Basic concepts in risk management

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7.10.2016

In the presentation of basic concepts of financial risks management we follow McNeil et al. (2005).

1 Risk factors and loss distributions

We introduce a general framework for modelling the change in the value of a given portfolio of assets depending on certain risk factors. The approach can be used in measuring and managing market risk. It corresponds to the methodology introduced and further developed in the commercial product RiskMetrics, a set of techniques and data used to measure the market risk in portfolios of fixed income instruments, equities, foreign exchange, commodities, and their derivatives.

We denote by $V(s)$ the value of a portfolio at time s. For a given time horizon Δ we define the loss over the period $[s, s+\Delta]$ by

$$L_{[s,s+\Delta]} = -\left(V(s+\Delta) - V(s)\right). \tag{1}$$

$L_{[s,s+\Delta]}$ is a random variable (from the viewpoint of time s) with certain probability distribution. The loss distribution is taken as contitional (given the information available up to time s) or unconditional.

Remark. R.v. $V(s+\Delta) - V(s)$ represents profit or loss over the period $[s, s+\Delta]$. In risk management we are mainly concerned with large losses. We therefore usually analyse the upper tail of the distibution of loss (1).

For a fixed time horizon Δ, we simplify the notation by using a time series $V_t = V(t\,\Delta)$. Then

$$L_{t+1} = L_{[t\,\Delta,(t+1)\,\Delta]} = -\left(V_{t+1} - V_t\right).$$

The value V_t is modelled as a function of time and of a random vector $\mathbf{Z}_t = (Z_{t,1}, \ldots, Z_{t,d})'$ of risk factors.

We have

$$V_t = f\left(t, \mathbf{Z}_t\right) \tag{2}$$

for some measurable function $f : \mathrm{R}_+ \times \mathrm{R}^d \mapsto \mathrm{R}$.

We assume that $\mathbf{Z}_t$ is observable at time t. Denoting by $\mathbf{X}_t = \mathbf{Z}_t - \mathbf{Z}_{t-1}$ the vector of risk-factor changes we obtain the expression

$$L_{t+1} = -\left(f\left(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}\right) - f\left(t, \mathbf{Z}_t\right)\right). \tag{3}$$

Since $\mathbf{Z}_t$ is known at time t, the loss distribution is determined by the distribution of the risk-factor changes $\mathbf{X}_{t+1}$. This fact is expressed by a new notation

$$L_{t+1} = l_{[t]} \left(\mathbf{X}_{t+1} \right), \tag{4}$$

where

$$l_{[t]}(\mathbf{x}) = - \left(f \left(t+1, \mathbf{Z}_t + \mathbf{x} \right) - f \left(t, \mathbf{Z}_t \right) \right), \ \mathbf{x} \in \mathrm{R}^d \, .$$

If f is differentiable, we approximate loss (3) by

$$L^{\Delta}_{t+1} = - \left(f_t \left(t, \mathbf{Z}_t \right) + \sum_{i=1}^{d} f_{z_i} \left(t, \mathbf{Z}_t \right) \, X_{t+1,i} \right), \tag{5}$$

where f_t, f_{z_i} denote partial derivatives of f. It means that we substitute the operator $l_{[t]}$ in (4) by a linear function of risk-factor changes

$$l^{\Delta}_{[t]}(\mathbf{x}) = - \left(f_t \left(t, \mathbf{Z}_t \right) + \sum_{i=1}^{d} f_{z_i} \left(t, \mathbf{Z}_t \right) \, x_i \right). \tag{6}$$

The approximation is acceptable for a short time horizon (small changes of risk factors) and if the function f has small second derivatives.

