HOMEWORK: NOVEMBER 3

Problem 1: The goal is to show the optimality of the function space $W^{1,2}(\Omega)$ in the definition of a weak solution. To say it differently, one can define weaker or stronger concept of solution requiring that it belongs to some $W^{1,p}(\Omega)$ with $p \neq 2$, but it may lead either to nonexistence or to nonuniqueness. It is important to remember that it might be the case only if the elliptic operator has just measurable (and discontinuous) coefficient.

Consider $\Omega = B_1(0) \subset \mathbb{R}^2$ and $p \in (1,2)$ be arbitrary. Find an elliptic matrix $\mathbb{A}(x)$ and nontrivial $\hat{u} \in W_0^{1,p}(\Omega)$ such that

$$\int_\Omega \mathbb{A}\nabla\hat{u} \cdot \nabla\varphi = 0 \qquad \text{for all } \varphi \in \mathcal{C}_0^1(\Omega). \tag{1.1}$$

Note that we know that $u \equiv 0$ is the unique weak solution! Hence, we cannot get the uniqueness in large class of functions and this is the true motivation for the choice $W^{1,2}(\Omega)$ as the correct space.

Hint: Consider the matrix $\mathbb{A}$ with the coefficient given

$$(\mathbb{A})_{ij} = \delta_{ij} + (a-1)\frac{x_i x_j}{|x|^2}$$

with some $a > 1$. Show that it is an elliptic matrix. Consider the function $u(x)$ given as (here $x \in \mathbb{R}^2$ and $\varepsilon \in (0,1)$)

$$\bar{u}(x) := x_1|x|^{-1-\varepsilon}.$$

- Find all p's for which $\hat{u} \in W^{1,p}(\Omega)$.
- Find a proper relation between a and ε such that $\bar{u}$ solves (1.1) - justify it carefully!.
- Find $u \in W^{1,2}(\Omega)$ fulfilling (1.1) such that $u - \bar{u} \in W_0^{1,1}(\Omega)$.
- Show that $\hat{u} := u - \bar{u}$ solves Problem 1. Show that by proper choice of a and $\varepsilon > 0$ one can get cover the whole range $p \in (1,2)$.

Solution: We start with the ellipticity of the matrix $\mathbb{A}$. Let $z \in \mathbb{R}^2$ be arbitrary, then (here "$\cdot$"denotes the scalar product in $\mathbb{R}^2$)

$$\mathbb{A}z \cdot z = \sum_{i,j=1}^{2} \left(\delta_{ij} + (a-1)\frac{x_i x_j}{|x|^2}\right) z_i z_j = |z|^2 + (a-1)\frac{(x\cdot z)^2}{|x|^2} \geq |z|^2.$$

For sure,the matrix is also bounded and measurable and therefore we got it is elliptic.

Next, we checkfor which p's we have $\tilde{u} \in W^{1,p}$. Since we are asked to do it carefully, we derive rigorously what is the weak derivative of $\tilde{u}$. First, we evaluate the classical derivative outside of the point 0:

$$\frac{\partial \tilde{u}}{\partial x_1} = |x|^{-1-\varepsilon} - (1+\varepsilon)x_1^2|x|^{-3-\varepsilon} =: f_1$$
$$\frac{\partial \tilde{u}}{\partial x_2} = -(1+\varepsilon)x_1 x_2|x|^{-3-\varepsilon} =: f_2$$

Note that the above functions are for sure measurable and we just check whether they are L^p integrable. By using the polar coordinates and the substitution, we see that

$$\int_\Omega |f_1|^p < \infty \quad \Leftrightarrow \quad \int_\Omega \left||x|^{-1-\varepsilon} - (1+\varepsilon)x_1^2|x|^{-3-\varepsilon}\right|^p < \infty \quad \Leftrightarrow \quad \int_0^1 r^{-p(1+\varepsilon)} r\, dr < \infty$$
$$\Leftrightarrow \quad -p(1+\varepsilon)+1 > -1 \quad \Leftrightarrow p < \frac{2}{1+\varepsilon}$$
$$\int_\Omega |f_2|^p < \infty \quad \Leftrightarrow \quad \int_\Omega \left|x_1x_2|x|^{-3-\varepsilon}\right|^p < \infty \quad \Leftrightarrow \quad \int_0^1 r^{-p(1+\varepsilon)} r\, dr < \infty$$
$$\Leftrightarrow \quad -p(1+\varepsilon)+1 > -1 \quad \Leftrightarrow p < \frac{2}{1+\varepsilon}$$

