Algebraic curves

RH8

Let $P:\mathbb{C}^2\to\mathbb{C}$ be a non-constant polynomial, then

(x) $P(z,w) = p_0(z)\,w^n + p_1(z)\,w^{n-1} + \dots + p_n(z)$,

where ~~kde~~ $p_j:\mathbb{C}\to\mathbb{C}$ are polynomials ~~jsou polynomy~~, $n\in\mathbb{N}_0$ a $p_0\not\equiv 0$.

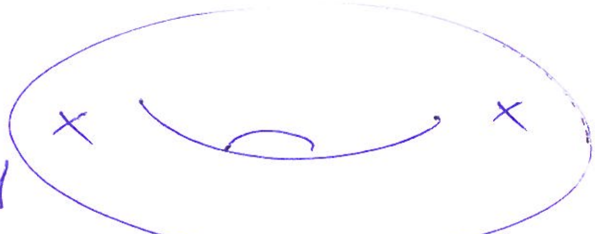
Consider the affine curve

$$X_P := \{(z,w)\in\mathbb{C}^2 \mid P(z,w)=0\}.$$

$\boxed{\text{Problem}}$ What is a topological shape of X_P?

$\boxed{\text{Example}}$ In Lecture 1, we showed that, for $P(z,w) = w^2-(z^2-1)\cdot(z^2-4)$, X_P is homeomorphic to the torus Σ_1 with 2 points at ∞ removed

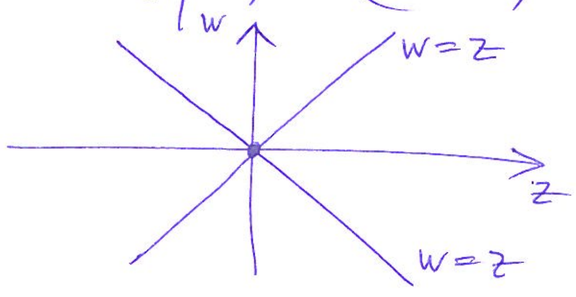
by cutting the plane and gluing the obtained sheets together.



Now we apply Riemann-Hurwitz theorem to do this.

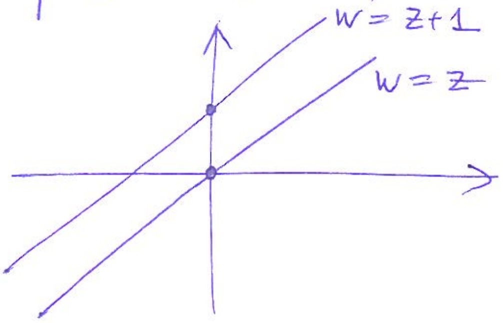
(1.) Let $P = Q\cdot R$ (*) for some non-constant polynomials Q, R in $\mathbb{C}^2$. Then $X_P = X_Q \cup X_R$.

$\boxed{\text{Example}}$ (i) $P(z,w) = (w-z)\cdot(w+z)$



X_P is not RS

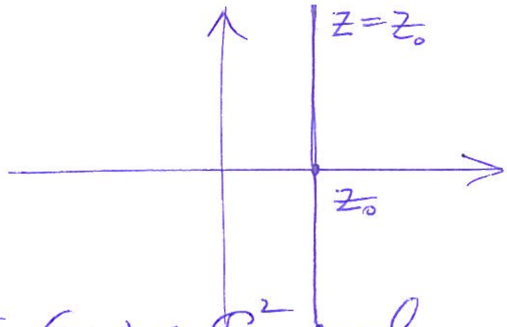
(ii) $P(z,w) = (w-z)\cdot(w-z-1)$ RH9



X_P is <u>not</u> connected

In what follows, we assume that P is <u>irreducible</u>, i.e., P cannot be factored as in (*).

Moreover, we assume that P is not of the form $c\cdot(z-z_0)$ with $c\neq 0$.



Then it is known*) that

(a) X_P is <u>connected</u>;

(b) there are only finitely points $(z,w)\in\mathbb{C}^2$ such that $P(z,w) = 0 = \frac{\partial P}{\partial w}(z,w)$.

2. We suppose that the affine curve X_P is <u>regular</u>, i.e., $\left(\frac{\partial P}{\partial z} \middle| \frac{\partial P}{\partial w}\right) \neq 0$ on $X := X_P$.

Then we know that X is a RS and the projection $\pi: X \longrightarrow \mathbb{C}$, $\pi(z,w) := z$ is holomorphic.

