

5. Three Great Theorems of FA

There is a nice (thick) book on Linear & Nonlinear FA by Philippe Ciarlet (*SIAM, 2013). The author worked in the fields like numerical mathematics, partial diff. equations, mathematics for deformation of elastic solids, calculus of variations. In Ciarlet's book, there is a special chapter called "The „Great Theorems“ of Linear FA", that are based on two fundamental results: Hahn-Banach theorem & Baire's theorem. Besides H-B theorem, the list of great theorems includes:

- Banach - Steinhaus theorem (alias uniform boundedness principle)
- Banach open mapping theorem
- Banach closed graph theorem

Theorem 5.1 (Banach-Steinhaus uniform boundedness principle)

Let X, Y be Banach spaces.

Let $\mathcal{F} \subset \mathcal{L}(X, Y)$ be arbitrary family of bdd linear operators.

Then,

Either \mathcal{F} is uniformly bdd, i.e.

$$\sup_{L \in \mathcal{F}} \|L\|_{\mathcal{L}(X, Y)} < \infty$$

Or There exists a dense set $S \subset X$ such that

$$\sup_{L \in \mathcal{F}} \|Lx\| = \infty \quad \forall x \in S$$

(Pf) For $m \in \mathbb{N}$ consider

$S_m := \{x \in X; \|Lx\|_Y > m \text{ for some } L \in \mathcal{F}\}$. Note S_m are open sets.

- ▷ If all these sets are dense in X , then their intersection is also dense in X by Baire's theorem, see below.
This proves part **Or** as for all $x \in S := \bigcap_{m \in \mathbb{N}} S_m$ and for all $m \in \mathbb{N}$ there is $L \in \mathcal{F}$: $\|Lx\|_Y > m$.
- ▷ If at least one of S_m is not dense in X , let us denote such set S_k , then there is a ball $B_r(x_0)$ such that $\overline{B_r(x_0)} \cap S_k = \emptyset$.

**) A subset $S \subset X$ is dense in X if $\overline{S} = X$; i.e. for $\forall x \in X \ \forall \varepsilon > 0 \ \exists y \in S \ |y - x|_X < \varepsilon$
It holds (verify!):

$$S \text{ is dense in } X \iff \forall \delta > 0 \ \exists r > 0 \ \forall x \in X \ \exists y \in S \quad |x - y|_X < r$$

*) SIAM ... publisher ... Society for Industrial and Applied Mathematics

****) See pages below

Hence, $\forall x \in B_r(x_0) : \|Lx\| \leq k$ for all $L \in \mathcal{F}$

It implies that $\sup_{L \in \mathcal{F}} \|L\| < +\infty$, which is the part [Either]

Indeed, if $\|x\|_X \leq r$, then

$$\|Lx\|_Y \leq \|L(x_0+x)\|_Y + \|Lx_0\|_Y \leq 2k$$

and consequently

$$\|L\|_{\mathcal{L}(X,Y)} = \sup_{\|x\|_X=1} \|Lx\|_Y = \sup_{\|rx\|_X=r} \|L(rx)\|_Y \leq \frac{2k}{r},$$

which implies .



NOTE Theorem 5.1 proves the following statement :

If $\mathcal{F} \subset \mathcal{L}(X,Y)$ is pointwise bounded,
then it is uniformly bounded,

or written differently :

$$\text{If } \sup_{L \in \mathcal{F}} \|Lx\|_Y < +\infty \quad \text{for each } x \in X : \|x\|_X = 1 \\ \text{then } \sup_{L \in \mathcal{F}} \|L\|_{\mathcal{L}(X,Y)} = \sup_{L \in \mathcal{F}} \sup_{\|x\|_X=1} \|Lx\|_Y < \infty$$

This why Theorem 5.1 is called uniform boundedness principle

Two applications

① If $\phi_n \xrightarrow{*} \phi$ $*$ -weakly in X^* , then $\sup_m \|\phi_m\|_{X^*} < +\infty$
i.e. $\{\phi_m\}_{m=1}^\infty$ is bdd.

(Pf) By definition of $*$ -weak convergence,
 $\langle \phi_m, x \rangle = \phi_m(x) \rightarrow \phi(x) = \langle \phi, x \rangle$, hence $\sup_{m \in \mathbb{N}} |\phi_m(x)| < \infty$

By Theorem 5.1, $\sup_{m \in \mathbb{N}} \|\phi_m\|_{X^*} = \sup_{m \in \mathbb{N}} \sup_{\|x\|=1} |\phi_m(x)| < +\infty$. □

Recall that we have already proved that ① implies :

If $x_m \rightarrow x$ weakly in X , then $\sup_{m \in \mathbb{N}} \|x_m\|_X < +\infty$
i.e. $\{x_m\}_{m=1}^\infty$ is bdd.

- (2) Let X, Y be Banach spaces.
- Let $\{L_m\}_{m=1}^{\infty} \subset \mathcal{L}(X, Y)$
 - Let $\lim_{n \rightarrow \infty} L_n x$ exist for all $x \in X$.
- Then L defined as $Lx = \lim_{n \rightarrow \infty} L_n x$ is bounded, linear operator, i.e. $L \in \mathcal{L}(X, Y)$.