Moreover, for $p < \frac{2}{1+\varepsilon}$ it is not difficult to check that $\int_\Omega |\tilde{u}|^p < \infty$. It remains to show, that the pointwise derivatives f_1, f_2 (which were derived only outside of the point 0, i.e., in $\Omega \setminus \{0\}$), are indeed weak derivative in Ω. To check it, let φbeasmooth compactly supported function. Then we use integration by parts formula for smooth functions and the Lebesgue dominated convergence theorem to see that

$$\begin{aligned}\int_\Omega f_1\varphi &= \lim_{\delta\to 0_+}\int_{\Omega\setminus B_\delta} f_1\varphi = \lim_{\delta\to 0_+}\int_{\Omega\setminus B_\delta} \frac{\partial \tilde{u}}{\partial x_1}\varphi \\ &\overset{by parts}{=} \lim_{\delta\to 0_+}\int_{\partial(\Omega\setminus B_\delta)} \tilde{u}\varphi n_1 - \int_{\Omega\setminus B_\delta}\frac{\partial\varphi}{\partial x_1}\tilde{u} \\ &= -\int_\Omega \frac{\partial\varphi}{\partial x_1}\tilde{u} + \lim_{\delta\to 0_+}\left(-\int_{\partial B_\delta}\varphi\frac{x_1^2}{\delta^{2+\varepsilon}} + \int_{B_\delta}\frac{\partial\varphi}{\partial x_1}\frac{x_1}{|x|^{1+\varepsilon}}\right)\end{aligned}$$

Since

$$\begin{aligned}&\left|-\int_{\partial B_\delta}\varphi\frac{x_1^2}{\delta^{2+\varepsilon}} + \int_{B_\delta}\frac{\partial\varphi}{\partial x_1}\frac{x_1}{|x|^{1+\varepsilon}}\right| \le \|\varphi\|_{1,\infty}\left(\int_{\partial B_\delta}\frac{1}{\delta^\varepsilon} + \int_{B_\delta}\frac{1}{|x|^\varepsilon}\right) \\ &\le C(\varepsilon)\|\varphi\|_{1,\infty}\left(\delta^{1-\varepsilon}+\delta^{2-\varepsilon}\right) \overset{\delta\to 0_+}{\to} 0,\end{aligned}$$

we see that

$$\int_\Omega f_1\varphi = -\int_\Omega \frac{\partial\varphi}{\partial x_1}\tilde{u}$$

and similarly we can deduce

$$\int_\Omega f_2\varphi = -\int_\Omega \frac{\partial\varphi}{\partial x_2}\tilde{u}.$$

Hence, it is nothing else than the definition of the weak derivatives and consequently f_1 is a weak derivative with respect to x_1 on Ω and f_2 is weak derivative with respect to x_2 on Ω.

Next, we choose ε and a so that (1.1) holds true. First, we take $\varphi \in \mathcal{C}_0^\infty(\Omega\setminus\{0\})$. Since $\mathbb{A}$ and $\tilde{u}$ are smooth on $\Omega \setminus \{0\}$, we can use standard integration by parts formula and deduce that (1.1) implies

$$\int_\Omega \operatorname{div}(\mathbb{A}\nabla u)\varphi = 0.$$

Since φ is arbitrary then the necessary condition for validity of (1.1) is that (here we are using the classical derivatives, since the exist outside of 0)

$$\operatorname{div}(\mathbb{A}\nabla u) = 0 \text{ in } \Omega \setminus \{0\}. \tag{1.2}$$

