MATEMATICKÉ METODY V MECHANICE NEWTONOVSKÝCH TEKUTIN

Matematické metody v mechanice nenewtonských tekutin

2/0 Zk

Zk { 50% DÚ
50% ÚSTNÍ POHOVOR

Prerekvisity
- PDR I, PDR II
- Mechanika & termodynamika nenewtonských tekutin

L1 Úvod

1) Rovnice a úlohy
2) NSR vs rovnice pro proudění nenewtonských tekutin
3) Cíle

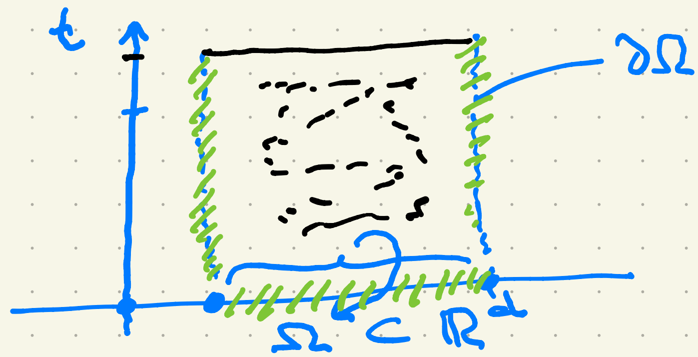
① Rovnice vs úlohy

- Bilanční rce
- Konstitutivní rce

} SYSTÉM NELINEÁRNÍCH PDR

IBVP
počáteční & okrajové úlohy

- POČÁTEČNÍ PODMÍNKY
- OKRAJOVÉ PODMÍNKY
  vesměs představují konstitutivní rce na hranice



OTEVŘENÉ

- vtoky / výtoky
- $\theta = \theta_{given}$ na $(0,T)\times\partial\Omega$

těžší

UZAVŘENÉ

$\vec{v}\cdot\vec{n} = 0 \quad (0,T)\times\partial\Omega$

$\vec{q}\cdot\vec{n} = 0$

lehčí

kontrola celkové energie

$$E = \frac{1}{2}|v|^2 + e$$

$\frac{1}{2}|v_0|^2 + e_0$

$$\int_\Omega \rho E(t,x)\,dx = \int_\Omega \rho_0 E_0(x)$$

VNITŘNÍ PROUDĚNÍ

$\boxed{\text{ROUNICE}}$ – Bilanční

hmoty $\quad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0 \qquad \vec{m} \qquad \nabla\rho\cdot\vec{v} + \rho \operatorname{div}\vec{v}$

hybnosti $\quad \frac{\partial(\rho\vec{v})}{\partial t} + \operatorname{div}(\rho\vec{v}\otimes\vec{v}) = \operatorname{div}\mathbb{T} + \rho\vec{b}$

energie $\quad \frac{\partial(\rho E)}{\partial t} + \operatorname{div}(\rho E\vec{v}) - \operatorname{div}\vec{j} = \operatorname{div}(\mathbb{T}\vec{v}) + \rho\vec{b}\cdot\vec{v}$

$E := \frac{1}{2}|v|^2 + e$

– —

2. zákon $\quad \frac{\partial(\rho\eta)}{\partial t} + \dots$

$\vec{v} = (v_1, v_2, v_3)$ vychlost

ρ hustota

e vnitřní energie $\quad$ (θ teplota)

$\vec{a}, \vec{b} \in \mathbb{R} \qquad (\vec{a}\otimes\vec{b})_{ij} = a_i b_j$

$\mathbb{T}$ Cauchyův tensor

$\vec{j}$ tok energie $\qquad \vec{b}$ dané objemové síly

$\oplus$ Celková energie se zachovává !!

Mechanika $\downarrow$ vs Termodynamika

$\boxed{\varrho, \vec{v}}$ $\quad \vec{z} = (z_1, z_2, z_3, z_4)$

- $(\varrho, \varrho v_1, \varrho v_2, \varrho v_3) =$

$$\mathrm{Div}_{t,x} := \frac{\partial}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}$$

- Bilance hmoty $\quad \boxed{\mathrm{Div}_{t,x}(\varrho, \varrho v_1, \varrho v_2, \varrho v_3) = 0}$

- Bilance hybnosti $\quad k = 1,2,3$

$$\boxed{\mathrm{Div}_{t,x}\left(\varrho v_k, \; \varrho v_k v_1 + \mathbb{T}_{k1}, \; \varrho v_k v_2 + \mathbb{T}_{k2}, \; \varrho v_k v_3 + \mathbb{T}_{k3}\right) = 0}$$

Stlačitelné

Nestlačitelné

$$\boxed{\operatorname{div} \vec{v} = 0}$$

$\boxed{\text{homogenní}}$ $\quad \varrho(t,x) = \varrho_* > 0$ (daná)

nehomogenní

$$\text{BHm} \Rightarrow \frac{\partial \varrho}{\partial t} + \nabla \varrho \cdot \vec{v} = 0$$

transportní rovnice pro ϱ

$$\mathbb{T} = \phi \mathbb{I} + \mathbb{S}$$

tensor napětí $\quad$ sférické napětí $\quad$ konstitutivní část

ϕ skalár – neznámá veličina, která vstupuje mezi neznámé veličiny PDR

$(t, x) \longmapsto \phi(t,x)$

$t \in (0,T)$, $x \in \Omega$

často: $\phi = -p$ $\neq$ termodynamický tlak!

