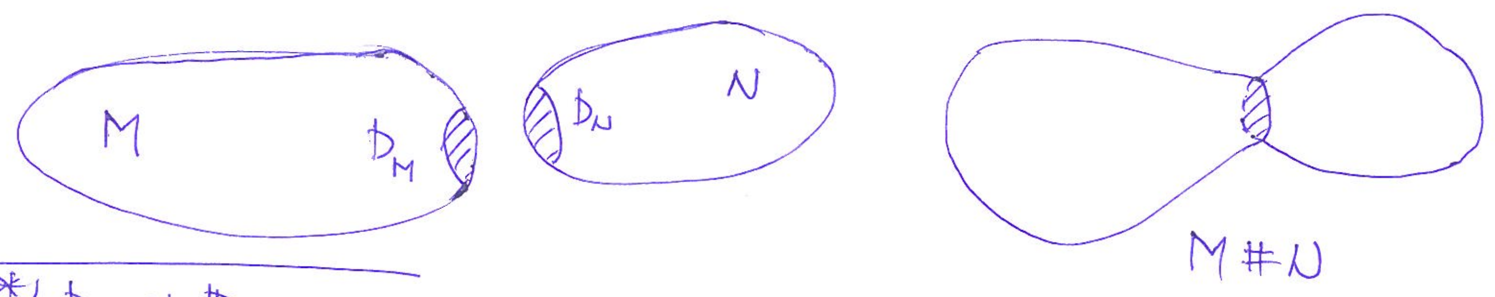
SURFACE TOPOLOGY

We give an informal account. For more details, see e.g. [HIRSCH M.W., Differential Topology, Springer, N.Y., 1994].

Let M, N be compact, connected and orientable surfaces. Here a surface means a topological surface (the same definition as for RS but we make no assumption on transition functions, they are just homeomorphisms).

We define the connected sum $M \# N$ as follows. We choose small discs D_M*) in M and D_N in N and cut them out. Then we glue the boundary circles of D_M and D_N together to form $M \# N$, see the figure below:



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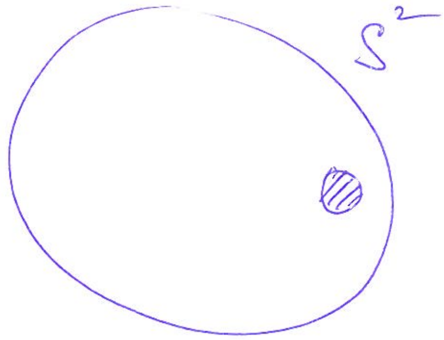
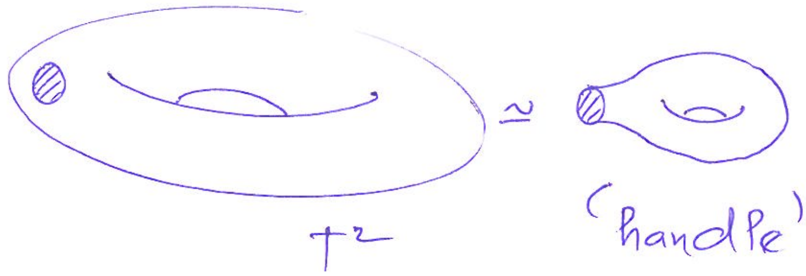
*) $D_M \simeq \mathbb{D}$

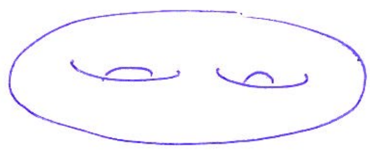
Examples

- $\Sigma_0 := S^2$ the unit sphere in $\mathbb{R}^3$



- $\Sigma_1 := S^2 \# T^2 \simeq T^2$ the torus in $\mathbb{R}^3$



- $\Sigma_2 := S^2 \# T^2 \# T^2 \simeq T^2 \# T^2 \simeq$ 

Define inductively $\Sigma_{g+1} := \Sigma_g \# T^2$, $g \in \mathbb{N}_0$.
So we get Σ_g by adding g handles to the sphere S^2.