Remark. In (2)-(5) we asssumed that time is measured in units of Δ. When we consider measuring of time in years we obtain

$$V_t = f \left(t, \mathbf{Z}_t \right) = g \left(t \, \Delta, \mathbf{Z}_t \right)$$

and

$$L_{t+1} = - \left(g \left((t+1) \, \Delta, \mathbf{Z}_t + \mathbf{X}_{t+1} \right) - g \left(t \, \Delta, \mathbf{Z}_t \right) \right),$$

where Δ gives the length of the time horizon in years. For the linear approximation (5) it then holds

$$L_{t+1}^{\Delta} = -\left(g_t \left(t\,\Delta, \mathbf{Z}_t \right) \, \Delta + \sum_{i=1}^{d} g_{z_i} \left(t\,\Delta, \mathbf{Z}_t \right) \, X_{t+1,i} \right). \tag{7}$$

For a short time horizon Δ the first term in the brackets on the right-hand side of (7) is small and is sometimes dropped in practice.

Conditional and unconditional loss distribution

The differences between the distribution of loss L_{t+1} conditioned on the information available at time t and the unconditional loss distribution are related to the properties of the time series $\{\mathbf{X}_t\}, t \in \mathrm{N}$. Suppose that the risk-factor changes form a (strictly) stationary time series with stationary distribution with the distribution function $F_{\mathbf{X}}$. Denote by $S_t = \sigma\left(\{X_s,\ s \leq t\}\right)$ the sigma field generated by risk-factor changes up to time t. Denote by $F_{\mathbf{X}_{t+1}|S_t}$ the d.f. of the conditional distribution of $\mathbf{X}_{t+1}$ given current information at time t. In general this distribution is different from the stationary distribution with d.f. $F_{\mathbf{X}}$. (If $\{\mathbf{X}_t\}$ is a sequence of i.i.d. random variables, both distributions coincide.)

The distribution function of the loss conditioned on the iformation available at time t is

$$F_{L_{t+1}|S_t}(l) = \mathrm{P}\left(L_{t+1} \leq l | S_t\right) = \mathrm{P}\left(l_{[t]}\left(\mathbf{X}_{t+1}\right) \leq l | S_t\right).$$

Distributions of this type are often used in market risk management.

The d. f. of the unconditional distribution,

$$F_{L_{t+1}}(l) = \mathrm{P}\left(L_{t+1} \leq l\right) = \mathrm{P}\left(l_{[t]}\left(\mathbf{X}_{t+1}\right) \leq l\right)$$

depends on the stationary distribution of the series $\{\mathbf{X}_t\}$. The unconditional distribution is more suitable for longer time horizons that are often considered in credit risk management and in insurance.

Example 1. Consider a fixed portfolio of d stocks and denote by λ_i the number of shares of stock i in the portfolio at time t. The process $\left(S_{t,i}\right)_{t\in\mathrm{N}}$ describes the development of the price of stock i in time. The risk factors are logarithmic prices

$$Z_{t,i} = \log S_{t,i},\ \ i = 1, \ldots, d.$$

The risk-factor changes are then the log-returns

$$X_{t+1,i} = \log S_{t+1,i} - \log S_{t,i}$$

and the value of the portfolio at time t is

$$V_t = \sum_{i=1}^{d} \lambda_i \exp\left(Z_{t,i}\right).$$

For the loss over the period $[t, t+1]$ it yields

$$L_{t+1} = -\left(V_{t+1} - V_t\right) = -\sum_{i=1}^{d} \lambda_i S_{t,i}\ \left(\exp\left(X_{t+1,i}\right) - 1\right).$$

The linear approximation (5) of the loss is given by

$$L_{t+1}^{\Delta} = -\sum_{i=1}^{d} \lambda_i \, S_{t,i} \, X_{t+1,i} = -V_t \sum_{i=1}^{d} w_{t,i} \, X_{t+1,i}, \tag{8}$$

where the weight $w_{t,i} = (\lambda_i \, S_{t,i}) \, / V_t$ gives the proportion of the portfolio value invested at time t in stock i.

