1

Parametrizujeme ako

$$\varphi = (a\sin\alpha\cos\beta, a\sin\alpha\sin\beta, a\cos\alpha) \tag{1}$$

$$\frac{\partial\varphi}{\partial\alpha} = (a\cos\alpha\cos\beta, a\cos\alpha\sin\beta, -a\sin\alpha) \tag{2}$$

$$\frac{\partial\varphi}{\partial\beta} = (-a\sin\alpha\sin\beta, a\sin\alpha\cos\beta, 0) \tag{3}$$

$$\left(\frac{\partial\varphi}{\partial\alpha}\mathrm{x}\frac{\partial\varphi}{\partial\beta}\right) = a^2\left(\sin^2\alpha\cos\beta, \sin^2\alpha\sin\beta, \cos\alpha\sin\alpha\cos^2\beta + \cos\alpha\sin\alpha\sin^2\beta\right) \tag{4}$$

$$= a^2\sin\alpha\left(\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha\right) \tag{5}$$

$$\left|\left|\frac{\partial\varphi}{\partial\alpha}\mathrm{x}\frac{\partial\varphi}{\partial\beta}\right|\right| = a^2\sin\alpha\sqrt{\sin^2\alpha\cos^2\beta + \sin^2\alpha\sin^2\beta + \cos^2\alpha} \tag{6}$$

$$a^2\sin\alpha\sqrt{\sin^2\alpha + \cos^2\alpha} = a^2\sin\alpha \tag{7}$$

Pre uhly platí $0 \leq \alpha \leq \pi$, $0 \leq \beta \leq 2\pi$ a pre konštantnú plošnú hustotu máme

$$U = G\rho\int_S \frac{\mathrm{d}S}{r} \tag{8}$$

$$= G\rho\int_0^{2\pi}\int_0^{\pi}\frac{a^2\sin\alpha}{\sqrt{a^2\sin^2\alpha\cos^2\beta + a^2\sin^2\alpha\sin^2\beta + (z_0 - a\cos\alpha)^2}}\mathrm{d}\alpha\mathrm{d}\beta \tag{9}$$

$$= G\rho\int_0^{2\pi}\int_0^{\pi}\frac{a^2\sin\alpha}{\sqrt{a^2\sin^2\alpha + (z_0 - a\cos\alpha)^2}}\mathrm{d}\alpha\mathrm{d}\beta \tag{10}$$

$$= G\rho a^2\int_0^{2\pi}\int_0^{\pi}\frac{\sin\alpha}{\sqrt{z_0^2 - 2z_0a\cos\alpha + a^2}}\mathrm{d}\alpha\mathrm{d}\beta \tag{11}$$

Substitujeme $u = z_0^2 - 2z_0a\cos\alpha + a^2$, $\frac{\mathrm{d}u}{\mathrm{d}\alpha} = 2z_0a\sin\alpha$

$$U = \frac{G\rho a^2}{2z_0a}\int_0^{2\pi}\mathrm{d}\beta\int_{(z_0-a)^2}^{(z_0+a)^2}\frac{\mathrm{d}u}{\sqrt{u}} = \frac{G\rho a\pi}{z_0}\int_{(z_0-a)^2}^{(z_0+a)^2}\frac{\mathrm{d}u}{\sqrt{u}} \tag{12}$$

$$= \frac{2G\rho a\pi}{z_0}\left[\sqrt{z_0^2 - 2z_0a\cos\alpha + a^2}\right]_0^{\pi} \tag{13}$$

$$= \frac{2G\rho a\pi}{z_0}\left(\sqrt{(z_0+a)^2} - \sqrt{(z_0-a)^2}\right) = \frac{2G\rho a\pi}{z_0}\left(|z_0+a| - |z_0-a|\right) \tag{14}$$

$$U = \frac{4G\rho a^2\pi}{z_0}\ ak\ |z_0| \geq |a|,\ U = 4G\rho a\pi\ pre\ |z_0| < |a| \tag{15}$$

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Ako prvé zistíme hmotnosť tohto objektu. Za predpokladu, že plošná hustota je vo všetkých miestach rovnaká. Rovnicu kruhového prierezu valca si vieme prepísať ako

$$x^2 - ax + y^2 = 0 = \left(x - \frac{a}{2}\right)^2 + y^2 - \frac{a^2}{4} \implies \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4} \tag{16}$$

Kde $\frac{a}{2}$ má význam polomeru tejto kružnice.
Zavedieme si parametrizáciu ako

$$\varphi = \left(\frac{a}{2} + r\cos t, r\sin t, \sqrt{\left(\frac{a}{2} + r\cos t\right)^2 + (r\sin t)^2}\right) \tag{17}$$