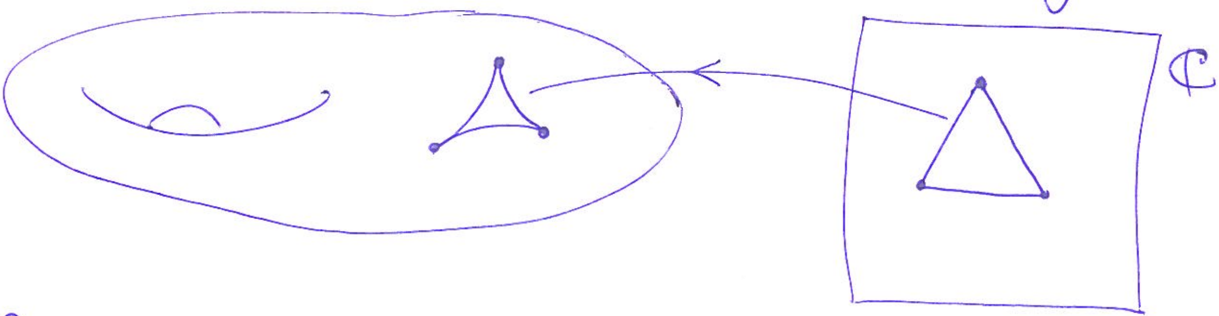
Theorem If X is a compact, connected and orientable surface then there is a <u>unique</u> $g \in \mathbb{N}_0$ such that $X \simeq \Sigma_g$. We write $g(X) := g$ and call it the <u>genus</u> of X. Here $\simeq$ means the topological equivalence (i.e., being homeomorphic to)

<u>Remark</u>: In particular, this is true for R.S.

The Euler characteristic

Let X be a compact, connected and orientable surface. It is known that X has a finite triangulation, i.e., there are 'triangles' $\Delta_1, \dots, \Delta_k$ on X such that $X = \bigcup_{j=1}^{k} \Delta_j$ and

- each Δ_j is homeomorphic to a closed non-trivial triangle in $\mathbb{C}$; so it is clear what are the vertices and the edges of Δ_j



- for $i \neq j$, $\Delta_i \cap \Delta_j$ is either $\emptyset$, the common vertex, or the common edge.
- every edge of the triangulation is common just to two triangles

Define $\boxed{\chi(X) := V - E + F}$

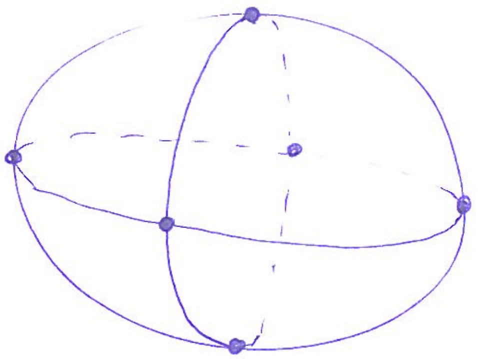
where V is the number of the vertices,
E —"— —"— edges, and
F —"— —"— triangles (faces)
of the given triangulation on X.

Theorem $\chi(X)$ does not depend on the choice of a triangulation of X, it is a topological invariant.

We call $\chi(X)$ the <u>Euler characteristic</u> of X.