From (8) it follows the form of the linear operator

$$l_{[t]}^{\Delta}(\mathbf{x}) = -V_t \, \mathbf{w}_t' \, \mathbf{x}. \tag{9}$$

Substituting in (9) a random vector $\mathbf{X}$ with a mean vector μ and a covariance matrix Σ, we obtain

$$\mathrm{E}\left(l_{[t]}^{\Delta}(\mathbf{X})\right) = -V_t \, \mathbf{w_t}' \mu, \tag{10}$$

$$\mathrm{Var}\left(l_{[t]}^{\Delta}(\mathbf{X})\right) = V_t^2 \, \mathbf{w_t}' \, \Sigma \, \mathbf{w_t}. \tag{11}$$

(10) a (11) are applicable for computing the first two moments in both the conditional and unconditional cases. In the conditional case we substitute for μ and Σ the mean vector μ_t and the covariance matrix Σ_t of the conditional distribution $F_{\mathbf{X}_{t+1}|S_t}$. In the unconditional case we use the mean vector and the covariance matrix of the unconditional distribution of $\mathbf{X}_{t+1}$ (stationary distribution $F_{\mathbf{X}}$).

Example 2. Consider a portfolio of d default-free zero-coupon bonds with maturity T_i and price $p(s, T_i)$ at time s, $i = 1, \ldots, d$. We set the face values

$p(T_i, T_i)$ equal to one. By λ_i we denote the number of bonds with maturity T_i in the portfolio.

The price at time s of a bond with maturity T is expressed by

$$p(s,T) = \exp\left(-(T-s)\, y(s,T)\right),$$

where $T \mapsto y(s,T)$ represents the yield curve at time s (continuously compounded). We take yields $y(s,T_i), i = 1, \ldots, d$, as risk factors. The value of the portfolio is then given by

$$V_t = \sum_{i=1}^{d} \lambda_i\, p(t\,\Delta, T_i) = \sum_{i=1}^{d} \lambda_i \, \exp\left(-(T_i - t\,\Delta)\, y(t\,\Delta, T_i)\right).$$

The linear approximation of the loss is, according to (7),

$$L^{\Delta}_{t+1} = -\sum_{i=1}^{d} \lambda_i\, p(t\,\Delta, T_i)\ \left(y(t\,\Delta, T_i)\,\Delta - (T_i - t\,\Delta)\, X_{t+1,i}\right), \tag{12}$$

where

$$X_{t+1,i} = y\left((t+1)\,\Delta, T_i\right) - y\left(t\,\Delta, T_i\right).$$

(12) is related to the concept of *duration*. Suppose that the yield curve is flat, i.e. $y(s,T) = y(s)$ and the only possible change is paralel shift of the yield curve so that $y(s+\Delta, T) = y(s) + \delta$ for all T. Then (12) is rewritten as

$$L^{\Delta}_{t+1} = -V_t\left(y(t\,\Delta)\,\Delta - D\,\delta\right),$$

where

$$D = \sum_{i=1}^{d} \frac{\lambda_i\, p(t\,\Delta, T_i)}{V_t}\,(T_i - t\,\Delta)$$

is a weighted average of the times to maturity of the different cash flows in the portfolio, with weights proportional to the discounted value of the cash flows (duration).

Example 3. We consider a portfolio of m loans, the exposure to counterparty i is denoted by e_i. We consider time horizon of one year ($\Delta = 1$). For simplicity we assume that all loans are repaid at the same time $T > t$. Let $Y_{t,i}$ be random variable for which $Y_{t,i} = 1$ if counterparty i defaults in the time period $[0, t]$ and $Y_{t,i} = 0$ otherwise. We assume that upon default the whole exposure is lost. The value of loan i at time t is

$$\exp\left(-(T-t)\ (y(t,T) + c_i(t,T))\right)\ e_i,$$

where $y(t,T)$ is risk-free yield and $c_i(t,T)$ is credit spread of counterparty i corresponding to the maturity T. We further use a simplifying assumption

$$c_i(t,T) = c(t,T), i = 1, \ldots, m.$$

Then it holds

$$V_t = \sum_{i=1}^{m} (1 - Y_{t,i}) \exp\left(-(T-t)\ (y(t,T) + c(t,T))\right)\ e_i.$$

The risk in the case of a loan portfolio consists of the risk of a counterparty default, the risk of the decrease of the value of future payments caused by the increase of interest rates and in the risk of the increase of the credit spread.