Thus, we evaluate this condition

$$\begin{aligned}
&-\operatorname{div}(\mathbb{A}\nabla u) = -\sum_{i,j=1} \frac{\partial}{\partial x_i}\left(\mathbb{A}_{ij}\frac{\partial \tilde{u}}{\partial x_j}\right)\\
&= \sum_{i,j=1} \frac{\partial}{\partial x_i}\left(\left(\delta_{ij} + (a-1)\frac{x_i x_j}{|x|^2}\right)\left((1+\varepsilon)x_1 x_j |x|^{-3-\varepsilon} - \delta_{1j}|x|^{-1-\varepsilon}\right)\right)\\
&= \sum_{i=1} \frac{\partial}{\partial x_i}\left((a\varepsilon+1)x_1 x_i |x|^{-3-\varepsilon} - \delta_{1i}|x|^{-1-\varepsilon}\right)\\
&= \sum_{i=1} (a\varepsilon+1)\left(\delta_{1i}x_i|x|^{-3-\varepsilon} + x_1|x|^{-3-\varepsilon} - (3+\varepsilon)x_1 x_i^2 |x|^{-5-\varepsilon}\right) + (1+\varepsilon)\delta_{1i}x_i|x|^{-3-\varepsilon}\\
&= x_1|x|^{-3-\varepsilon}\left(1-\varepsilon^2 a\right)
\end{aligned}$$

Consequently, (1.2) holds if and only if

$$\varepsilon^2 a = 1 \tag{1.3}$$

Hence, we have just shown that (1.3) is a necessary condition for the validity of (1.1).

Next, we show that it is indeed sufficient. To do so, we use the Lebesgue dominated convergence theorem, integration by parts (outside of zero, where all functions are smooth), the fact that φ has compact support and also the point-wise relation (1.2). Here,$\varphi \in \mathcal{C}_0^1(\Omega)$

$$\begin{aligned}
&\int_\Omega \mathbb{A}\nabla\tilde{u}\cdot\nabla\varphi = \lim_{\delta\to 0_+}\int_{\Omega\setminus B_\delta(0)} \mathbb{A}\nabla\tilde{u}\cdot\nabla\varphi\\
&\overset{by parts}{=} \lim_{\delta\to 0_+} -\int_{\Omega\setminus B_\delta(0)} \operatorname{div}(\mathbb{A}\nabla\tilde{u})\varphi + \int_{\partial(\Omega\setminus B_\delta(0))}\varphi\mathbb{A}\nabla\tilde{u}\cdot n\\
&\overset{(1.2)}{=} \lim_{\delta\to 0_+}\delta^{-1}\int_{\partial B_\delta(0)}\varphi(x)\sum_{i=1}^{2}\underbrace{\left((a\varepsilon+1)x_1 x_i|x|^{-3-\varepsilon} - \delta_{1i}|x|^{-1-\varepsilon}\right)}_{(-\mathbb{A}\nabla\tilde{u})_i} x_i\\
&= a\varepsilon\lim_{\delta\to 0_+}\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}\varphi(x)x_1.
\end{aligned}$$

Hence, we need to show that the last limit is equal to zero.

BE CAREFUL: If you used just brutal force then you would not get the result, e.g.,

$$\left|\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}\varphi(x)x_1\right| \le \|\varphi\|_\infty\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}|x| = 2\pi\|\varphi\|_\infty\delta^{-\varepsilon} \overset{\delta\to 0_+}{\to} \infty.$$

So we must proceed differently and use a certain cancelation effect.

$$\begin{aligned}\left|\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}\varphi(x)x_1\right| &= \left|\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}(\varphi(x)-\varphi(0))x_1+\delta^{-2-\varepsilon}\varphi(0)\int_{\partial B_\delta(0)}x_1\right| \\ &= \left|\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}(\varphi(x)-\varphi(0))x_1\right| \\ &\leq \|\nabla\varphi\|_\infty\delta^{-2-\varepsilon}\int_{\partial B_\delta(0)}|x|^2=\|\nabla\varphi\|_\infty\delta^{1-\varepsilon}\overset{\delta\to 0_+}{\to}0,\end{aligned}$$

provided that $\varepsilon<1$, which is however always the case since due to the fact that $\varepsilon^2 a=1$ and $a>1$.

Thus is we set $\varepsilon:=\sqrt{a^{-1}}$, we see that

$$\frac{2}{1+\varepsilon}\overset{a\to\infty}{\to}2,\qquad \frac{2}{1+\varepsilon}\overset{a\to 1_+}{\to}1,$$

thus going back to the condition for $\tilde{u}$ being in $W^{1,p}$, we see that by proper choice of $a>1$ we can cover the whole interval $(1,2)$.