často: $\frac{1}{3} \operatorname{tr} \mathbb{T} = \phi$ $\quad (\operatorname{tr} \mathbb{S} = 0)$

průměr normálových napětí

$$(*)\quad \boxed{\begin{aligned} &\operatorname{div} v = 0 \qquad\qquad\qquad\qquad\qquad \mathbb{S} = \mathbb{S}^T \\ &\varrho_*\left(\partial_t \vec{v} + \operatorname{div}(\vec{v}\otimes\vec{v})\right) = \underline{\nabla\phi} + \underline{\operatorname{div} \mathbb{S}} + \underline{\varrho_* \vec{b}} \end{aligned}}$$

4 rovnice pro $\vec{v} = (v_1, v_2, v_3)$, ϕ

$\mathbb{S}$ vstupuje do dalších materiálových rovnic, kterým se říká KONSTITUTIVNÍ ROVNICE

Energetická bilance pro (*)

$\boxed{(*)_2 \cdot \vec{v}}$

- $\partial_t \vec{v} \cdot \vec{v} = \frac{1}{2}\partial_t |\vec{v}|^2 = \partial_t\left(\frac{|\vec{v}|^2}{2}\right)$ $\qquad |\vec{v}|^2 = \sum_{i=1}^3 v_i v_i$

- $\vec{v}\cdot\operatorname{div}(\vec{v}\otimes\vec{v}) \underset{(*)_1}{=} \vec{v}\cdot\sum_{k=1}^3 v_k \frac{\partial\vec{v}}{\partial x_k} = \sum_{k=1}^3 v_k \frac{\partial}{\partial x_k}\left(\frac{|v|^2}{2}\right)$

$$\underset{(*)_1}{=} \underline{\operatorname{div}\left(\frac{|\vec{v}|^2}{2}\vec{v}\right)}$$

- $\nabla\phi\cdot\vec{v} \underset{(*)_1}{=} \operatorname{div}(\phi\vec{v})$

- $\operatorname{div}\mathbb{S}\cdot\vec{v} = \operatorname{div}(\mathbb{S}\vec{v}) + \underbrace{\mathbb{S}\cdot\nabla\vec{v}}_{(\mathbb{S})_{ij}\frac{\partial v_i}{\partial x_j}}$ $\quad$ symetrický

$$= \operatorname{div}(\mathbb{S}\vec{v}) + \mathbb{S}\cdot \mathbb{D}v$$

- $\varrho_* \vec{b}\cdot\vec{v}$ $\qquad\qquad \mathbb{D}\vec{v} := \dfrac{\nabla v + (\nabla v)^T}{2}$

Celkově

$$(E_{koc})\qquad \partial_t\left(\rho_* \frac{|v|^2}{2}\right) + \mathrm{div}\left(\rho_* \frac{|v|^2}{2}\vec{v}\right) - \mathrm{div}\left(S\vec{v}\right) + S:\mathbb{D}v = \mathrm{div}\left(\phi\vec{v}\right) + \rho_*\vec{b}\cdot\vec{v}$$

Vnitřní proudění: $\vec{v}\cdot\vec{n} = 0$ na $(0,t)\times\partial\Omega$

$\int_\Omega (E_{koc})\,dx$ + Gaussova věta

$$(E_{glob})\qquad \frac{d}{dt}\int_\Omega \rho_* \frac{|\vec{v}|^2}{2}\,dx + \int_{\partial\Omega} \rho_* \frac{|v|^2}{2}\underbrace{(\vec{v}\cdot\vec{n})}_{=0}\,dS + \int_{\partial\Omega}(-S\vec{v})\cdot\vec{n}\,dS + \int_\Omega S:\mathbb{D}v\,dx = \int_{\partial\Omega}\phi\,\underbrace{\vec{v}\cdot\vec{n}}_{=0}\,dS + \int_\Omega \rho_*\vec{b}\cdot\vec{v}\,dx$$

$$-S\vec{v}\cdot\vec{n} = -S\cdot(\vec{v}\otimes\vec{n}) \overset{\text{sym } S}{=} -S\cdot(\vec{n}\otimes\vec{v}) = -S\vec{n}\cdot\vec{v} =$$

$\vec{z} \to (\vec{z}\cdot\vec{n})\vec{n}$ normálový směr

$\vec{z} \to \vec{z}_\tau := \vec{z} - (\vec{z}\cdot\vec{n})\vec{n} \dashrightarrow \vec{z}_\tau\cdot\vec{n} = 0$

$$\left[-(S\vec{n})_\tau - (S\vec{n}\cdot\vec{n})\vec{n}\right]\cdot\left[\vec{v}_\tau + \underbrace{(\vec{v}\cdot\vec{n})}_{=0}\vec{n}\right] = \underbrace{-(S\vec{n})_\tau}_{=\vec{s}}\cdot\vec{v}_\tau$$

DÚ Ukažte, že platí

$$(\mathbb{T}\vec{n})_\tau = (S\vec{n})_\tau \quad !$$

Do pondělí! 12.10. 12:00

(E)

$$\boxed{\frac{d}{dt}\int_\Omega \varrho_* \frac{|\vec v|^2}{2} + \int_\Omega \mathbb{S}\cdot\mathbb{D}v + \int_{\partial\Omega} \vec s\cdot\vec v_\tau\, dS = \int_\Omega \varrho\, \vec b\cdot\vec v}$$