We take as the vector of risk factors

$$\mathbf{Z}_t = (Y_{t,1}, \ldots, Y_{t,m}, y(t,T), c(t,T))'.$$

In credit risk management the linear approximation of loss is not suitable due to the discrete nature of the default indicators and the long time horizon. The key issue is finding a model for the joint distribution of $Y_{t+1,i}, i = 1, \ldots, m$.

2 Risk measurement

In this section we recall risk measures most widely used in management of financial risks.

2.1 Value at risk

Denote by $F_L(l)$ the d.f. of loss for a fixed time horizon Δ.

Definition. For $\alpha \in (0,1)$ the value at risk at level α is given by

$$\mathrm{VaR}_\alpha = \inf\{l \in \mathrm{R} : \mathrm{P}(L > l) \leq 1 - \alpha\} \tag{13}$$

$$= \inf\{l \in \mathrm{R} : F_L(l) \geq \alpha\}. \tag{14}$$

Examples. If we assume that L has normal distribution with mean μ and variance σ^2, we obtain

$$\mathrm{VaR}_\alpha = \mu + \sigma\, \Phi^{-1}(\alpha), \tag{15}$$

where Φ is d.f. of $N(0,1)$.

Similarly, if $\frac{L-\mu}{\sigma}$ has a standard t-distribution with ν degrees of freedom and with d.f. t_ν, we have

$$\mathrm{VaR}_\alpha = \mu + \sigma\, t_\nu^{-1}(\alpha).$$

Note that in this case it holds $\mathrm{E}\, L = \mu$, $\mathrm{Var}\, L = \frac{\nu\sigma^2}{\nu-2}$ for $\nu > 2$.

Remark. VaR is not subaditive, i.e. for the aggregate loss $L = L_1 + L_2$ from two portfolios it does not always hold

$$q_\alpha(F_L) \leq q_\alpha(F_{L_1}) + q_\alpha(F_{L_2}). \tag{16}$$

Example 1 continued. The vector of risk-factor changes $\mathbf{X}$ is assumed to have a multivariate normal distribution $N_d(\mu, \Sigma)$. Then the linearized loss (9) has a univariate normal distribution with mean (10) and variance (11).

VaR may be calculated using (15). We insert (10) for μ and (11) for σ^2 there. To obtain those values we need estimates of μ and σ based on historical data $X_{t-n+1}, \ldots, X_t$.

In market risk management the usual approach is using the conditional distribution. We consider conditional mean μ_{t+1} and conditional variance σ_{t+1} of X_{t+1} given the information to time t.

Various techniques are used in practice for forecasting the conditional moments, usually based on an appropriate time series model. Multivariate GARCH models are particularly popular.

Remark. The method described in the above example is known as Variance-Covariance metod used for measuring market risk over short time intervals.

2.2 Other risk measures based on risk distributions

Variance. Variance is used as a measure of risk in the portfolio theory of Markowitz. For working with variance we need to assume the existence of a finite second moment of the loss. Variance is not a good measure of risk for losses with skewed distributions.

(Upper) partial moment is a characteristic based on the upper tail of the loss distribution:

$$UPM(k,q) = \int_q^\infty (1-q)^k \, dF_L(l). \tag{17}$$

For $k = 0$ we obtain $\mathrm{P}(L \geq q)$, for $k = 1$ (17) equals $\mathrm{E}\left((L-q)\, I_{L \geq q]}\right)$ and for $k = 2$ and $q = \mathrm{E}\, L$ we obtain the upper semivariance of L.

The higher the value of k, the more conservative the risk measure is since it gives more weight to large deviations from q.

Expected shortfall

Definition. For a loss L with $\mathrm{E}\,|L|, < \infty$ and for $\alpha \in (0,1)$, the expected shortfall at level α is

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 q_u(F_L) du, \tag{18}$$

where $q_u(F_L)$ is the quantile function corresponding to the d.f. F_L.

From (18) it follows the relation between ES_α and VaR_α:

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L) du.$$

Obviously

$$\mathrm{ES}_\alpha \geq \mathrm{VaR}_\alpha .$$

For a continuous distribution d.f. F_L we have

$$\mathrm{ES}_\alpha = \frac{\mathrm{E}[L; L \geq q_\alpha(F_L)]}{1-\alpha} = \mathrm{E}\left(L | L \geq \mathrm{VaR}_\alpha\right), \tag{19}$$

where $\mathrm{E}[X; A] = \mathrm{E}(X\, I_A)$.