Finally, it is woth noticing that if we set $\varepsilon:=-\sqrt{a^{-1}}$ then (1.3) holds. Moreover, if we denote for such ε the function $u:=x_1|x|^{-1+\sqrt{a^{-1}}}$ then by above computation we can justify that $u\in W^{1,2}(\Omega)$ solves (1.1) and in addition $u=\tilde{u}$ on $\partial\Omega$. Consequently $\hat{u}:=u-\tilde{u}$ has zero trace solves (1.1) but $\hat{u}\notin W^{1,2}$.

□

Problem 2: The goal is to show that the maximal regularity[1] cannot hold in Lipschitz domains or when changing the type of boundary conditions. Let $\varphi_0\in(0,2\pi)$ be arbitrary and consider $\Omega\subset\mathbb{R}^2$ given by[2]

$$\Omega:=\{(r,\varphi):\ r\in(0,1),\ \varphi\in(0,\varphi_0)\}.$$

Denote $\Gamma_i\subset\partial\Omega$ in the following way (in polar coordinates (r,φ): $\Gamma_1:=(r,0)$, $\Gamma_2:=(r,\varphi_0)$, $\Gamma_3:=(1,\varphi)$.

Consider two functions

$$\begin{aligned}u_1(r,\varphi)&:=r^{\alpha_1}\sin\left(\frac{\varphi\pi}{\varphi_0}\right),\\ u_2(r,\varphi)&:=r^{\alpha_2}\sin\left(\frac{\varphi\pi}{2\varphi_0}\right)\end{aligned}$$

- Find the condition on α_i so that $u_i\in W^{1,2}(\Omega)$ - find an explicit formula for ∇u_i - and prove that it is really the weak derivative!
- Find the proper condition on α_i so that u_i solves the problem

$$-\Delta u_1=0 \text{ in } \Omega,\quad u_1=0 \text{ on } \Gamma_1\cup\Gamma_2,\quad u_1=\sin\left(\frac{\varphi\phi}{\varphi_0}\right) \text{ on } \Gamma_3$$

$$-\Delta u_2=0 \text{ in } \Omega,\quad u_2=0 \text{ on } \Gamma_1,\quad u_2=\sin\left(\frac{\varphi\phi}{\varphi_0}\right) \text{ on } \Gamma_3,\quad \nabla u_2\cdot n=0 \text{ on } \Gamma_2$$

[1] Maximal regularity statement means that if

$$\Delta u=f$$

then $f\in L^p(\Omega)\implies u\in W^{2,p}(\Omega)$. The goal of the homework is to show that this is not true on domains with corners.

[2] We use polar coordinates, i.e. $x_1=r\cos\varphi$, $x_2=r\sin\varphi$

CHECK in details that for such α_i's the weak formulation of the above elliptic equations hold!

- Find all p's for which $u_i \in W^{2,p}(\Omega)$. What is the criterium on α_i so that $u_i \in W^{2,2}(\Omega)$?
- With the help of the above computation, find $f_i \in L^2(\Omega)$ such that the problems with homogeneous boundary conditions, i.e.,

$$-\Delta v_1 = f_1 \text{ in } \Omega, \quad v_1 = 0 \text{ on } \partial\Omega,$$
$$-\Delta v_2 = f_2 \text{ in } \Omega, \quad v_2 = 0 \text{ on } \Gamma_1 \cup \Gamma_3, \nabla v_2 \cdot n = 0 \text{ on } \Gamma_2$$

posses unique weak solutions $v_i \in W^{1,2}(\Omega)$ but $v_1 \notin W^{2,2}(\Omega)$ if $\varphi_0 > \phi$ and $v_2 \notin W^{2,2}(\Omega)$ for $\varphi_0 > \frac{\pi}{2}$.

- **REMEMBER:** On domains with corner - the $W^{2,2}$ regularity statement does not hold for Dirichlet problem for angels greater than π and does not hold when changing Dirrichlet to Neumann problems on corners with angle greater than $\pi/2$. In general $W^{2,2}$ regularity for Dirichlet problems holds in any dimension either for convex domains or for domains with $\mathcal{C}^{1,1}$ boundary.

