{Exercise}}$

In particular, let G be 2-connected.

[Recall that a domain $G \subset \mathbb{C}$ is called k-connected ($k \in \mathbb{N}$) if $\mathbb{S} \setminus G$ has k components. In particular, simply connected means 1-connected.] Then it is known that

(a) If both the components of $\mathbb{S} \setminus G$ are one-point sets then $G \simeq \mathbb{C} \setminus \{0\}$.

(b) If the only one component of $\mathbb{S} \setminus G$ is a one-point set then $G \simeq \mathbb{D} \setminus \{0\}$.

(c) If none of the components of $\hat{\mathbb{C}} \setminus G$ is a one-point set, then

$$G \simeq \mathbb{P}(0, r, 1) \text{ for some } r \in (0, 1).$$

Here $\mathbb{P}(z_0, r, R) := \{ z \in \mathbb{C} \mid r < |z - z_0| < R \}$, $0 < r < R < +\infty$, is an <u>annulus</u> in $\mathbb{C}$. Moreover,

$$\mathbb{P}(z_0, r_1, R_1) \simeq \mathbb{P}(w_0, r_2, R_2) \text{ iff } \frac{r_1}{R_1} = \frac{r_2}{R_2} \, (= r).$$

$$\left(\simeq \mathbb{P}(0, r, 1) \right)$$

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