3. We assume that $p_0 \equiv 1$ in (x).

Then $\pi: X \longrightarrow \mathbb{C}$ is a non-constant proper holomorphic map. Moreover, $\deg(\pi) = n$ and X is <u>not</u> <u>compact</u>.

*) See DONALDSON S., Riemann Surfaces, OXFORD, 2011; Section 4.2.3 and 11.1.1.

Pf: (i) π is proper: since π is continuous we need to prove that, for a given bounded $K \subset \mathbb{C}$, $\pi^{-1}(K)$ is bounded. Choose $R > 0$ so big that $\forall z \in K\ \forall w \in \partial U(0,R)$: $|P(z,w) - w^n| < |w^n| = R^n$.

By ROUCHÉ's theorem, we know that, for each $z \in K$, the polynomial $P(z,\cdot)$ has in $U(0,R)$ n roots if multiplicities are counted, so in $U(0,R)$ there are all of its roots.

Hence we get $\pi^{-1}(K) \subset K \times U(0,R)$.

(ii) By (1.) (b), for all but finitely many $z \in \mathbb{C}$, $\pi^{-1}(z)$ has n points. Indeed, let $z_0 \in \mathbb{C}$ be such that $\frac{\partial P}{\partial w} \neq 0$ on $\pi^{-1}(z_0)$. Then

$$\pi^{-1}(z_0) = \{w_1, \dots, w_n\}$$

where $w_1, \dots, w_n$ are the simple roots of $P(z_0, \cdot)$.

(iii) X is not compact: Otherwise, π should be constant.

(4.) Assume that we can compactify X, i.e., we can extend π to a meromorphic function on a compact and connected RS $\overline{X}$ such that $\overline{X} \setminus X = \pi^{-1}(\infty)$ is the set of $N \in \mathbb{N}$ points at ∞.

[Remark] It is always possible. ⌈Actually, every compact RS is a compactified algebraic curve.⌋

Then we have $\pi : \overline{X} \xrightarrow{\text{onto}} S = \mathbb{C} \cup \{\infty\}$
is a non-constant holomorphic map.
By Riemann–Hurwitz theorem, we have

$$\chi(\overline{X}) = 2n - b(\pi)$$

and $b(\pi) = b_{fin} + (n - N)$ where

$$b_{fin} := \sum_{y \in \mathbb{C}} \left(\deg(\pi) - |\pi^{-1}(y)| \right)$$

is the total branching index of π for finite values. So we get $\chi(\overline{X}) = n + N - b_{fin}$ and

$$g := g(\overline{X}) = 1 - \chi(\overline{X})/2 = 1 + \tfrac{1}{2} b_{fin} - \tfrac{1}{2} n - \tfrac{1}{2} N.$$

Conclusion X is homeomorphic to Σ_g with N points at ∞ removed.

Hyperelliptic RS

Let $w^2 = p(z)$ with a polynomial p of degree $k \geq 1$ having just simple roots.
Putting $P(z,w) := w^2 - p(z)$, consider the algebraic curve $X_P =: X$.

1. X is regular: Indeed, $\frac{\partial P}{\partial w} = 2w \neq 0$ if $w \neq 0$. If $w = 0$ and $p(z) = 0$, then

$$\frac{\partial P}{\partial z}(z,0) = -p'(z) \neq 0$$

because z is a simple root of p.

2. Let $\pi : X \longrightarrow \mathbb{C}$ be the z-projection, i.e.,

$$\pi(z,w) := z, \quad (z,w) \in X.$$

If z is not a root of p, then

$$\pi^{-1}(z) = \{(z, \pm p(z)^{1/2})\}.$$

Every root z of p is a critical value (branching point) of π. Thus we have

$$\boxed{\deg \pi = 2} \quad \text{and} \quad \boxed{b_{fin} = k}$$

3. Points at ∞: Let $p(z) = a_0 z^k + \cdots + a_k$, $a_0 \neq 0$.

(a) Let $\boxed{k = 2m}$. Then there is $R > 0$ big enough such that the multivalued function $w^2 = p(z)$

has two different holomorphic branches

$$w_{\pm}(z) := \pm \sqrt{a_0}\, z^m \cdot \left(1 + \frac{a_1}{a_0 z} + \cdots + \frac{a_k}{a_0 z^k}\right)^{1/2},$$

$|z| > R$.

Of course, both $w_\pm$ have a pole at ∞ of degree m.