Konstitutivní rovnice

v objemu : spojují $\mathbb{S}$ a $\mathbb{D}v$

na hranici —"— $\vec s$ a $\vec v_\tau$

Příklad **Lineární vztahy**

→ $\boxed{\mathbb{S} = 2\nu\,\mathbb{D}v}$ Navier–Stokesova tekutina

→ $\boxed{\vec s = \gamma_* \vec v_\tau}$ Navierův skluz

$$\vec s := -(\mathbb{S}n)_\tau = -(\mathbb{T}n)_\tau$$

NS tekutina

$\Updownarrow$

Newtonská tekutina

$$\boxed{\mathbb{S} = 2\nu\,\mathbb{D} \underset{\nu>0}{\iff} \mathbb{D} = \frac{1}{2\nu}\,\mathbb{S}}$$

viskozita (ν) — tekutost ($\frac{1}{2\nu}$)

$$\boxed{\mathbb{G}(\mathbb{A},\mathbb{B}) := \mathbb{A} - 2\nu\mathbb{B}}$$

Klasifikace nenewtonovských tekutin

1) $\mathbb{G}(\mathbb{S},\mathbb{D}) = \mathbb{O}$
2) $\mathbb{G}(\overset{*}{\mathbb{S}},\overset{*}{\mathbb{D}},\mathbb{S},\mathbb{D}) = \mathbb{O}$
3) $\mathbb{G}(\overset{*}{\mathbb{S}},\overset{*}{\mathbb{D}},\mathbb{S},\mathbb{D}) - \Delta\mathbb{S} = \mathbb{O}$
4) $\mathbb{G}(\overset{**}{\mathbb{S}},\overset{**}{\mathbb{D}},\overset{*}{\mathbb{S}},\overset{*}{\mathbb{D}},\mathbb{S},\mathbb{D}) = \mathbb{O}$
5) —||— $- \Delta\mathbb{S} = \mathbb{O}$

$\mathbb{G}$ nelineární spojitá funkce

$\overset{*}{\mathbb{A}}$ nějaká objektivní derivace $\mathbb{A}$

$\boxed{\text{Ad 1)}}$

$$\mathbb{G}(\mathbb{S},\mathbb{D}) = \mathbb{O}$$

implicitní konstit. tekutiny

$$\mathbb{S} = 2\nu(|\mathbb{D}|^2)\,\mathbb{D} \qquad \mathbb{D} = \alpha(|\mathbb{S}|^2)\,\mathbb{S}$$

$$\boxed{\mathbb{D} = \mathbb{D}v}$$

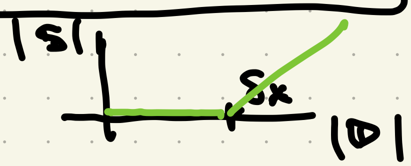
$$\boxed{\alpha(|\mathbb{S}|^2,|\mathbb{D}|^2)\,\mathbb{S} = \beta(|\mathbb{S}|^2,|\mathbb{D}|^2)\,\mathbb{D}}$$

$$\mathbb{S} = |\mathbb{D}|^{p-2}\mathbb{D} \underset{\text{ekvi.}}{\iff} \mathbb{D} = |\mathbb{S}|^{p'-2}\mathbb{S}$$

$p \in (1,+\infty)$, $p' = p/(p-1)$

$$\mathbb{S} = (1+|\mathbb{D}|^2)^{\frac{p-2}{2}}\mathbb{D} \longleftrightarrow \mathbb{D} = (1+|\mathbb{S}|^2)^{\frac{p'-2}{2}}\mathbb{S}$$

$$\boxed{\frac{1}{2\nu}\mathbb{S} = \frac{(|\mathbb{D}|-\delta_*)^+}{|\mathbb{D}|}\mathbb{D}} \longleftrightarrow \boxed{2\nu\,\mathbb{D} = \frac{(|\mathbb{S}|-\tau_*)^+}{|\mathbb{S}|}\mathbb{S}}$$



mají schopnost popsat jevy

- zesílení / zeslabení smyku
- —||—
- přítomnost aktivačních napětí / kritérií.

Ad 2)–5) schopnost popsat
- normal stress diff.
- napěťovou relaxaci
- nelineární creep.

a také zesilování/zeslabení smyku.

2) Oldroyd B, Maxwell
3) s napěť. relax.
4) Burgersův
5) —"—

El-Kareh, Leal 1989

Pohled PDR-matematika na mat. teorii pro modely nestlač. tekutin

FYZ | MAT

NSR 1821–1845

Oseen 1921/22

formulace bilančních rovnic v integrálním tvaru

$$C^2 \longrightarrow C^1$$

Leray 1929 – 1933

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu\|\nabla v\|_2^2 = 0$$

$$v \in L^\infty(0,T;L^2(\Omega)^3)$$
$$\nabla v \in L^2(0,T;L^2(\Omega)^{3\times 3})$$

∃ slabého řeš.
- pro Cauchyho úlohu v 3D
- pro okraj. úlohu na zom. omez. $\Omega \subset \mathbb{R}^2$ v 2D

- (+) ROBUSTNÍ TEORIE
- (+) základ num. metod
  MKP, MKO, spektrální metody

1949 · Hopf (3D teorie v omez. oblasti)

1951 · Ladyzhenskaya, Kiselev

! ve 2D

· ! ve 3D OPEN

· hladkost ve 3D 2000 OPEN

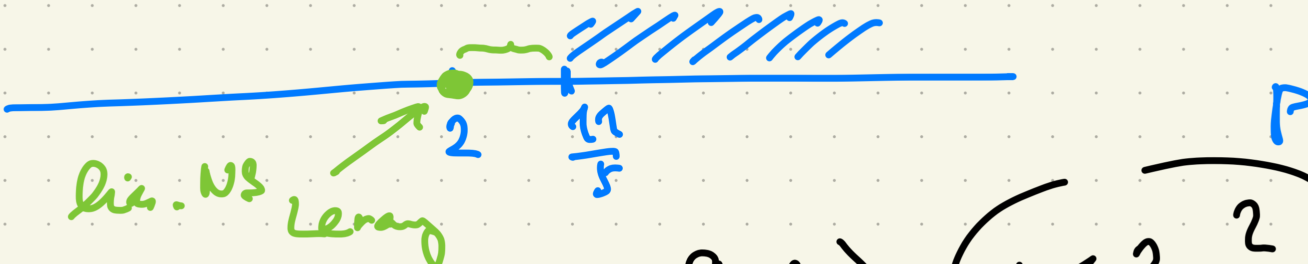
1965–70 $\mathbb{S} = 2(\nu_0 + \nu_1 |Du|^{p-2}) Du$

redukuje na NSR $p=2$ nebo $\nu_1 = 0$

$\exists!$ slabého řeš. $p \geq \frac{11}{5}$ $\quad p-2 = \frac{1}{5}$

1970 * Ladyzh. $p \geq \frac{5}{2}$ (J-L. Lions)

2019 * Bulíček, Kaplický, Pražák $p \geq \frac{11}{5}$



$p < 2$?