"Pointwise limit of bounded linear operators is bounded linear operator."

Pf Since $Lx := \lim_{n \rightarrow \infty} L_n x$ exists for all x , then

$$\sup_{n \in \mathbb{N}} \|L_n\|_{\mathcal{L}(X, Y)} < \infty. \text{ Hence}$$

$$\begin{aligned} \|L\|_{\mathcal{L}(X, Y)} &= \sup_{\substack{\|x\|=1 \\ X}} \|Lx\|_{\mathcal{L}(X, Y)} = \sup_{\substack{\|x\|=1 \\ X}} (\lim_n \|L_n x\|_{\mathcal{L}(X, Y)}) \\ &\leq \sup_{\|x\|=1} (\lim_{n \rightarrow \infty} \|L_n\|_{\mathcal{L}(X, Y)} \|x\|_X) \\ &\leq \sup_{n \in \mathbb{N}} \|L_n\|_{\mathcal{L}(X, Y)} < +\infty. \end{aligned}$$

Thus L is bounded (and linearity follows from pointwise convergence of linear operators). 

Before formulating the Open mapping theorem, we recall the definition of open mapping: this is a mapping that maps open sets on open sets, i.e. $\forall U \subset X \text{ open: } F(U) \subset Y \text{ is open}$

This condition is the same as: $(\forall B_r(x)) (\exists B_\delta(f(x))) B_\delta(f(x)) \subset F(B_r(x))$
Note $L: \mathbb{R} \rightarrow \mathbb{R}$ $Lx = ax$ maps (A, B) onto (aA, aB) , which is open if $a \neq 0$.

Theorem 5.2 (Open mapping) Let X, Y be Banach spaces.

Let $L \in \mathcal{L}(X, Y)$ be surjective (onto). Then L is open.

Pf Step 1 As L is linear, we have

$$L(B_r(x)) = Lx + L(B_r(0)) = Lx + rL(B_1(0)).$$

Thus, to prove Open mapping theorem it suffices to show that (as $Lx = 0$) there exist $B_\delta(0)$ such that $B_\delta(0) \subset L(B_1(0))$. We will show this with $\delta = \frac{1}{2}$.

Step 2 Since L is onto, $Y = \bigcup_{n=1}^{\infty} L(B_n(0))$. As Y is complete, by Banach's theorem, at least one of the closures $\overline{L(B_n(0))} \subset Y$ has nonempty interior.

By rescaling: $\overline{L(B_1(0))} = \frac{1}{m} \overline{L(B_m)} \text{ must have nonempty interior.}$

Hence, $\boxed{\exists y_0 \in Y \exists r > 0 B_r(y_0) \subset \overline{L(B_1(0))}}$,

Since $B_1(0)$ is convex & symmetric ($a \in B_1(0) \Rightarrow -a \in B_1(0)$),

then $L(B_1(0))$ —||—

and $\overline{L(B_1(0))}$ —||—

In particular, $\boxed{B_r(-y_0) \subset \overline{L(B_1(0))}}$

Due to convexity

$$B_r(0) = \frac{1}{2} B_r(y_0) + \frac{1}{2} B_r(-y_0) \subset \overline{L(B_1(0))}$$

By rescaling, due to linearity of L ,

** $\boxed{B_{\frac{r}{2^m}}(0) \subset \overline{L(\frac{B_1(0)}{2^m})} \text{ for all } m \in \mathbb{N}}$

Step 3

We show that $\boxed{B_{\frac{r}{2}}(0) \subset \overline{L(B_1(0))}}$ (see Step 1)

Let $y \in B_{\frac{r}{2}}(0)$ be arbitrary. Iteratively:

- by ** $\exists x_1 \in B_{\frac{1}{2}}(0) : \|y - Lx_1\|_Y \leq \frac{r}{2^2}$

- by ** again

$$\exists x_2 \in B_{\frac{1}{2^2}}(0) : \|(y - Lx_1) - Lx_2\|_Y \leq \frac{r}{2^3}$$

- and so on, for each $m \in \mathbb{N}$,

$$\cdot y - \sum_{j=1}^{m-1} Lx_j \in B_{\frac{r}{2^m}}(0) \subset \overline{L(\frac{B_1(0)}{2^m})}.$$

and there is $x_m \in B_{\frac{1}{2^m}}(0)$ such that

$$\|(y - \sum_{j=1}^m Lx_j) - Lx_m\|_Y \leq \frac{r}{2^{m+1}}$$

Since X is Banach, $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$, then $\exists x \in X : \sum_{n=1}^{\infty} x_n = x \in X$

Furthermore, $\|x\|_X \leq \sum_{n=1}^{\infty} \|x_n\|_X \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$, $Lx = \lim_{n \rightarrow \infty} \sum_{j=1}^n Lx_j = y$.

Hence, $\overline{L(B_1(0))}$ contains all $y \in B_{\frac{r}{2}}(0)$.

Q.E.D.