Pre parametre platí $0 \leq r \leq \frac{a}{2},\ 0 \leq t \leq \frac{a}{2}$. Pre parciálne derivácie máme

$$\frac{\partial\varphi}{\partial r} = \left(\cos t, \sin t, \frac{a\cos t + 2r}{2z}\right) \tag{18}$$

$$\frac{\partial\varphi}{\partial t} = \left(-r\sin t, r\cos t, \frac{-ar\sin t}{2z}\right) \tag{19}$$

$$\left(\frac{\partial\varphi}{\partial r} \mathrm{x} \frac{\partial\varphi}{\partial t}\right) = \frac{r}{2z}\left(-(a + 2r\cos t), -2r\sin t, 2z\right) \tag{20}$$

$$\left|\left|\frac{\partial\varphi}{\partial r} \mathrm{x} \frac{\partial\varphi}{\partial t}\right|\right| = \frac{r}{2z}\sqrt{a^2 + 4ar\cos t + 4r^2\cos^2 t + 4r^2\sin t + 4z^2} \tag{21}$$

$$= \frac{r}{2z}\sqrt{4z^2 + 4z^2} = r\sqrt{2} \tag{22}$$

Plochu vypočítame ako

$$S = \sqrt{2}\int_0^{2\pi}\int_0^{\frac{a}{2}} r\mathrm{d}r\mathrm{d}t = 2\pi\sqrt{2}\left[\frac{r^2}{2}\right]_0^{\frac{a}{2}} = \frac{\pi a^2}{2\sqrt{2}} \tag{23}$$

Odtiaľ máme pre hmotnosť $m = \rho\frac{\pi a^2}{2\sqrt{2}}$. Súradnice hmotného stredu máme ako

$$X = \frac{1}{m}\int_S x\rho\mathrm{d}S = \frac{2\sqrt{2}}{\pi a^2}\int_0^{2\pi}\int_0^{\frac{a}{2}}\left(\frac{a}{2} + r\cos t\right)\sqrt{2}r\mathrm{d}r\mathrm{d}t \tag{24}$$

$$= \frac{4}{\pi a^2}\int_0^{2\pi}\left[\frac{ar^2}{4} + \frac{r^3\cos t}{3}\right]_0^{\frac{a}{2}}\mathrm{d}t = \frac{4}{\pi a^2}\left(\frac{a^3}{16}\int_0^{2\pi}\mathrm{d}t + \frac{a^3}{24}\int_0^{2\pi}\cos t\mathrm{d}t\right) \tag{25}$$

$$= \frac{4}{\pi a^2}\left(\frac{\pi a^3}{8}\right) = \frac{a}{2} \tag{26}$$

Ďalej

$$Y = \frac{1}{m}\int_S y\rho \mathrm{d}S = \frac{2\sqrt{2}}{\pi a^2}\int_0^{2\pi}\int_0^{\frac{a}{2}}(r\sin t)\sqrt{2}r\mathrm{d}r\mathrm{d}t \tag{27}$$

$$= \frac{4}{\pi a^2}\int_0^{2\pi}\left[\frac{r^3}{3}\sin t\right]_0^{\frac{a}{2}}\mathrm{d}t = \frac{4}{\pi a^2}\frac{a^3}{24}\left[-\cos t\right]_0^{2\pi} = 0 \tag{28}$$

Pre poslednú súradnicu máme

$$Z = \frac{1}{m}\int_S z\rho \mathrm{d}S = \frac{2\sqrt{2}}{\pi a^2}\int_0^{2\pi}\int_0^{\frac{a}{2}}\sqrt{\left(\frac{a}{2}+r\cos t\right)^2+(r\sin t)^2}\sqrt{2}r\mathrm{d}r\mathrm{d}t \tag{29}$$

Výpočet tohoto integrálu nie je triviálny preto si zavedieme súradnice pre kužeľ φ' a máme pre ne

$$\varphi' = (r\cos t, r\sin t, r) \tag{30}$$

kde pre r máme $0 \leq r \leq a\cos t$, $\frac{-\pi}{2} \leq t \leq \frac{\pi}{2}$ (volíme t tak aby $r\cos t$ bolo kladné).

$$\frac{\partial\varphi'}{\partial r} = (\cos t, \sin t, 1) \tag{31}$$

$$\frac{\partial\varphi'}{\partial t} = (-r\sin t, r\cos t, 0) \tag{32}$$

$$\left(\frac{\partial\varphi'}{\partial r}\mathrm{x}\frac{\partial\varphi'}{\partial t}\right) = (r\cos t, -r\sin t, r) \tag{33}$$

$$\left|\left|\frac{\partial\varphi'}{\partial r}\mathrm{x}\frac{\partial\varphi'}{\partial t}\right|\right| = r\sqrt{\cos^2 t+\sin^2 t+1} = r\sqrt{2} \tag{34}$$

