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Integral Representation Theory

Applications to Convexity,
Banach Spaces and Potential Theory

Studies in Mathematics

35

de Gruyter

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