Solution: First of all, one should be able to derive the formula for derivatives in polar coordinates. Since our change of coordinates is given by

$$x_1 = r\cos\varphi, \qquad x_2 = r\sin\varphi,$$

Then for any $\mathcal{C}^1$ function $f(r,\varphi)$, we have (for $r \in (0,1)$ and $\varphi \in (0,2\pi)$)

$$\frac{\partial f(r,\varphi)}{\partial x_1} = \cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}$$
$$\frac{\partial f(r,\varphi)}{\partial x_2} = \sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}$$

Consequently, we can also deduce that

$$\begin{aligned}
\Delta f(r,\varphi) &= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2} \\
&= \frac{\partial}{\partial x_1}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) + \frac{\partial}{\partial x_2}\left(\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) \\
&= \cos\varphi\frac{\partial}{\partial r}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) - \frac{\sin\varphi}{r}\frac{\partial}{\partial \varphi}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r} - \frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) \\
&\quad + \sin\varphi\frac{\partial}{\partial r}\left(\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right) + \frac{\cos\varphi}{r}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial f(r,\varphi)}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial \varphi}\right)
\end{aligned}$$

Hence, evaluating the right hand side, we get that

$$\begin{aligned}
\Delta f(r,\varphi) &= \cos\varphi\left(\cos\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2}+\frac{\sin\varphi}{r^2}\frac{\partial f(r,\varphi)}{\partial\varphi}-\frac{\sin\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi\partial r}\right)\\
&\quad-\frac{\sin\varphi}{r}\left(-\sin\varphi\frac{\partial f(r,\varphi)}{\partial r}+\cos\varphi\frac{\partial^2 f(r,\varphi)}{\partial r\partial\varphi}-\frac{\cos\varphi}{r}\frac{\partial f(r,\varphi)}{\partial\varphi}-\frac{\sin\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right)\\
&\quad+\sin\varphi\left(\sin\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2}-\frac{\cos\varphi}{r^2}\frac{\partial f(r,\varphi)}{\partial\varphi}+\frac{\cos\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi\partial r}\right)\\
&\quad+\frac{\cos\varphi}{r}\left(\cos\varphi\frac{\partial f(r,\varphi)}{\partial r}+\sin\varphi\frac{\partial^2 f(r,\varphi)}{\partial r\partial\varphi}-\frac{\sin\varphi}{r}\frac{\partial f(r,\varphi)}{\partial\varphi}+\frac{\cos\varphi}{r}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\right)\\
&=\cos^2\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2}+\frac{\sin^2\varphi}{r}\frac{\partial f(r,\varphi)}{\partial r}+\frac{\sin^2\varphi}{r^2}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\\
&\quad+\sin^2\varphi\frac{\partial^2 f(r,\varphi)}{\partial r^2}+\frac{\cos^2\varphi}{r}\frac{\partial f(r,\varphi)}{\partial r}+\frac{\cos^2\varphi}{r^2}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}\\
&=\frac{\partial^2 f(r,\varphi)}{\partial r^2}+\frac{1}{r}\frac{\partial f(r,\varphi)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 f(r,\varphi)}{\partial\varphi^2}
\end{aligned}$$

Next, we use the hint an check for which $A, B \in \mathbb{R}$, we have that

$$\Delta(r^A\sin(B\varphi)) = 0 \qquad \text{in } \Omega.$$

Note that such function is smooth outside of the origin. Consequently, using the above computation, we see that for $u_{AB} := r^A\sin(B\varphi)$

$$\begin{aligned}
\Delta u_{AB} &= A(A-1)r^{A-2}\sin(B\varphi)+Ar^{A-2}\sin(B\varphi)-r^{A-2}B^2\sin(B\varphi)\\
&=\left(A^2-B^2\right)r^{A-2}\sin(B\varphi).
\end{aligned}$$