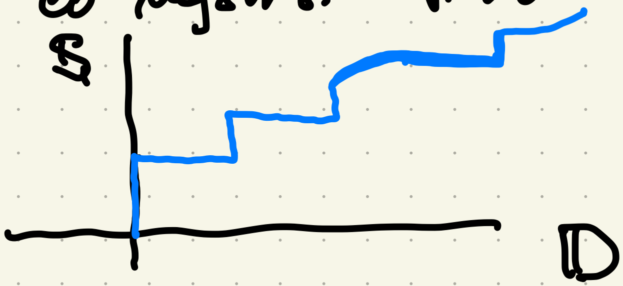
Aplikace (shear-thinning fluids)

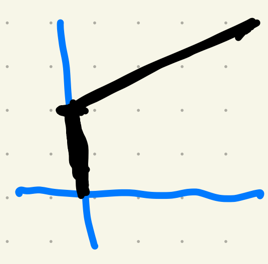
1988 Nečas & spol. Téma. 1985

Cíl: teorii à la Leray a) pro co největší interval p

b) pro co nejširší třídu tekutin

$G(S,D) = 0$





Pokud $G(S, D) = 0$ generuje maximální monotónní graf splňující

$$S : D \geq c_1 (|S|^{p'} + |D|^p) - c_2$$

$p \in (1, +\infty)$

$p > \frac{6}{5}$ ve 3D $\Rightarrow$ $\exists$ slabé řeš. à la Leray

Bulíček, Málek, Rajagopal
SIAM J. Math. Anal. 2000

- $p \in \langle 1, \frac{6}{5} \rangle$

Abbatiello, Feireisl
2000

obecnější pojem disipativního řeš.
existuje $\forall$ data
a je jediné ve třídě silných řeš.

- Buckmaster, Modena, Székelyhidi – ArXiv 2020

Leray-Hopf řeš. je nejednoznačné pro
$1 \leq p < \frac{6}{5}$

Teorie à la Leray pro modely 2) – 5) (ve 3D)

- 2000 Lions (PL), Masmoudi
  speciální model Oldroydova typu
  s korotační derivací
- 2011 Masmoudi Gieseküsův a FENE-P model

až "doposud" žádný výsledek pro modely s napět. relaxací

výsledek | Bathory, Bulíček, Málek 2021 Adv. in Nonl. Analysis.

MINULE

→ Nestlačitelné kom. tekutiny

$$\underbrace{\frac{1}{2}\frac{d}{dt}\|v\|_2^2 + \int_\Omega \mathbb{S}:\mathbb{D} + \int_{\partial\Omega} s\cdot v_\tau \, dS = 0}$$

→ $\dot{\varrho} = -\varrho \operatorname{div}\vec{v}$

$\varrho\dot{v} = \operatorname{div}\mathbb{T}$ $\quad E := \frac{|v|^2}{2} + e$

$\varrho\dot{E} = \operatorname{div}(\mathbb{T}v - \vec{j}_e)$ $\quad$, $\mathbb{T} = \mathbb{T}^T$ $\quad \rightarrow \quad \boxed{\varrho\dot{e} = \mathbb{T}:\mathbb{D} - \operatorname{div}\vec{j}_e}$

$\varrho\dot{\eta} - \operatorname{div}\vec{j}_\eta =: \zeta$ a $\zeta \geq 0$

Isotermální proces: θ konst.

$$\boxed{\psi = e - \theta\eta}$$

$$\varrho\dot{\psi} = \varrho\dot{e} - \theta\varrho\dot{\eta} = \mathbb{T}:\mathbb{D} - \operatorname{div}\vec{j}_e - \theta\zeta + \operatorname{div}(\theta\vec{j}_\eta)$$
$$= \mathbb{T}:\mathbb{D} - \theta\zeta + \operatorname{div}\underbrace{(\theta\vec{j}_\eta - \vec{j}_e)}_{=0}$$

$$\boxed{\vec{j}_\eta = \frac{\vec{j}_e}{\theta}}$$

stress power

$$\boxed{\mathbb{T}:\mathbb{D} - \varrho\dot{\psi} = \xi}$$

rychlost disipace $\quad \boxed{\xi := \theta\zeta}$

redukovaná termodynamická identita

KT: $\psi = \psi(\varrho) \Rightarrow$ stlač. NSR

ϱ konst. $\Rightarrow \dot{\psi} = 0$ $\quad$ + nestl. $\quad \boxed{\operatorname{div} v = 0}$

$$\boxed{\text{RT1}} \Rightarrow \boxed{\mathbb{S}:\mathbb{D} = \xi}$$

$\boxed{\text{Přednáška 3}}$ Mat. teorie

$$(P)\quad \left\{\begin{aligned} &\frac{\partial v}{\partial t} + \operatorname{div}(v\otimes v) - \operatorname{div} S = -\nabla p \\ &\qquad\qquad \operatorname{div} v = 0 \\ &\quad S = S^T \qquad G(S, D) = 0 \end{aligned}\right\} \text{ in } (0,T)\times\Omega$$

$$\left[\ \vec{v}\cdot\vec{n} = 0 \quad \text{a} \quad \boxed{\theta v_\tau + (1-\theta)\gamma_*(Sn)_\tau = 0}\right. \quad \text{na } (0,T)\times\partial\Omega$$

$$\Downarrow$$

$$\boxed{(Sn)_\tau = \frac{\theta}{(1-\theta)\gamma_*} v_\tau} \qquad \theta\in[0,1)$$

$$\left[\ v(0,\cdot) = v_0 \right. \quad \text{in } \Omega$$

$$(E)\quad \frac{1}{2}\frac{d}{dt}\int_\Omega |v|^2 + \int_\Omega S : Dv + \int_{\partial\Omega} (-Sn)_\tau \cdot v_\tau \, dS = 0$$

