Name and surname: ______________________________

| Problem | 1 | 2 | 3 | 4 | **Total points** |
|---|---|---|---|---|---|
| Points | 7 | 7 | 10 | 6 | 30 |
| Points earned | | | | | |

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[7] 1. **Operator and its properties**

Let the operator $L : L^2((0,1)) \to L^2((0,1))$ be defined through

$$(Lu)(x) := \int_0^x u(s)\,\mathrm{d}s.$$

1. Show that for every $u \in L^2((0,1))$, the function $Lu$ is Hölder continuous; in fact you should show that $u \in C^{0,\frac{1}{2}}((0,1))$. Is then $L \in \mathcal{L}(L^2((0,1)))$?
2. Show that $L : L^2((0,1)) \to L^2((0,1))$ is compact.
3. Find the adjoint operator $L^*$ to $L$.
4. Does the equation $u - Lu = g$ have a unique solution for a given $g \in L^2((0,1))$? Explain in detail. If $g$ is differentiable, what ODE is satisfied by such a solution?

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[7] 2. **Spectrum**

Consider $L : \ell^\infty \to \ell^\infty$ defined through

$$L : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, \dots). \tag{1}$$

1. Show that $L \in \mathcal{L}(\ell^\infty)$ and its norm equals to 1. Is $L$ onto? Is $L$ one-to-one?
2. Define the definition of spectrum, point spectrum, essential spectrum, continuous spectrum and residual spectrum (for any $L \in \mathcal{L}(X)$, $X$ being a Banach space). Give definition of spectral radius, its characterization and the upper bound.
3. Determine all these spectra for the operator $L$ from (1).

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[10] 3. **Weakly converging sequence**

Let $X$ be a Banach space.

1. Define $X^*$ and explain why $X^*$ is a Banach space.
2. Give the definitions of $\{x_n\}$ converges to $x$ (i) strongly, (ii) in the norm, (iii) weakly, (iv) *-weakly.
3. Explain correctness of the concept of weak convergence.
4. Show that if $\{\Phi_n\}$ converges to $\Phi$ in $X^*$, then $\{\Phi_n\}$ converges to $\Phi$ *-weakly.
5. Show that weakly converging sequence is bounded.
6. Give an explicit description of $\{f_n\}$ converges to $f$ weakly in $L^p(\Omega)$.
7. Is $\{\sin(nx)\}$ converging weakly in $L^2((0,1))$? If so, what is the (weak) limit?
8. Is $\{\sqrt{n}\sin(nx)\}$ converging weakly in $L^2((0,1))$?

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[6] 4. **Weak formulation**

Consider the problem: given $f \in L^2(\Omega)$ and two functions $a, c \in L^\infty(\Omega)$, find $u : \Omega \to \mathbb{R}$ satisfying

$$\begin{aligned} -\mathrm{div}(a(x)\nabla u) + c(x)u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

1. Give the definition of weak solution to (2).
2. Find sufficient (but general) conditions on $a$ and $c$ so that you can guarantee the existence and uniqueness of weak solution. Provide the explanation.
3. What conditions on $u$ guarantee that weak solution satisfies the first equation in (2) almost everywhere (pointwise)? Give explanation.