Hence, we require $A^2 = B^2$ in what follows. Next, we also check for which A, B we have $u_{AB} \in W^{1,2}(\Omega)$. Since, the classical derivatives exist in Ω, we know that the weak derivative also exists and we just need to specify the conditions on A, B so that

$$\int_\Omega |\nabla u_{AB}|^2 < \infty.$$

Using the transformation into the polar coordinates and the substitution theorem, we have

$$\begin{aligned}
\int_\Omega |\nabla u_{AB}|^2\,\mathrm{dx} &= \int_\Omega \left(\frac{\partial u_{AB}}{\partial x_1}\right)^2+\left(\frac{\partial u_{AB}}{\partial x_2}\right)^2\mathrm{dx}\\
&=\int_0^1\int_0^{\varphi_0}\left(\cos\varphi\frac{\partial u_{AB}}{\partial r}-\frac{\sin\varphi}{r}\frac{\partial u_{AB}}{\partial\varphi}\right)^2 r+\left(\sin\varphi\frac{\partial u_{AB}}{\partial r}+\frac{\cos\varphi}{r}\frac{\partial u_{AB}}{\partial\varphi}\right)^2 r\,d\varphi\,dr\\
&=\int_0^1\int_0^{\varphi_0}\left(\left(\frac{\partial u_{AB}}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u_{AB}}{\partial\varphi}\right)^2\right)r\,d\varphi\,dr\\
&=\int_0^1\int_0^{\varphi_0}\left(\left(Ar^{A-1}\sin(B\varphi)\right)^2+\frac{1}{r^2}\left(Br^A\cos(B\varphi)\right)^2\right)r\,d\varphi\,dr\\
&=\int_0^1\int_0^{\varphi_0} r^{2A-1}\left(A^2\sin^2(B\varphi)+B^2\cos^2(B\varphi)\right)\,d\varphi\,dr\\
&=\int_0^1\int_0^{\varphi_0} A^2r^{2A-1}\,d\varphi\,dr = A^2\varphi_0\int_0^1 r^{2A-1}\,dr,
\end{aligned}$$

where we used the fact that $A^2 = B^2$. Consequently, if we want to have the above integral finite, we must impose the condition $A > 0$. Since the regularity of the solution does not depend on the sign of B, we assume in what follows only the case $A = B > 0$.

Next, we show for which $p's$ the function $u_{AB} \in W^{2,p}(\Omega)$. For that reasons, we compute the second derivatives (in term of variables (r, φ))

$$\begin{aligned}\frac{\partial^2 u_{AB}}{\partial x_1^2} &= \cos^2\varphi \frac{\partial^2 u_{AB}}{\partial r^2} + \frac{2\cos\varphi\sin\varphi}{r^2}\frac{\partial u_{AB}}{\partial \varphi} - \frac{2\cos\varphi\sin\varphi}{r}\frac{\partial^2 u_{AB}}{\partial\varphi\partial r} \\ &+ \frac{\sin^2\varphi}{r}\frac{\partial u_{AB}}{\partial r} + \frac{\sin^2\varphi}{r^2}\frac{\partial^2 u_{AB}}{\partial\varphi^2} \\ &= A(A-1)r^{A-2}\cos^2\varphi\sin(A\varphi) + 4Ar^{A-2}\cos\varphi\sin\varphi\cos(A\varphi) \\ &+ Ar^{A-2}\sin^2\varphi\sin(A\varphi) - A^2r^{A-2}\sin^2\varphi\sin(A\varphi)\end{aligned}$$

In the same manner we shall estimate other second derivatives to finally conclude that

$$\int_\Omega |\nabla^2 u_{AB}|^p \,\mathrm{dx} \sim \int_0^1 r^{(A-2)p} r\, dr$$

and wee that the integral is finite if and only if (for $0 \le A < 2$, since for $A \ge 2$ it is always finite)

$$p < \frac{2}{2-A}.$$

Finally, we apply everything to the functions u_1 and u_2, which are thus given as

$$u_1 = r^{\frac{\pi}{\varphi_0}} \sin\left(\frac{\varphi\pi}{\varphi_0}\right), \qquad u_2 = r^{\frac{\pi}{2\varphi_0}} \sin\left(\frac{\varphi\pi}{2\varphi_0}\right)$$