Hence $\boxed{N = 2}$ and $g = 1 + m - \frac{1}{2} \cdot 2 - \frac{1}{2} \cdot 2$

$$\boxed{g = m - 1}.$$

$\boxed{\text{Conclusion}}$ $X \simeq \Sigma_{m-1}$ with 2 points at ∞ removed

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(b) Let $\boxed{k = 2m+1}$. Then there is $R > 0$ such that $w^2 = p(z)$ has branches

$$w_{\pm}(z) := \pm \sqrt{a_0}\, z^m \cdot z^{1/2} \cdot \left(1 + \frac{a_1}{a_0 z} + \cdots + \frac{a_k}{a_0 z^k}\right)^{1/2},$$

$|z| > R$ & $z \notin (-\infty, 0]$.

As we know (see Lecture 1) we can 'glue together' these branches across the cut to get one (double) point at ∞.

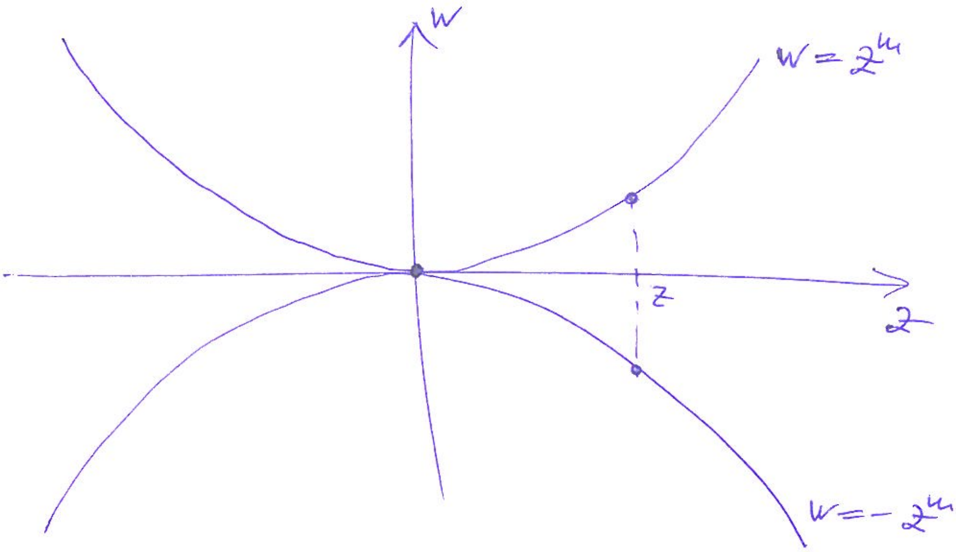
Hence $\boxed{N = 1}$ and $g = 1 + m + \frac{1}{2} - \frac{1}{2} \cdot 2 - \frac{1}{2} \Rightarrow$

$$\boxed{g = m}.$$

$\boxed{\text{Conclusion}}$ $X \simeq \Sigma_m$ with 1 point at ∞ removed

Examples

(1) $w^2 = z^{2n}$: $\quad (w - z^n)\cdot(w + z^n) = 0 \qquad w = \pm z^{2n}$



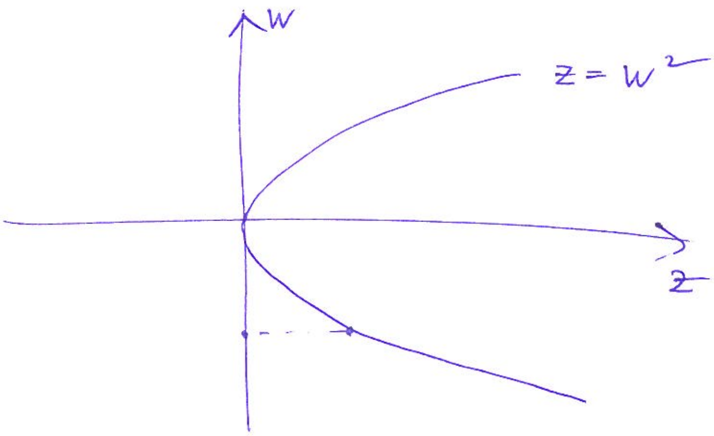
for real z, w

(2) $w^2 = z$: $\quad X = \{(w^2, w) \mid w \in \mathbb{C}\} \simeq \mathbb{C}$

the square root

$p: X \longrightarrow \mathbb{C}$ ... w-projection

$(w^2, w) \longrightarrow w$



for real z, w