$\Downarrow$ NS Navier-stokes

$$(E^*)\quad \boxed{\frac{1}{2}\int_\Omega |v(t)|^2 + \nu \int_0^t\!\!\int_\Omega |Dv|^2 + \frac{\theta}{(1-\theta)\gamma_*}\int_0^t\!\!\int_{\partial\Omega} |v_\tau|^2 dS = \frac{1}{2}\|v_0\|_2^2}$$

$\boxed{\nabla v \in L^2}$ $\qquad L^2(0,T; L^2(\Omega))$, $L^2(Q_T)$

$$\boxed{v_0 \in L^2, \ \underset{\in L^2}{\operatorname{div} v_0} = 0 \ \text{ a } \ v_0\cdot n = 0 \text{ na } \partial\Omega \quad \in (W^{1/2,2}(\partial\Omega))^*}$$

[Constantin, Foias – NSEs, 1988
Sekce 2, 3, 5, 6, 7, 8 teorie pro NSEs]

Prostory funkcí

* $\boxed{\text{No-slip}}$ $C_0^\infty = \{v: \Omega \to \mathbb{R}^3,\ v \text{ hladké},\ \operatorname{supp} v \subset \Omega\}$
$C^\infty_{0,div} = \{v \in C_0^\infty;\ \operatorname{div} v = 0 \ v\ \Omega\}$

$\Omega \subset \mathbb{R}^3$ omezená, otevř., souvislé

$W_0^{1,p} := \overline{C_0^\infty}^{\|\cdot\|_{1,p}}$

$W^{1,p}_{0,div} := \overline{C^\infty_{0,div}}^{\|\cdot\|_{1,p}}$

$L^p_{n,div} := \overline{C^\infty_{0,div}}^{\|\cdot\|_p}$

Ω je lipschitz.

$W_0^{1,p} = \{v \in \{W^{1,p}(\Omega)\}^3;\ v = 0 \text{ na } \partial\Omega\}$

$W^{1,p}_{0,div} = \{v \in W_0^{1,p};\ \operatorname{div} v = 0 \text{ na } \Omega\}$

* $\boxed{\text{Skluzové podmínky}}$ Ω předpokládám: ot., om., souvis.
+ Lipschitz hranici
$\Downarrow$
$n: \partial\Omega \to \mathbb{R}^3$ je definován s.v. na $\partial\Omega$.

$W_n^{3,2} := \{v \in (W^{3,2}(\Omega))^3;\ \underline{v} \cdot \underline{n} = 0 \text{ na } \partial\Omega\}$ $\quad W^{3,2}_{n,div} = \{v \in W^{3,2}_n;\ \operatorname{div} v \overset{\shortparallel}{0}\}$

$W^{1,p}_{n,div} := \overline{W^{3,2}_{n,div}}^{\|\cdot\|_{1,p}}$

$W_n^{3,2} \overset{\shortparallel}{\cap} L^2_{n,div}$

$\boxed{\text{V obou situacích platí Poincarého} \leq}$

$\boxed{* \text{ No-slip}}$ $p \in (1,+\infty)$:

$$\underline{\|Dv\|_p \leq \|\nabla v\|_p} \leq \|v\|_{1,p} \underset{\text{Poincaré}}{\leq} c_p\|\nabla v\|_p \underset{\text{Korn}}{\leq} c_p c_K \|Dv\|_p$$

* Okrajové podmínky

Poincaré ≤ platí, než stačí $\vec{v}\cdot\vec{n}=0$ na $\partial\Omega$

$$\|\nabla v\|_p \le \|v\|_{1,p} \le c_p \|Dv\|_p$$

$\boxed{\text{Korn} \le}$ Rozlišit evol. vs stacionární problém

- lze využít $\underline{\int_{\partial\Omega} |v|^2 dS < C}$

$$\|\nabla v\|_p \le C_K \left\{ \|Dv\|_p + \|v\|_{2,\partial\Omega} \right\}$$

- Pokud Ω není axisymetrická, $1<p<+\infty$

$$\|\nabla v\|_p \le C_p \|Dv\|_p$$

Ω je axisymetrická $\equiv$ $\exists\, v \in W^{1,\infty}_n$ taková, že $Dv = 0$ a $\nabla v \neq 0$

$x_0 \in \mathbb{R}^3$

$v(x) = Q(x-x_0)$

$\boxed{\text{cylinder}}$

Platí

$$\|\nabla v\|_{p,\Omega} \le C_K \left\{ \|Dv\|_{p,\Omega} + \|v\|_{1,\Omega} \right\}$$

$\boxed{\text{Rychlo-kurs k teorii NSR}}$ – Lerayův program

$(P)_{NS} := (P)$ kde $G(\mathbb{S},\mathbb{D})$ je nahrazena pomocí $\boxed{\mathbb{S} = 2\nu\,\mathbb{D}v}$ $\left[\nu = \frac{\mu_*}{\varrho_*} > 0\right]$

$$(\operatorname{div}\mathbb{S})_i = \nu\frac{\partial}{\partial x_j}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \nu\Delta v_i + \nu\frac{\partial}{\partial x_i}\operatorname{div} v = \nu\Delta v_i$$

$$\boxed{\text{NSR}\qquad \frac{\partial v}{\partial t} + \operatorname{div}(v\otimes v) - \nu\Delta v = -\nabla p}$$

