HOMEWORK: DECEMBER 8

Problem: Prove rigorously the finite speed of propagation of weak solution to linear hyperbolic equations of the second order.

Let $\Omega \subset \mathbb{R}^d$ be an open set fulfilling $B_1(0) \subset \Omega$. Assume that $\mathbb{A} \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{sym})$ be elliptic and that u is a weak solution to

$$\partial_{tt} u - \operatorname{div}(\mathbb{A}\nabla u) = 0 \qquad \text{in } Q := (0,T)\times\Omega,$$

i.e.,

$$u \in L^2(0,T; W^{1,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega)) \cap W^{2,2}(0,T; (W^{1,2}_0(\Omega))^*) \tag{1.1}$$

satisfies for almost all $t \in (0,T)$ and all $w \in W^{1,2}_0(\Omega)$

$$\langle \partial_{tt} u, w\rangle + \int_\Omega \mathbb{A}\nabla u \cdot \nabla w = 0 \tag{1.2}$$

GOAL: Find proper/optimal relation[1] between $\Omega_0 \subset B_1(0)$ and $Q_0 \subset Q$ such that the following implication holds true

$$\boxed{u(0) = \partial_t u(0) = 0 \text{ in } \Omega_0 \quad \Longrightarrow \quad u = 0 \text{ in } Q_0.}$$

Subgoal1 (obligatory): Show the result for constant matrix $\mathbb{A}$.
Subgoal2 (not obligatory but recommended): Show it for general $\mathbb{A}$.

Hint: Please, do everything rigorously, i.e., assume just (1.1)–(1.2). For the first subgoal, I would suggest two options. Either follow the lecture and on the places where we used $|x|$ use a different norm in $\mathbb{R}^d$, that will depend on $\mathbb{A}$, or try to find a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ such that the function $\tilde{u}(t,x) := u(t,Tx)$ is a weak solution to the classical wave equation and then use the result of the lecture.

For the second subgoal: Try to prove everything formally and then try to justify it rigorously. The formal procedure can be: Multiply (1.1) by $\partial_t u$ to get

$$\frac{1}{2}\partial_t(|\partial_t u|^2 + \mathbb{A}\nabla u \cdot \nabla u) - \operatorname{div}(\mathbb{A}\nabla u \partial_t u) = 0$$

Integrate the result over Q_0 and then use the integration by parts formula (in $\mathbb{R}^{d+1}$) to create an integral over ∂Q_0. Try to find a proper shape of the boundary ∂Q_0 such that this procedure formally leads to $\partial_t u$ and ∇u are zero on ∂Q_0. This will then help you to show the desired result. Bu you should do everything rigorously!

[1]This relation depends essentially on the structure of $\mathbb{A}$