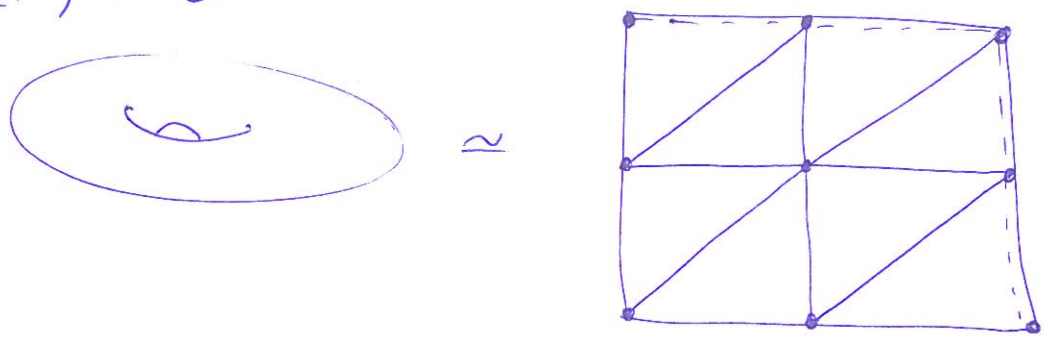
Examples

1. $\chi(S^2) = 2$



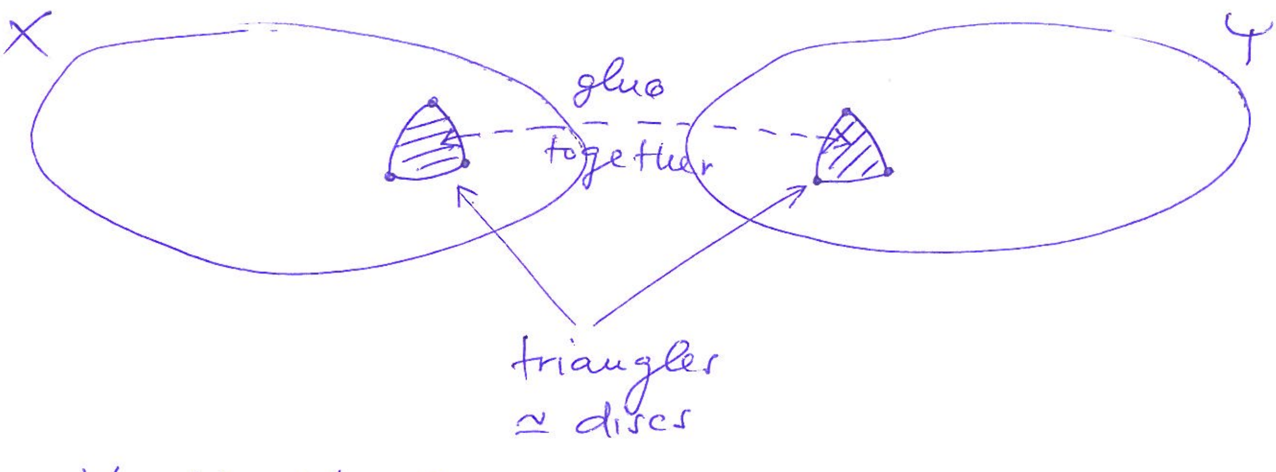
$V=6$
$E=12$
$F=8$

2. $\chi(T^2) = 0$



$V=4$
$E=12$
$F=8$

3. $\chi(X \# Y) = \chi(X) + \chi(Y) - 2$



$V = V_X + V_Y - 3$
$E = E_X + E_Y - 3$
$F = F_X + F_Y - 2$

4. $\boxed{\chi(\Sigma_g) = 2-2g}$ RHT

Indeed, by 3. we have $\chi(\Sigma_g) = \chi(\Sigma_{g-1}) - 2$

———— x ————

Riemann – HURWITZ THEOREM

$\boxed{\text{THEOREM}}$ Let $f: X \longrightarrow Y$ be a non-constant holomorphic map between compact and connected RSs. Then

$$\text{(RH)} \quad \chi(X) = \deg(f)\,\chi(Y) - b(f)$$

where the total branching index $b(f)$ is defined by

$$b(f) := \sum_{x \in X} (n_f(x) - 1).$$

Here $n_f(x)$ is the multiplicity f attains its value at x.

Remark: i) $b(f) = \sum_{\substack{x \in X \\ x \text{ is a critical point for } f \\ \text{(only finitely many!)}}} (n_f(x)-1) =$

