SS: 2021/2022 NMMO302 B/3 Josef MÁLEK

Functional analysis for physicists 4/2

GENERAL INFORMATION AND SYLLABUS

LITERATURE

A. Bressan: *Lecture notes on functional analysis: with applications to linear partial differential equations*, American Mathematical Society, Providence, 2013.

Ph. Ciarlet: *Linear and nonlinear functional analysis with applications*, SIAM, Philadelphia, 2013.

A.N. Kolmogorov, S.V. Fomin: *Elements of the theory of functions and functional analysis*, Dover publications, 1999.

H. W. Alt: *Linear functional analysis (an application-oriented introduction*, Universitext, Springer, 2016.

A. Friedman: *Foundations of modern analysis*, Dover publicatiosn, New York, 1982.

J. Málek: *Hand-written notes*
`http://www.karlin.mff.cuni.cz/∼malek`

EXAM, CREDITS FOR HOMEWORKS

The exam consists of a written part and an oral exam (questions may include abstract Fourier series). The written part contains problems to be solved as well as questions of a theoretical nature. Length 90-120 minutes, maximum number of points 30. If your score is below 12 points, your grade is F (fail).

Evaluation of the examination (which includes 10 points from the oral examination):

29 – 40 pts	excellent
23 – 28,9 pts	very good
17 – 22,9 pts	good
méně než 17 pts	fail

Syllabus of the lecture course NMMO302 (Functional analysis for physicists)

An introductory course to functional analysis focused on the extension of the results of linear algebra to infinite-dimensional spaces and on applications of general theoretical results within partial differential equations.

- *Introduction.* What is functional analysis (FA)? Why "functional" analysis? Linear vs nonlinear FA.
  Basic concepts: vector spaces, pre-Hilbert (inner product) spaces, normed spaces, metric spaces, topological spaces. Convergence. Completeness (Hilbert, Banach). Continuity. Examples: $\mathbb{R}^d$, $\mathbb{C}^d$, $C^k(\overline{\Omega})$, ℓ^p, $L^p(\Omega)$, $W^{k,p}(\Omega)$.
  Seminorms. Sequence of separating seminorms leads to a distance, Fréchet spaces (metric spaces that are complete with a well designed distance). Examples: $C(\Omega)$ and $L^p_{loc}(\Omega)$ for an open sets $\Omega \subset \mathbb{R}^d$.

- *Linear mappings (operators).* Linearity. Continuity and boundedness - their equivalence for linear mappings. The spaces $\mathcal{L}(X,Y)$, $\mathcal{L}(X)$ and $X^* = X'$. Finite-dimensional normed spaces (are Banach, all norms are equivalent, characterization: each bounded sequence contains converging subsequence). Equivalent description of the operator norm (see Assertion 1 in the notes to Section 6).

- *Hahn-Banach theorem.* Algebraic and topological Hahn-Banach theorems. Applications: there are enough linear functionals to separate points from closed subspaces, to separate points (and consequently to justify that the concept of weak convergence is well defined), to separate two nonempty disjoint convex subsets when additionally (i) one is open, or (ii) one is compact and the other closed. Dual description of the norm in a Banach space (see Assertion 2 in the notes to Section 6).

- *Dual spaces, reflexivity and weak and weak-* convergences.* For Banach spaces, the definition of its dual (and why the dual is is Banach space) and its second dual (via a canonical embedding). Reflexivity, examples of reflexive spaces. Weakly and weakly-* converging sequences, their relationship in the reflexive spaces and their boundedness (following from Banach-Steinhaus uniform convergence principle). Relation to the concept of strong convergence. Weak lower-semicontinuity of the norm. Banach-Alaoglu theorem (with the proof for separable Banach spaces); weak sequential compactness of the closed unit ball in the reflexive Banach spaces. Banach-Eberlein-Šmuljan theorem.

- *Baire's theorem.* Cantor's intersection theorem. Baire's theorem. Applications: non-existence of a countable Hamel basis, why the space of all polynomials cannot be equipped with the norm so that the space of all polynomials is Banach. Sets of the first category (meager sets); sets of the second category, and the theorems of the next section.

- *The great theorems of FA.* Banach-Steinhaus uniform principle and its applications (boundedness of weakly converging sequences, pointwise limit of linear bounded operators leads to a bounded linear operator). Open mapping theorem and its applications (Inverse of any linear continuous bijection is continuous, a sufficient condition for the equivalence of the norms in Banach spaces). Closed graph theorem.

- *Adjoint operators. Compact operators.* Orthogonal sets to subspaces in a Banach space or its dual. Definition of the adjoint operator to a linear bounded operator from X to Y and its properties. Compact operators: definition, characterizations, examples. Arzelà-Ascoli theorem and Kolmogorov criterium of compactness in L^p-spaces (Riesz theorem). Adjoint of a compact operator is compact and vice versa. Compact operators generated by integral operators (continuous kernel).

- *Linear operators in Hilbert spaces.* Definition of the Hilbert spaces over $\mathbb{K}$. Decomposition of a Hilbert space into the direct sum of a closed subspace and its orthogonal (always closed) subspace. Riesz representation theorem. Riesz map. Reflexivity of Hilbert spaces. Well-posedness for positive definite operators. Lax-Milgram lemma over $\mathbb{K}$. Application of Lax-Milgram lemma: Dirichlet and Neumann problems for linear elliptic operators. Sobolev spaces - definition. Poincaré inequality. Concept of weak solution and its compatibility with the concept of classical solution. Relation to the the global minimization of certain quadratic functionals. Weak convergence in Hilbert spaces. Weak convergence composed with a compact operator.

- *Fredholm theory/Fredholm alternative.* Fredholm theorem. Why are the operators of the form $I - K$, where K is compact important? Fredholm alternative in finite-dimensional spaces (over $\mathbb{R}$ and $\mathbb{C}$).

- *Spectrum. An introduction to spectral theory.* Invertible operator to a continuous linear operator in the Banach space and its continuity. Sufficient condition for the existence of invertible operators in the form $I - T$(Neumann series); consequences. Resolvent set. Resolvent of an operator T in λ: $R(\lambda, T)$. Assertions: resolvent set is open in $\mathbb{K}$, the mapping $\lambda \to R(\lambda, T)$ is holomorphic. Spectrum (nonempty for any $T \in \mathcal{L}(X)$, X nontrivial Banach space; spectral radius and its characterization, "Cauchy integral" formula for T^j). Point spectrum (=set of eigenvalues). Continuous spectrum. Residual spectrum. The set of invertible operators is open in the space of linear bounded operators. Spectrum of a compact operator on a Hilbert space. Selfadjoint operators and bounds on their spectrum. Hellinger-Toeplitz theorem. Spectrum of selfadjoint compact operators. Normal operators. Spectral theory for linear bounded normal operators.

Complements (summary):

- A. Compact sets and Arzelà-Ascoli theorem.
- C. Basis in vectors spaces.
- F. Abstract Fourier series.
- L. Lebesgue spaces (an overview) and important theorems of the theory of Lebesgue integral (that were not mentioned/proved before).