Summary

- For reasonable data $v_0 \in L^2_{u,div}$, $f \in L^2(0,T;\Omega) \subset \mathbb{R}^3$, $T \in [0,+\infty)$: $\exists$ global-in-time Leray-Hopf weak sol. to NSE on $[0,T]$.
- If data smooth(er) $v_0 \in W^{1,2}_{0,div}$: $\exists$ strong sol. on $[0,t^*)$ $t^* \in [0,T]$
  $v_0 \in W^{2,2}(\Omega) \cap \forall \Omega \in \mathbb{N}$ $\exists$ smooth sol. on $[0,t^*)$

W.S. $BC_{weak}([0,T);L^2(\Omega)) \cap L^2(0,T;W^{1,2}_{0,div})$

S.S. $L^\infty(0,t^*;W^{1,2}_0(\Omega)) \cap L^2(0,t^*;W^{2,2})$

Note: weak sol. $\Rightarrow$ $\forall t \in [0,T] - E$ ← meas. zero $\quad v(t,\cdot) \in W^{1,2}$

Evenmore $\Rightarrow$ Strong sol. is unique in the class of strong Leray-Hopf weak solutions (satisfying energy ineq.)

Recall that ▷ uniqueness of (suitable) Leray-Hopf w.s. to the NS in 3D is open

▷ Global-in-time regularity/smoothness of solutions to 3D NSE is open

However: if instead of $S = 2\mu * Dv$, consider $S = (2\mu_* + 2\mu_1 |Dv|^{r-2}) Dv$

then $\exists!$ of weak solution is available

$$\begin{cases} \frac{\partial u}{\partial t} - div(|\nabla u|^{r-2}\nabla u) = 0 \\ u(0,\cdot) = u_0 \\ u|_{\partial\Omega} = 0 \end{cases} \quad (*)$$

Monotone-operator approach $\Rightarrow$ $\exists$ weak solution to $(*)$

$r \geq \frac{5}{2}$ if $v_0 \in L^2(\Omega)$

$r \geq \frac{11}{5}$ if $v_0 \in W^{1,r}_{0,div}$

Open: Is this solution smooth? One can get $L^\infty(0,T;W^{2,2+\varepsilon}(\Omega))$? How to go higher $C^{1,\alpha}$ $d=2$ $d=3$?

The same approach works for $(GNS)_r$ for $r \geq \frac{11}{5}$ and provides the $\exists$ of w. solution.

w.s. to $(GNS)_r$, $r \geq \frac{11}{5}$: $v \in C([0,T]; L^2_{u,div}) \cap L^r(0,T; W^{1,r}_{0,div})$

$\frac{\partial v}{\partial t} \in X^*$ $\quad X = W^{1,r}_{0,div} \hookrightarrow L^2_{u,div} \left(= L^{2}_{u,div}\right)^* \subset \left(W^{1,r}_{0,div}\right)^*$

(WF)
$$\left\langle \frac{\partial v}{\partial t} \middle| \psi \right\rangle + \int_\Omega v_k \frac{\partial v}{\partial x_k} \cdot \psi + \int_\Omega \mathbb{S} \cdot D\psi = 0 \qquad \forall \psi \in W^{1,r}_{0,div} \text{ a.a. } t \in (0,T)$$
$$v(0,\cdot) = v_0$$

Uniqueness v^1, v^2 starting from v_0 Aim: $v^1 = v^2$ in Q a.e.

((WF) for v_1 − (WF) for v_2) and test $\psi = w := v_1 - v_2$

$$\frac{1}{2}\frac{d}{dt}\|w\|_2^2 + \tilde{c}\|\nabla w\|_2^2 \text{ (something + non-negative)} \leq$$
$$\left| \int_\Omega \left(v^1_k \frac{\partial v^1}{\partial x_k} - v^2_k \frac{\partial v^2}{\partial x_k} \right) \cdot w \, dx \right|$$
$$\pm v^1_k \frac{\partial v^2}{\partial x_k}$$
$$= \left| \int_\Omega \underbrace{v^1_k \frac{\partial}{\partial x_k} \frac{|w|^2}{2}}_{=0} + w_k \frac{\partial v^2}{\partial x_k} \cdot w \, dx \right| \leq$$

Hölder
$$\int_\Omega \underbrace{|w|^2}_{\frac{r}{r-1}} \underbrace{|\nabla v^2|}_{r} \leq \|\nabla v^2\|_r \|w\|^2_{\frac{2r}{r-1}} \leq c\|\nabla v^2\|_r \|w\|_2^{2\lambda} \|\nabla w\|_2^{2(1-\lambda)}$$

