Příklad Řešení vlnové rovnice na kouli (v $\mathbb{R}^3$)

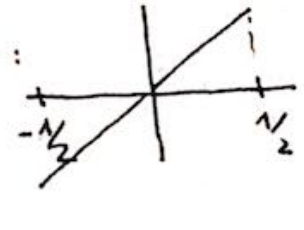
Řešení $u(t,x_1,x_2,x_3)$ rovnice $\frac{\partial^2 u}{\partial t^2} - k^2 \Delta u = 0$ hledejme ve tvaru $u(t,x) = v(t,|x|)$.

Pak $\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial r}\frac{x_i}{|x|}$ a $\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial^2 v}{\partial r^2}\frac{x_i}{|x|}\frac{x_i}{|x|} + \frac{\partial v}{\partial r}\left(\frac{1}{|x|} - \frac{x_i x_i}{|x|^3}\right)$

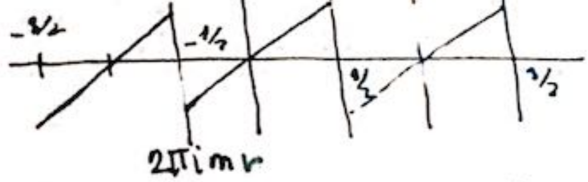
Tedy $\Delta u = \frac{\partial^2 v}{\partial r^2} + \frac{2}{r}\frac{\partial v}{\partial r} = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rv)$, kde $v = v(t,r)$, $r \in (0,R)$

Řešení
$$\boxed{\begin{aligned} &\Box u = 0 \quad v\ (0,+\infty)\times B_R(0) \\ &u(0,x) = 0,\ \frac{\partial u}{\partial t}(0,x) = 1 \quad v\ B_R(0) \\ &u(t,x) = 0 \quad na\ (0,+\infty)\times \partial B_R(0) \end{aligned}}$$
jsme redukovali na úlohu
$$\left\{\begin{aligned} &\frac{\partial^2 v}{\partial t^2} - \frac{k^2}{r}\frac{\partial^2}{\partial r^2}(rv) = 0 \quad v\ (0,+\infty)\times(0,R) \\ &v(0,|x|) = 0,\ \frac{\partial v}{\partial t}(0,\cdot) = 1 \quad v\ (0,R) \\ &v(t,R) = 0 \quad v\ (0,+\infty) \end{aligned}\right.$$
Vydělíme-li rovnici r a označíme $w := rv(t,r)$, pak pro w máme
$$\left[\begin{aligned} &\frac{\partial^2 w}{\partial t^2} - k^2\frac{\partial^2 w}{\partial r^2} = 0 && v\ (0,+\infty)\times(0,R) \\ &w(0,r) = 0,\ \frac{\partial w}{\partial t}(0,r) = r && v\ (0,R) \\ &w(t,0) = w(t,R) = 0 && v\ (0,+\infty) \end{aligned}\right]$$
Tuto úlohu vyřešíme metodou periodizace: pro jednoduchost $R = \frac{1}{2}$. Pak liché prodloužení:

$\tilde{w}_p$:



konvolucí s $\delta_{\mathbb{Z}}$ dostáváme:



a
$$\tilde{w} = \sum_{m\in\mathbb{Z}} \frac{\sin 2\pi k|m|t}{2\pi k|m|} c_m e^{2\pi i m r}, \quad \text{kde } c_m = \langle \tilde{w}_p, e^{-2\pi i m r}\rangle = \int_{-1/2}^{1/2} \tilde{w}_p e^{-2\pi i m r}\, dr$$

Tedy
$$c_n = \int_{-1/2}^{1/2} r\, e^{-2\pi i n r}\, dr = \int_{-1/2}^{1/2} \underbrace{\underset{\text{lichá}}{r}\,(\underset{\text{sudá}}{\cos 2\pi n r}}_{=0} - i \underset{\text{lichá}}{\sin 2\pi n r})\, dr =$$

$$= -2i \int_0^{1/2} \underset{\downarrow \; 1}{r}\, \underset{\downarrow \; -\frac{\cos 2\pi n r}{2\pi n}}{\sin 2\pi n r}\, dr$$

per partes
$$= + 2i \left[r \frac{\cos 2\pi n r}{2\pi n} \right]_0^{1/2} + 2i \int_0^{1/2} \frac{\cos 2\pi n r}{2\pi n}\, dr$$

$$= + \frac{(-1)^n i}{2\pi n} + 2i \left[\frac{\sin 2\pi n r}{(2\pi n)^2} \right]_0^{1/2} = \frac{(-1)^{n+1}}{2\pi n} i$$

Tedy $(c_0 = 0)$
$$\underbrace{\tilde{w}(t, r) = \sum_{n\in\mathbb{Z}} \underbrace{\frac{\sin 2\pi k |n| t}{2\pi k |n|}}_{\text{sudá v } n} \underset{\downarrow \; \text{lichá v } n}{\frac{(-1)^n}{2\pi n}} i \,(\underset{\downarrow \; \text{sudá}}{\cos 2\pi n r} + i \underset{\downarrow \; \text{lichá}}{\sin 2\pi n r})}$$

$$= \sum_{n=1}^{\infty} \frac{\sin 2\pi k |n| t}{2\pi k |n|} \frac{(-1)^{n+1}}{\pi n} \sin 2\pi n r$$

$$\Rightarrow \boxed{ v(t, r) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin 2\pi k n t}{2\pi^2 k\, n^2} \cdot \frac{\sin 2\pi n r}{r} }$$