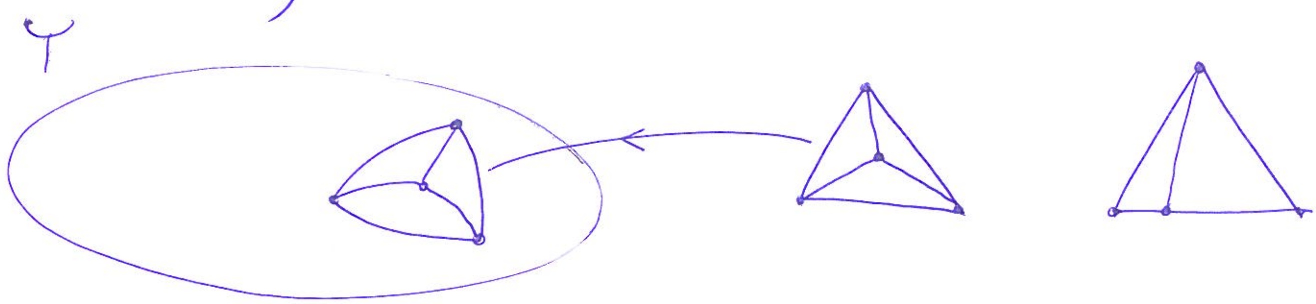
$$= \sum_{\substack{y \in Y \\ y \text{ is a critical value for } f}} \sum_{x \in f^{-1}(y)} (n_f(x)-1) = \sum_{y \in Y} \left(\deg(f) - |f^{-1}(y)|\right).$$

Here $|A|$ is the number of elements of a set A. Hence, for a given $y \in Y$, $\deg(f)$ is the number of solutions of $y = f(x)$ if multiplicities are counted but $|f^{-1}(y)|$ is the number of 'real' solutions of the same equation. To summarise $b(f)$ is the total number of 'missing' solutions. RH6

(ii) By Theorem, $b(f)$ is <u>even</u>.

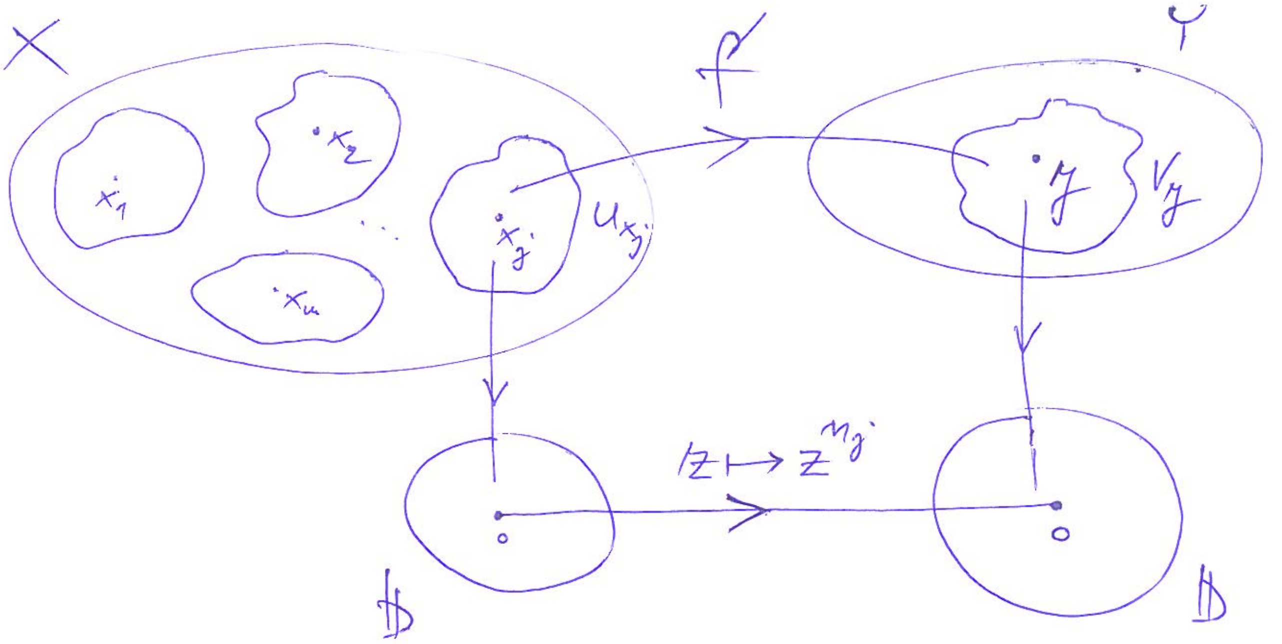
<u>Proof (sketch)</u>: Let us consider a triangulation $\Delta_1, \dots, \Delta_K$ of Y.