(HW) complete the proof following these hints:

(i) $\|w\|^2_{\frac{2r}{r-1}} \leq \|w\|_2^{\lambda} \|w\|_6^{1-\lambda}$

(ii) Young's ineq. $\leq \frac{\tilde{c}}{2}\|\nabla w\|_2^2 + \tilde{c}\|\nabla v^2\|_r^q \|w\|_2^2$

(iii) apply Gronwall ineq. for $y(t) = \|w(t)\|_2^2$ $\quad \int_0^t$

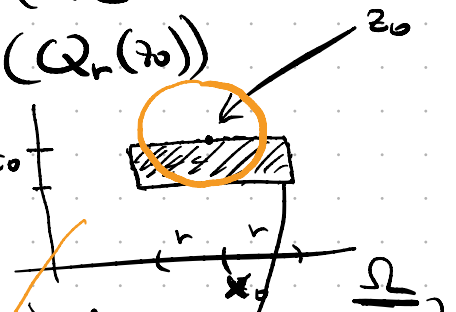
The λ which comes from $q \leq r \Rightarrow r \geq \frac{5}{2}$.

$$\|\nabla v^2\|_r^q \in L^1(0,T)$$

by Monday 19, 12:00

▷ **Partial regularity approach**: estimation of the size of the (hypothetical) singular sets, (instead of proving full reg.)
- Morrey, Almgren — minimal surface problem
- — nonlinear elliptic systems 1955
- NSEs — Scheffer
- — Caffarelli, Kohn, Nirenberg 1982

Def. A [L-Hopf weak sol] is **regular** at $z_0=(t_0,x_0)\in(0,T]\times\overline{\Omega}$ if
$$v\in L^\infty_{t,x}(Q_r(z_0))$$
Otherwise v is **singular** at z_0



$$\Sigma := \{t\in(0,T];\ v \text{ is singular at } (t,x_0) \text{ for some } x_0\in\overline{\Omega}\}$$
$$S := \{z_0\in(0,T]\times\Omega;\ \text{— " — at } z_0\}$$
$$S^* = \{z_0\in(0,T]\times\overline{\Omega},\ \text{— " — " — }\}$$

Clearly $S\subset S^*$ and projection of S^* on time axes is Σ.

$$h^\alpha(E) = \lim_{\delta\to 0+}\ \inf\Big\{\sum_{j=1}^{\infty} r_j^\alpha,\ E\subset\bigcup_{j=1}^{\infty} B_{r_j}(z_j),\ r_j\le\delta\Big\}$$
$$p^\alpha(E) = \lim_{\delta\to 0+}\ \inf\Big\{\sum_{j=1}^{\infty} r_j^\alpha,\ E\subset\bigcup_{j=1}^{\infty} Q_{r_j}(z_j),\ r_j\le\delta\Big\}$$
$$E\subset\mathbb{R}^{m+1}$$

$$Q_r(z_0) := (t_0-r^2,t_0]\times B_r(x_0) \qquad z_0=(t_0,x_0)$$

It holds:
- $h^\alpha(E)\le c\,p^\alpha(E)$
- $h^\alpha(E)=0$ if $\alpha>d+1$
- $p^\alpha(E)=0$ if $\alpha>d+2$

In certain sense (similarity invariance)

3D evolutionary NSE are similar to **5D elliptic NSE**

$p\in L^{5/3}(Q_T)$ if $d=3$
$v\in L^{10/3}(Q_T)$

$p\in L^{5/3}(\Omega)$ if $d=5$
$v\in L^{10/3}(\Omega)$

Existence of smooth solutions for smooth data for 5D NSE is known (Frehse, Reinače, Struwe)

At the end of WSI lecture: Theorem (Leray): $\mathcal{H}^{1/2}(\Sigma) = 0$.

Theorem (CKN '82)
- $\Omega \subset \mathbb{R}^3$ any domain
- $0 < T \leq +\infty$
- (v, p) is suitable weak sol. to 3D NSE in $(0,T) \times \Omega$.

$\Rightarrow$

$$\mathcal{P}^1(S) = 0$$

Serrin: $\mathcal{P}^1(S^* \cap \Sigma_D) = 0$ if $v|_{\Sigma_D} = 0$

$\Sigma_D := (0,T) \times \Gamma_D$

Γ_D: Dirichlet part of $\partial\Omega$.

either Γ_D is flat
or Γ_D is C^2