Remark. In case when F_L is not continuous (19) does not hold for all values of α, we have instead the expression

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \left(\mathrm{E}(L; L \geq q_\alpha) + q_\alpha \left(1 - \alpha - \mathrm{P}(L \geq q_\alpha)\right)\right).$$

Examples. Let the loss distribution be $N(\mu, \sigma^2)$. Then

$$\mathrm{ES}_\alpha = \mu + \sigma \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{1-\alpha},$$

where ϕ is the density of $N(0,1)$.

Assume that L has the t-distribution with a location parameter μ and a scale parameter σ, i.e. $\tilde{L} = \frac{L-\mu}{\sigma}$ has the standard t-distribution with ν degrees of freedom. It again holds

$$\mathrm{ES}_\alpha = \mu + \sigma \ \mathrm{ES}_\alpha(\tilde{L}).$$

From (19) we derive

$$\mathrm{ES}_\alpha(\tilde{L}) = \frac{g_\nu\left(t_\nu^{-1}(\alpha)\right)}{1-\alpha}\left(\frac{\nu + \left(t_\nu^{-1}(\alpha)\right)^2}{\nu - 1}\right).$$

For the measure ES we have the following version of the law of large numbers:

Lemma. For a sequence $\{L_i\}_{i\in \mathrm{N}}$ of i.i.d. random variables with d.f. F_L we have

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{[n(1-\alpha)]} L_{i,n}}{[n(1-\alpha)]} = \mathrm{ES}_\alpha \quad s.j., \tag{20}$$

where $L_{1,n} \geq \cdots \geq L_{n,n}$ and $[n(1-\alpha)]$ is the larger integer not exceeding $n(1-\alpha)$.

The above lemma shows that ES_α can be thought of as the limiting average of the $[n(1-\alpha)]$ upper order statistics from a sample of size n. (20) can be used for estimation of ES_α in the situation when we have large data samples and $[n(1-\alpha)]$ is a relatively large number.

Expected shortfall (ES) is a coherent risk measure.

To prove its subaditivity we consider a sequence of random variables $L_1, \ldots, L_n$ with corresponding order statistics $L_{1,n} \geq \cdots \geq L_{n,n}$. For any m satisfying $1 \leq m \leq n$ we have

$$\sum_{i=1}^{m} L_{i,n} = \sup\{L_{i_1} + \cdots + L_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}.$$

Let L and $\tilde{L}$ be random variables with the joint d.f. F and let

$$(L_1, \tilde{L}_1), \ldots, (L_n, \tilde{L}_n)$$

be vectors with the same d.f. F. Denote by $(L + \tilde{L})_i = L_i + \tilde{L}_i$ and by $(L + \tilde{L})_{i,n}$ order statistics of the sequence $(L + \tilde{L})_1, \ldots, (L + \tilde{L})_n$. We have

$$\sum_{i=1}^{m} (L + \tilde{L})_{i,n} = \sup\{(L + \tilde{L})_{i_1} + \cdots + (L + \tilde{L})_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}$$

$$\leq \sup\{L_{i_1} + \cdots + L_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}$$

$$+ \sup\{\tilde{L}_{i_1} + \cdots + \tilde{L}_{i_m} : 1 \leq i_1 < \cdots < i_m \leq n\}$$

$$= \sum_{i=1}^{n} L_{i,n} + \sum_{i=1}^{n} \tilde{L}_{i,n}.$$

Setting $m = [n(1-\alpha)]$, it follows from (20)

$$\mathrm{ES}_\alpha(L + \tilde{L}) \leq \mathrm{ES}_\alpha(L) + \mathrm{ES}_\alpha(\tilde{L}).$$

Literature: A.J.McNeil, R.Frey, P.Embrechts, Quantitative Risk Management. Concepts, Technics and Tools. Princeton University Press, 2005.