Pre výpočet ťažiska máme

$$Z = \frac{1}{m}\int_S z\rho \mathrm{d}S = \frac{\rho}{m}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{a\cos x} r^2\sqrt{2}\mathrm{d}r\mathrm{d}t = \frac{\rho\sqrt{2}}{m}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{a^3\cos^3 t}{3}\mathrm{d}t \tag{35}$$

Tu musíme spočítať integrál

$$\int \cos^3 t\mathrm{d}t = \sin t\cos^2 t - \int -2\sin^2 t\cos t\mathrm{d}t \tag{36}$$

Subst. $\sin t = u, \quad \mathrm{d}u = \cos t\mathrm{d}t$

$$\int u^2 du = \frac{u^3}{3} = \frac{\sin^3 t}{3}, \quad \int \cos^3 t\mathrm{d}t = \sin t\left(\cos^2 t + \frac{2\sin^2 t}{3}\right) \tag{37}$$

A môžme písať

$$Z = \frac{\rho\sqrt{2}}{m}\frac{a^3}{3}\left[\sin t\left(\cos^2 t+\frac{2\sin^2 t}{3}\right)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{16a}{9\pi} \tag{38}$$

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Zavedieme súradnice ako

$$\varphi = (r\cos t, r\sin t, z) \tag{39}$$

$$\frac{\partial\varphi}{\partial r} = (\cos t, \sin t, 0) \tag{40}$$

$$\frac{\partial\varphi}{\partial t} = (-r\sin t, r\cos t, 0) \tag{41}$$

$$\left(\frac{\partial\varphi}{\partial r}\text{x}\frac{\partial\varphi}{\partial t}\right) = \left(0, 0, r\cos^2 t + r\sin^2 t\right) \tag{42}$$

$$\left|\left|\frac{\partial\varphi}{\partial r}\text{x}\frac{\partial\varphi}{\partial t}\right|\right| = r \tag{43}$$

Ako prvé spočítame polárny moment spodnej podstavy I_{P0} kedy $z = 0$ máme pre parametre $0 \leq r \leq R$, $0 \leq t \leq 2\pi$

$$I_{P0} = \rho\int_0^{2\pi}\int_0^R r(r^2\cos^2 t + r^2\sin^2 t + 0^2)\mathrm{d}r\mathrm{d}t = \rho\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^R \mathrm{d}t = \frac{\pi\rho R^4}{2} \tag{44}$$

Pre hornú podstavu máme $z = H$

$$I_{Ph} = \rho\int_0^{2\pi}\int_0^R r(r^2\cos^2 t + r^2\sin^2 t + H^2)\mathrm{d}r\mathrm{d}t = \rho\int_0^{2\pi}\left[\frac{r^4}{4} + \frac{r^2H^2}{2}\right]_0^R \mathrm{d}t \tag{45}$$

$$= \frac{\pi\rho R^4}{2} + \pi\rho R^2H^2 = \pi\rho R^2\left(\frac{R^2}{2} + H^2\right) \tag{46}$$

Ostáva nám určiť polárny moment plášťa. Vieme, že $x^2 + y^2 = R$. Teraz je dôležité uvedomiť akú parametrizáciu máme

$$\varphi' = (R\cos t, R\sin t, z) \tag{47}$$

$$\frac{\partial\varphi'}{\partial t} = (-R\sin t, R\cos t, 0) \tag{48}$$

$$\frac{\partial\varphi'}{\partial z} = (0, 0, 1) \tag{49}$$

$$\left(\frac{\partial\varphi'}{\partial t}\text{x}\frac{\partial\varphi'}{\partial z}\right) = (R\cos t, R\sin t, 0) \tag{50}$$

$$\left|\left|\frac{\partial\varphi'}{\partial t}\text{x}\frac{\partial\varphi'}{\partial z}\right|\right| = R \tag{51}$$

Paramtre sú obmedzené $0 \leq t \leq 2\pi$, $0 \leq z \leq H$

$$I_{Pla} = \rho\int_0^H\int_0^{2\pi}(R^2+z^2)R\ \mathrm{d}t\mathrm{d}z = 2\pi\rho R\left(HR^2+\frac{H^3}{3}\right) \tag{52}$$

Celkový polárny moment bude

$$I = I_{P0}+I_{Ph}+I_{Pla} = \pi\rho R\left(R^3+RH^2+2HR^2+\frac{2H^3}{3}\right) \tag{53}$$