NEVHODNÝ POČ. TVAR RCIC PRO SLIPOVÉ PODMÍNKY

Lépe

$$\frac{\partial v}{\partial t} + \operatorname{div}_j(v\otimes v) - \operatorname{div}_j\underbrace{(2\nu\,\mathbb{D}v)}_{\mathbb{S}} = -\nabla p \qquad \Big|\cdot\varphi + \int_\Omega dx$$

NO SLIP $\varphi = 0$ MOHU NEMUSÍM $\operatorname{div}\varphi = 0$

SLIP $\varphi\cdot n = 0$

$$\int_\Omega \frac{\partial v}{\partial t}\cdot\varphi - \int_\Omega (v\otimes v)\cdot\nabla\varphi\,dx + \cancel{\int_{\partial\Omega} v_i \underbrace{v_j n_j}_{\vec v\cdot\vec n}\varphi_i\,dS}$$

$$+ \int_\Omega 2\nu\,\mathbb{D}v\cdot\nabla\varphi - \int_{\partial\Omega} S_{ij}n_j\varphi_i = \cancel{\int_\Omega p\operatorname{div}\varphi\,dx} - \cancel{\int_{\partial\Omega} p\underbrace{\varphi_i n_i}_{\vec\varphi\cdot\vec n}dS}$$

$= 0$ pokud no-slip neboť $\varphi = 0$ na $\partial\Omega$

$= (\mathbb{S}n)_\tau\cdot\varphi_\tau$ pokud slip neboť $\varphi\cdot n = 0$ na $\partial\Omega$

$= \frac{-\theta}{(1-\theta)\gamma_*}v_\tau\cdot\varphi_\tau$ $\qquad \int_{\partial\Omega}\frac{\theta}{(1-\theta)\gamma_*}v_\tau\cdot\varphi_\tau\,dS$

$$(E)_{NS}\left[\ \operatorname{ess\,sup}_{t} \frac{\|v(t)\|_2^2}{2} + 2\nu\int_0^t\int_\Omega |Dv|^2 + \frac{\theta}{(1-\theta)\gamma_*}\int_0^t\int_{\partial\Omega}|v_\tau|^2 \le \frac{\|v_0\|_2^2}{2}\right.$$

$$(WF)\quad -\int_0^t\int_\Omega v\cdot\frac{\partial\varphi}{\partial t}\,dx - \int_0^t\int_\Omega (v\otimes v)\cdot\nabla\varphi \qquad \boxed{\varphi(T,\cdot)=0}$$

$$+\int_0^t\int_\Omega 2\nu\, Dv\cdot D\varphi + \int_0^t\int_{\partial\Omega} c(\theta,\gamma_*)\, v_\tau\cdot\varphi_\tau\, dS$$

$$= \int_\Omega \underbrace{v(0,\cdot)}_{v_0(x)}\cdot\varphi(0,\cdot)\,dx$$

Def. slabé řešení $(P)_{NS}$ Řekneme, že v je slabé řeš. $(P)_{NS}$

pokud

- $v\in L^\infty(0,T;L^2)\cap L^2(0,T;W^{1,2})$ $\to W^{1,2}_{0,div}\big|_{v=0}$, $W^{1,2}_{n,div}\big|_{v\cdot n=0}$
- WF platí $\forall\varphi\in C^\infty$ $\to C^\infty_{0,div}$, $C^\infty_{n,div}$
- platí $(E)_{NS}$
- $\lim_{t\to 0+}\|v(t,\cdot)-v_0\|_2=0$
- v je slabě spojité v čase v prostoru L^2

test

Def. slabé řešení $(P)_{NS}$

Řekneme, že v je slabé ř.š. $(P)_{NS}$ pokud

- $v \in L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2})$ $\to W^{1,2}_{0,div}\,|_{v=0}$, $W^{1,2}_{n,div}\,|_{v\cdot n=0}$
- WF platí $\forall \varphi \in C^\infty$ $\to C^\infty_{0,div}$, $C^\infty_{n,div}$
- platí $(E)_{NS}$
- $\lim_{t\to 0+} \|v(t,\cdot)-v_0\|_2 = 0$
- v je nějak spojité v čase v prostoru L^2

L4 Leray's theory for NSE (Leray-Hopf)

$v = (v_1, \dots, v_d)$
p

$$(P) \begin{cases} \operatorname{div} v = 0 \\ \dfrac{\partial v}{\partial t} + v_k \dfrac{\partial v}{\partial x_k} - \nu \Delta v = -\nabla p \end{cases} \quad \text{in } \underbrace{(0,T)\times\Omega}_{Q_T}, \ \Omega \subset \mathbb{R}^d, \ d \geq 2$$

$$v = 0 \quad \text{on } (0,T)\times\partial\Omega$$

$$v(0,\cdot) = v_0 \quad \text{in } \Omega$$

$W_0^{1,2} = \text{closure } \overline{C_0^\infty}^{\|\cdot\|_{1,2}}$ | $W^{1,2}_{0,div}$ | $L^2_{n,div} = \overline{C^\infty_{0,div}}^{\|\cdot\|_2}$

Definition (Let $v_0 \in L^2_{n,div}$) We say that v is weak solution to (P) if

- $v \in L^\infty(0,T; L^2_{n,div}(\Omega)) \cap L^2(0,T; W^{1,2}_{0,div}) \cap C([0,T]; L^2_{weak}(\Omega))$
- $\dfrac{\partial v}{\partial t} \in L^\alpha(0,T; (W^{1,2}_{0,div})^*)$ for $\alpha > 1$; $\alpha = \frac{4}{3}$ if $d=3$, 2 if $d=2$
- $\left\langle \dfrac{\partial v}{\partial t}, \varphi \right\rangle_{(W^{1,2}_{0,div})^*} - (v \otimes v, \nabla \varphi) + \nu (\nabla v, \nabla \varphi) = 0 \quad \forall \varphi \in W^{1,2}_{0,div}$ for a.a. $t \in [0,T]$.
- $\frac{1}{2}\|v(t)\|_2^2 + \nu \int_0^t \|\nabla v\|_2^2 \, ds = \frac{1}{2}\|v_0\|_2^2$
- $\lim_{t\to 0+} \|v(t,\cdot) - v_0\|_2^2 = 0$