and due to the above computations we get that $u_1, u_2 \in W^{1,2}(\Omega)$ and

$$u_1 \in \left\{\begin{matrix} W^{2,p}(\Omega) & \text{for } p < \dfrac{2}{2-\frac{\pi}{\varphi_0}} \text{ and } \varphi_0 > \dfrac{\pi}{2} \\ W^{2,\infty}(\Omega) & \text{for } \varphi_0 \le \dfrac{\pi}{2} \end{matrix}\right\} \implies u_1 \in W^{2,2}(\Omega) \text{ if } \varphi_0 < \pi,$$

$$u_2 \in \left\{\begin{matrix} W^{2,p}(\Omega) & \text{for } p < \dfrac{2}{2-\frac{\pi}{2\varphi_0}} \text{ and } \varphi_0 > \dfrac{\pi}{4} \\ W^{2,\infty}(\Omega) & \text{for } \varphi_0 \le \dfrac{\pi}{4} \end{matrix}\right\} \implies u_2 \in W^{2,2}(\Omega) \text{ if } \varphi_0 < \frac{\pi}{2}.$$

Finally, we check that u_1 and u_2 are the solutions of the corresponding problem. Since, u_1 is continuous in $\overline{\Omega}$ and also $u_1 \in W^{1,2}(\Omega)$, then evidently the trace of u_1 is zero. Next, for any smooth compactly supported function v we get by integration by parts (we ca apply that because u_1 is smooth in the interior of Ω)

$$\int_\Omega \nabla u_1 \cdot \nabla v = -\int_\Omega \Delta u_1 v = 0,$$

where we used the fact that u_1 is harmonic. Finally since the space $C_0^\infty(\Omega)$ is dense in $W_0^{1,2}(\Omega)$ we can generalize the above relation also for any $v \in W_0^{1,2}(\Omega)$. Indeed, for any $v \in W_0^{1,2}(\Omega)$, we can find a sequence (by density) $\{v^n\} \subset \mathcal{C}_0^\infty(\Omega)$ such that $v^n \to v$ in $W^{1,2}(\Omega)$ and then

$$\int_\Omega \nabla u_1 \cdot \nabla v = \lim_{n\to\infty} \int_\Omega \nabla u_1 \cdot \nabla v^n = 0. \tag{1.4}$$

Finally, let w be a smooth function on $\mathbb{R}^2$ fulfilling $w = 1$ in $B_{\frac{1}{2}}(0)$ and $w = 0$ on $\mathbb{R}^2 \setminus B_{\frac{3}{4}}(0)$ and define $v_1 := u_1 w$. Then evidently $v_1 \in W^{1,2}(\Omega) \cap W^{2,p}(\Omega)$ with p specified above, $v_1 = 0$ on $\partial\Omega$ and we have for any $z \in W^{1,2}_0(\Omega)$

$$\begin{aligned}\int_\Omega \nabla v_1 \cdot \nabla z &= \int_\Omega w\nabla u_1 \cdot \nabla z + u_1 \nabla w \cdot \nabla z \\ &= \underbrace{\int_\Omega \nabla u_1 \cdot \nabla (wz)}_{(1.4)} - \int_\Omega z\nabla u_1 \cdot \nabla w - u_1 \nabla w \cdot \nabla z = - \int_\Omega z\nabla u \cdot \nabla w + \operatorname{div}(u_1 \nabla w) z,\end{aligned}$$

which is the weak formulation of

$$-\Delta v_1 = -\nabla u_1 \cdot \nabla w - \operatorname{div}(u_1 \nabla w) =: f_1$$

Note that thanks to the presence of w, the function f_1 is smooth since u_1 is regular outside of 0 but w is constant near zero.

For v_2 we could use exactly the same arguments to get the result, but since we did not formulate any result concerning the density of functions vanishing only near Γ_1. Thus, we proceed differently. Let $Q : [0,\infty) \to [0,\infty)$ be smooth function fulfilling $Q = 1$ on $[0, 1/4]$ and $Q = 0$ on $[1,\infty)$ and let $R : [0,\infty) \to [0,\infty)$ be smooth nondecreasing function fulfilling $R = 0$ on $[0, 1/2]$ and $R = 1$ on $[1,\infty)$. Next, we define

