Domácí úkol # 4 **Termín odevzdání: Čt 14/11 8:09**

1. Spočtěte

$$\lim_{x\to\frac{\pi}{3}} \frac{\sin\left(x-\frac{\pi}{3}\right)}{1-2\cos x}.$$

Řešení

Pro $y = x - \frac{\pi}{3}$ máme

$L := \lim_{y\to 0} \frac{\sin y}{1-2\cos\left(y+\frac{\pi}{3}\right)}$. Protože $\cos\left(y+\frac{\pi}{3}\right) = \cos y \cos\frac{\pi}{3} - \sin y \sin\frac{\pi}{3}$ a $\cos\frac{\pi}{3} = \frac{1}{2}$, $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

tak $L = \lim_{y\to 0} \frac{\sin y}{1-\cos y + \sqrt{3}\sin y} = \lim_{y\to 0} \frac{\overbrace{\frac{\sin y}{y}}^{\to 1}}{\underbrace{\frac{1-\cos y}{y^2}}_{\to \frac{1}{2}}\underbrace{y}_{\to 0} + \sqrt{3}\underbrace{\frac{\sin y}{y}}_{\to 1}} = \frac{1}{\sqrt{3}}$

Dle vět o limitě podílu, součtu a [illegible]

$\lim_{y\to 0} \frac{\sin y}{y} = 1$ & $\lim_{y\to 0} \frac{1-\cos y}{y^2} = \frac{1}{2}$

2. Spočtěte

$$\lim_{x\to 1} (2-x)^{\left(\cos\frac{\pi x}{2}\right)^{-1}}. \qquad \text{Zde } z^{-1} := \frac{1}{z}.$$

$(2-x)^{\left(\cos\frac{\pi x}{2}\right)^{-1}} = \exp\left(\frac{1}{\cos\frac{\pi x}{2}} \ln(2-x)\right).$

Pro $y = x-1$: $\frac{1}{\cos\left(\frac{\pi}{2}y + \frac{\pi}{2}\right)} \ln(1-y) = \frac{1}{-\sin\frac{\pi}{2}y} \ln(1+(-y)) = \frac{\frac{2}{\pi}\frac{\pi}{2}y}{\sin\frac{\pi}{2}y} \frac{\ln(1+(-y))}{-y}$

$= \frac{2}{\pi} \underbrace{\frac{\frac{\pi}{2}y}{\sin\frac{\pi}{2}y}}_{\to 1} \underbrace{\frac{\ln(1+(-y))}{-y}}_{\to 1} \to \frac{2}{\pi}$

Výsledek $\boxed{e^{\frac{2}{\pi}}}$

3. Zjistěte, zda existují (a pokud existují čemu se rovnají) $\lim_{x\to 0+} f(x)$, $\lim_{x\to 0-} f(x)$, $\lim_{x\to 0} f(x)$, kde

(a) $f(x) = \frac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}}$,

(b) $f(x) = \frac{1}{x^{1/3}\operatorname{tg} x} \frac{1-\sqrt{\cos x}}{1-\cos(x^{1/3})}$,

(c) $f(x) = \frac{x}{\sqrt{1-e^{-x^2}}}$.

Ad (a) $\frac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}}$ je definován jen pro $\boxed{x>0}$ a $\boxed{\cos\sqrt{x} \neq 1}$. Tedy má smysl studovat $\lim_{x\to 0+} \frac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}} =: L$. Protože

$\frac{1-\sqrt{\cos x}}{1-\cos\sqrt{x}} = \frac{1-\cos x}{\frac{1-\cos\sqrt{x}}{x}} \frac{1}{(1+\sqrt{\cos x})} \frac{1}{x} = \underbrace{\frac{1-\cos x}{x^2}}_{\to \frac{1}{2}} \frac{\overbrace{x}^{\to 0}}{\underbrace{1+\sqrt{\cos x}}_{\to 2}} \cdot \frac{1}{\underbrace{\frac{1-\cos\sqrt{x}}{x}}_{\to \frac{1}{2}}} \to \underline{\underline{0}}$ pro $x\to 0+$

je $L = 0$.

Ad (b) $f(x)$ je definována na $P_\delta(0)$. Podobný postup

$f(x) = \frac{1-\cos x}{1+\sqrt{\cos x}} \frac{1}{\frac{1-\cos x^{1/3}}{x^{2/3}}} \frac{1}{x^2 \frac{\operatorname{tg} x}{x}} = \underbrace{\frac{1-\cos x}{x^2}}_{\to \frac{1}{2}} \underbrace{\frac{1}{1+\sqrt{\cos x}}}_{\to \frac{1}{2}} \underbrace{\frac{1}{\underbrace{\frac{1-\cos x^{1/3}}{x^{2/3}}}_{\to \frac{1}{2}}}}_{\to 2} \underbrace{\frac{x}{\operatorname{tg} x}}_{\to 1} \to \frac{1}{2}$

Tedy $\lim_{x\to 0} f(x) = \frac{1}{2}$ (a také pro jednostranné limity)

Ad (c) $f(x)$ je definována na $P_\delta(0)$. Máme: $1-e^{-x^2} = -\frac{e^{-x^2}-1}{-x^2}(-x^2) = \frac{e^{-x^2}-1}{x^2} x^2$

Tedy

$f(x) = \frac{x}{\sqrt{\frac{e^{-x^2}-1}{x^2} x^2}} = \overbrace{\frac{x}{|x|}}^{\operatorname{sgn} x} \frac{1}{\sqrt{\underbrace{\frac{e^{-x^2}-1}{x^2}}_{\to 1}}} \to \begin{cases} 1 & \text{pro } x\to 0+ \\ -1 & \text{pro } x\to 0- \end{cases} \Rightarrow$

$\boxed{\lim_{x\to 0} f(x) \text{ neexistuje}}$