Goal: to show existence of weak solution

Notes: In 2D:
- $\dfrac{\partial v}{\partial t} \in L^2(0,T; (W^{1,2}_{0,div})^*)$
- v is admissible test function in weak form
- weak sol. is unique

In $d \geq 3$: uniqueness open

- PDE theory of weak sol. - done in 2 steps

1) Stability of PDE or its weak formulation w.r.t. weakly converging subsequences

and the convergence take place in function spaces in which we have apriori estimates

2) complete proof starting from (some) suitable (close) approximations, showing the existence of solutions for these approximations and passing to the limit from approximations to original problem.

Ad 1) for 3D NSEs Let $\{v^\varepsilon\}_{\varepsilon>0}$ be weak solution of our problem (P). a.a. $\forall t \in [0,T]$

In particular, $(\bullet)$ $\frac{1}{2}\|v^\varepsilon(t)\|_2^2 + \nu_* \int_0^t \|\nabla v^\varepsilon(t)\|_2^2 \le \frac{1}{2}\|v_0^\varepsilon\|_2^2$

$(*)$ $\left\langle \frac{\partial v^\varepsilon}{\partial t}, \varphi \right\rangle - (v^\varepsilon \otimes v^\varepsilon, \nabla\varphi) + \nu_* (\nabla v^\varepsilon, \nabla\varphi) = 0$ $\quad \forall \varphi \in W^{1,2}_{0,div}$

$\boxed{v^\varepsilon(0,\cdot) = v_0^\varepsilon \text{ in } \Omega}$

$\|v_0^\varepsilon - v_0\|_2 \to 0$, $\varepsilon \to 0$

$\{v^\varepsilon\}$ is bdd in $L^\infty(0,T;L^2) \cap L^2(0,T;W^{1,2}_{0,div})$

since $\sup_{t\in[0,T]} \|v^\varepsilon(t)\|_2^2 \le \|v_0\|_2^2$

$\nu_* \int_0^T \|\nabla v^\varepsilon\|_2^2 \le \|v_0\|_2^2$

there is $v \in L^\infty(0,T;L^2) \cap L^2(0,T;W_0^{1,2})$

$v^\varepsilon \rightharpoonup v$ $*$-weakly $L^\infty(0,T;L^2)$

weakly $L^2(0,T;W^{1,2}_{0,div})$

Q: Is v weak solution to (P)

Steps

1) $\left\{\frac{\partial v^\varepsilon}{\partial t}\right\}$ is bounded in $L^{4/3}(0,T,(W^{1,2}_{0,div})^*)$
2) $v^\varepsilon \to v$ strongly in $L^2(0,T;L^2(\Omega))$ which suffices to take the limit in $(*)$.
3) $v \in C_{weak}([0,T],L^2_{n,div})$
4) v fulfills the energy ineq. $(\bullet)$
5) $\lim \|v(t) - v_0\|_2^2$

$\boxed{\text{Ad 1)}}$ From weak formulation, for smooth function in time, we conclude

$$\int_0^T \left\langle \frac{\partial v^\varepsilon}{\partial t}, \psi \right\rangle = \underbrace{\int_0^T (v^\varepsilon \otimes v^\varepsilon, \nabla\psi)} - \nu_* \int_0^T (\underbrace{\nabla v^\varepsilon}_{\in L^2}, \nabla\psi)$$

$\forall \psi \in C^\infty(0,T; W^{1,2}_{0,\mathrm{div}})$

L^2

$$\boxed{\left\| \frac{\partial v^\varepsilon}{\partial t} \right\|_{X^*} := \sup_{\|\psi\|_X \le 1} \left| \left\langle \frac{\partial v^\varepsilon}{\partial t}, \psi \right\rangle_{X_*, X_*} \right|}$$

$$\le \nu_* \underbrace{\left(\int_0^T \|\nabla v^\varepsilon\|_2^2 \right)^{1/2}}_{< \frac{1}{2}\|v_0\|_2^2} \underbrace{\left(\int_0^T \|\nabla\psi\|_2^2 \right)^{1/2}}_{\le 1}$$

$$\int_0^T \underbrace{\|v^\varepsilon\|_4^2} \|\nabla\psi\|_2 \le \int_0^T \|\nabla v^\varepsilon\|_2^2 \|\nabla\psi\|_2$$

$W^{1,2} \hookrightarrow L^6$ Imbedding Poincaré

$\boxed{d=3}$

$\boxed{d=4}$ Homework: $\boxed{\text{(1) Do derivat. of } \frac{\partial v^\varepsilon}{\partial t} \text{ if } \underline{d=4}}$ (3)

(2) Show that $v^\varepsilon \to v$ strongly in L^2

Better via interpolation: by Hölder ineq.

$$\|z\|_q \le \|z\|_{p_1}^\lambda \|z\|_{p_2}^{1-\lambda} \qquad 1 \le p_1 \le q \le p_2 \le +\infty$$

$$\lambda \in [0,1] \qquad \boxed{\frac{1}{q} = \frac{\lambda}{p_1} + \frac{1-\lambda}{p_2}}$$

$$\boxed{\|v^\varepsilon\|_4 \le \|v^\varepsilon\|_2^{1/4} \|v^\varepsilon\|_6^{3/4}}$$