$$v_2(r, \varphi) := u_2(r, \varphi) Q(r^2)$$

Then it is easy to check that $v_2 = 0$ on $\Gamma_1 \cup \Gamma_3$. Next we check what kind of problem v_2 satisfies. We use also the function R to cut everything near zero in order to be able to use integration by parts. In addition, it is also a direct consequence of the definition that $\nabla v_2 \cdot n = 0$ on Γ_2. Hence, let $z \in W^{1,2}(\Omega)$be arbitrary fulfilling (in sense of traces) $z = 0$ on $\Gamma_1 \cup \Gamma_3$

$$\begin{aligned}\int_\Omega \nabla v_2 \cdot \nabla z &= \lim_{\varepsilon\to 0_+} \int_\Omega \nabla v_2 \cdot \nabla z R(r^2/\varepsilon^2) \\ &= - \lim_{\varepsilon\to 0_+} \int_\Omega \operatorname{div}(R(r^2/\varepsilon^2)\nabla v_2) z - \lim_{\varepsilon\to 0_+} \int_{\partial\Omega} \underbrace{\nabla v_1 \cdot n}_{=0 \text{ on } \Gamma_2} \quad \underbrace{zR(r^2/\varepsilon^2)}_{=0 \text{ on } \Gamma_1\cap\Gamma_3} \\ &= - \lim_{\varepsilon\to 0_+} \int_\Omega (\nabla R(r^2/\varepsilon^2) \cdot \nabla v_2 + R(r^2/\varepsilon^2) \underbrace{\operatorname{div}(\nabla v_2)}_{\Delta(u_2 Q)}) z \\ &= - \int_\Omega (u_2 \Delta Q + \nabla u_2 \cdot \nabla Q) z - \lim_{\varepsilon\to 0_+} 2\varepsilon^{-2} \int_{B_\varepsilon(0)\cap\Omega} R'(r^2/\varepsilon) x \cdot \nabla v_2 z\end{aligned}$$

Thus, if we show that the last limit is equal to zero, we see that v_2 is a weak solution to the desired problem with $f_2 := -(u_2 \Delta Q + \nabla u_2 \cdot \nabla Q)$ which is a smooth function.

To estimate the limit, we recall the Poincaré inequality and for all $\tilde{z} \in W^{1,2}(\Omega)$ being equal to zero on Γ_1 there holds $\|\tilde{z}\|_2 \le C \|\nabla \tilde{z}\|_2$. Hence, if we define the particular $\tilde{z}$ as

$$\tilde{z}(x) := z(\varepsilon x)$$

and use the substitution and Poincaré inequality on Ω we have

(1.5)
$$\begin{aligned}\int_{\Omega\cap B_\varepsilon(0)} |z(x)|^2\,\mathrm{dx} &= \int_{\Omega\cap B_\varepsilon(0)} |\tilde{z}(x/\varepsilon)|^2\,\mathrm{dx} = \varepsilon^2\int_\Omega |\tilde{z}(x)|^2\,\mathrm{dx} \le C\varepsilon^2\int_\Omega |\nabla\tilde{z}(x)|^2\,\mathrm{dx}\\ &= C\varepsilon^4\int_\Omega |\nabla z(\varepsilon x)|^2\,\mathrm{dx} = C\varepsilon^2\int_{\Omega\cap B_\varepsilon(0)} |\nabla z(x)|^2\,\mathrm{dx}\end{aligned}$$

Hence, by the Hölder inequality and theabove proven re-scaled Poincaréinequality, we have

$$\left|\varepsilon^{-2}\int_{B_\varepsilon(0)\cap\Omega} R'(r^2/\varepsilon)x\cdot\nabla v_2 z\right| \le \|R'\|_\infty\left(\frac{\int_{\Omega\cap B_\varepsilon(0)}|z(x)|^2\,\mathrm{dx}}{\varepsilon^2}\right)^{\frac{1}{2}}\left(\int_{\Omega\cap B_\varepsilon(0)}|\nabla v_2|^2\right)^{\frac{1}{2}}$$

$$\le C\|R'\|_\infty\|\nabla z\|_2\left(\int_{\Omega\cap B_\varepsilon(0)}|\nabla v_2|^2\right)^{\frac{1}{2}} \overset{\varepsilon\to 0_+}{\to} 0$$

where the last limit holds since $v_2 \in W^{1,2}(\Omega)$.

$\square$