(i) WLOG, we can assume that all critical values of f are vertices of the given triangulation. Otherwise, we refine it.

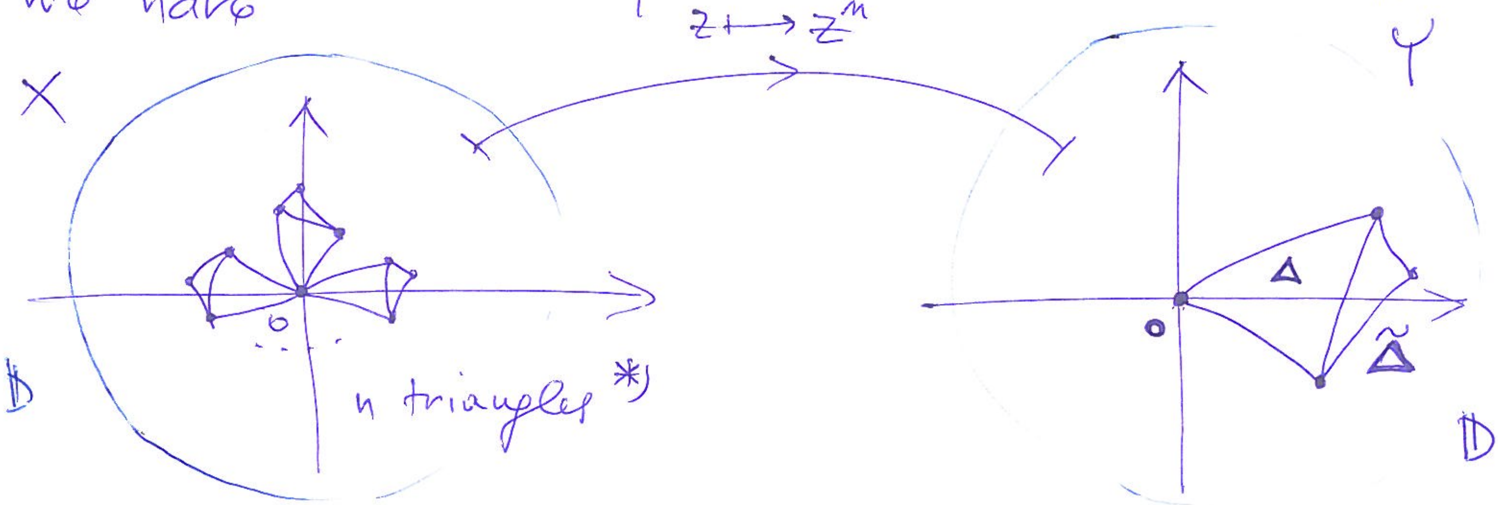


(ii) Let $y \in Y$ and $f^{-1}(y) = \{x_1, \dots, x_m\}$. There are an open set V_y in Y and pairwise disjoint open sets $U_{x_1}, \dots, U_{x_m}$ in X such that $y \in V_y$, $x_j \in U_{x_j}$ and each $f|_{U_{x_j}} : U_{x_j} \xrightarrow{\text{onto}} V_y$ looks like $z \mapsto z^{n_j}$ for some $n_j \in \mathbb{N}$. See a proof of Theorem on degree.

RH7



Choose a finite subcover $V_{y_1}, \ldots, V_{y_r}$ of Y.
WLOG, we can assume that every triangle of the given triangulation of Y lies in a certain V_{y_j}.
(iii) Then the preimage of the triangulation of Y under f gives a triangulation of X. Locally, we have



Then $F_X = \deg(f)\, F_Y$, $E_X = \deg(f)\, E_Y$,

$$V_X = \deg(f)\, V_Y - b(f).$$

Indeed, $f^{-1}(\Delta)$ has $2 \cdot \deg(f) + |f^{-1}(y)| = 3 \cdot \deg(f) - (\deg(f) - |f^{-1}(y)|)$ vertices. ∎

*) We can lift ~~ ~~ simply connected spaces.