$\frac{1}{4} = \frac{\lambda}{2} + \frac{1-\lambda}{6}$

$3 = 6\lambda + 2 - 2\lambda$

$\lambda = 1/4 \quad 1-\lambda = \frac{3}{4}$

$$\int_0^T \|v^\varepsilon\|_4^2 \|\nabla\psi\|_2 \le \int_0^T \|v^\varepsilon\|_2^{1/2} \|v^\varepsilon\|_6^{3/2} \|\nabla\psi\|_2 \le$$

Hölder in time

$$\le C \sup_t \|v^\varepsilon(t)\|_2^{1/2} \int_0^T \|\nabla v^\varepsilon\|_2^{3/2} \|\nabla\psi\|_2 \le$$

$\delta = \frac{4}{3} \quad \delta' = 4$

$$\le \underbrace{C(\|v_0\|_2)^{1/2}} \Big(\underbrace{\int_0^T \|\nabla v^\varepsilon\|_2^2}_{\le \|v_0\|_2^2} \Big)^{3/4} \underbrace{\left(\int_0^T \|\nabla\psi\|_2^4 \right)^{1/4}}_{\le 1} < +\infty$$

$\left\{\frac{\partial v^\varepsilon}{\partial t}\right\}_\varepsilon$ bdd in $L^{4/3}(0,T;(W^{1,2}_{0,\mathrm{div}})^*)$

$\boxed{\text{Ad 2)}}$

$$X_0 \hookrightarrow\hookrightarrow X_1 \hookrightarrow X_2$$

$$W^{1,2}_{0,\mathrm{div}} \hookrightarrow\hookrightarrow L^2_{n,\mathrm{div}} = (L^2_{n,\mathrm{div}})^* \hookrightarrow (W^{1,2}_{0,\mathrm{div}})^*$$

$, p \geq 1$

$$\left\{u \in L^2(0,T;W^{1,2}_{0,\mathrm{div}});\ \frac{\partial u}{\partial t} \in L^p(0,T;(W^{1,2}_{0,\mathrm{div}})^*)\right\} \hookrightarrow\hookrightarrow \underline{L^2(0,T;L^2_{n,\mathrm{div}})}$$

Having $v^\varepsilon \rightharpoonup v$ weakly in $L^2(0,T;W^{1,2}_{0,\mathrm{div}})$

$\frac{\partial v^\varepsilon}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}$ — " — $L^{4/3}(0,T;(W^{1,2}_{0,\mathrm{div}})^*)$

$\Longrightarrow$ $\underline{v^\varepsilon \to v \text{ STRONGLY}}$ in $L^2(0,T;L^2_{n,\mathrm{div}})$

HW part 3: Show that if $d=3,4$

$$\int_0^T\int_\Omega v^\varepsilon \otimes v^\varepsilon : \nabla\psi \longrightarrow \int_0^T\int_\Omega v\otimes v : \nabla\psi$$

$\boxed{\text{Ad 3)}}$ $\boxed{v \in C_{weak}(0,T;L^2_{n,\mathrm{div}})}$

Take weak form for v^ε write in the following way: and integrate per partes w.r.t. time

$\psi \in C^\infty(0,T;W^{1,2}_{n,\mathrm{div}})$

$\psi(T,\cdot)=0$

$$-\int_0^T\left(v^\varepsilon, \frac{\partial\psi}{\partial t}\right) - \int_0^T (v^\varepsilon\otimes v^\varepsilon, \nabla\psi) + \nu_*\int_0^T (\nabla v^\varepsilon, \nabla\psi) = (v_0^\varepsilon, \psi(0,\cdot))$$

$\varepsilon\to 0$ $\downarrow$

$$-\int_0^T\left(v, \frac{\partial\psi}{\partial t}\right) - \int_0^T (v\otimes v, \nabla\psi) + \nu_*\int_0^T (\nabla v, \nabla\psi) = (v_0, \psi(0,\cdot))$$

$$\lim_{t\to t_0} \underbrace{(v(t,\cdot) - v(t_0,\cdot), \varphi)}_{} = 0 \quad \forall\, \varphi \in L^2_{n,\mathrm{div}}$$

$\overset{def}{=}$ $v \in C_{weak}([0,T], L^2_{n,\mathrm{div}})$

v satisfies W.F. for $\psi \in L^4(0,T; W^{1,2}_{0,div})$

Taking $\psi(t,x) = \varphi(x)\chi_{[t_0,t]}$ $\qquad \underline{\varphi \in W^{1,2}_{0,div}}$

$t > t_0$ $\quad \int_0^T \left\langle \frac{\partial v}{\partial t}, \psi(t) \right\rangle dt = \int_{t_0}^{t} \left\langle \frac{\partial v}{\partial t}, \varphi \right\rangle dt = \left(v(t,\cdot) - v(t_0,\cdot), \varphi \right)$

$\|$ weak formulation

$$- \nu \underbrace{\int_{t_0}^{t} (\nabla v, \nabla \varphi)} + \int_0^t \left(\frac{v \otimes v}{2}, \frac{\nabla \varphi}{2} \right)$$

$$\leq \int_{t_0}^{t} \|\nabla v\|_2 \cdot \|\nabla \varphi\|_2 \, 1 \leq \overset{|v^\varepsilon|^4}{\|\nabla \varphi\|_2} \underbrace{\left(\int_{t_0}^{t} \|\nabla v\|_2^2 \right)^{1/2}}_{< C} |t - t_0|^{1/2} \xrightarrow[t \to t_0]{} 0$$

$$\Rightarrow \lim_{t \to t_0} \left(\underbrace{v(t,\cdot) - v(t_0,\cdot)}_{L^\infty(0,T;}, \varphi \right) = 0 \qquad \underline{\forall \varphi \in W^{1,2}_{0,div}}$$

$\downarrow^2$

$\forall \varphi \in L^2_{n,div}$ by density