FOURIEROVY ŘADY

H – Hilbertův prostor – má skalární součin a prostor je úplný

$\{w_i\}_{i=1}^{\infty}$ – je ortogonální systém, pokud $(w_i, w_j)_H = 0 \quad \forall i \neq j$

– je ortonormální systém, pokud $(w_i, w_j)_H = \delta_{ij}$

– úplný, pokud $\left(\forall i \ (u, w_i)_H = 0 \right) \Rightarrow u = 0$

Pro úplný ortonormální systém definujeme abstraktní Fourierovu řadu

$$u \sim \sum_{i=1}^{\infty} a_i w_i, \quad \text{kde} \quad a_i := (u, w_i)_H$$

Platí: $$\lim_{n \to \infty} \left\| u - \sum_{i=1}^{n} a_i w_i \right\|_H = 0$$

$$\|u\|_H^2 = \sum_{i=1}^{\infty} a_i^2 \qquad \text{(Parseval)}$$

TRIGONOMETRICKÉ ŘADY

Pro $f \in L^2(a, a+l)$ definujeme

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(\frac{2k\pi}{l} x\right) + b_k \sin\left(\frac{2k\pi}{l} x\right)$$

kde $$a_k = \frac{2}{l} \int_a^{a+l} f \cos\left(\frac{2k\pi}{l} x\right)$$

$$b_k = \frac{2}{l} \int_a^{a+l} f \sin\left(\frac{2k\pi}{l} x\right)$$

ODPOVÍDAJÍCÍ ORTOGONÁLNÍ SYSTÉM

$$w_0 = \frac{1}{\sqrt{l}}, \quad w_i \sim \frac{1}{\sqrt{l}} \sin \frac{2k\pi}{l} x, \quad \frac{1}{\sqrt{l}} \cos \frac{2k\pi}{l} x$$

ZNAČÍME $$F_f = \frac{a_0}{2} + \sum_k a_k \cos\left(\frac{2k\pi}{l} x\right) + b_k \sin\left(\frac{2k\pi}{l} x\right)$$

Věta: Buď f po částech spojitě diferencovatelná na $(a, a+l)$, pak

1) $\forall x$ $$\frac{f(x_+) + f(x_-)}{2} = F_f(x)$$

2) pokud $f \in C(a, a+l)$ pak $F_f \overset{lok}{\rightrightarrows} f$ na $(a, a+l)$

3) řadu lze integrovat člen po členu

4) pokud f je po částech C^1, pak lze derivovat člen po členu

Př: rozvijte $f(x) = \operatorname{sgn} x$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sign} x = 0 \qquad a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sign} x \cos(kx) = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \operatorname{sign} x \sin kx = \frac{2}{\pi} \int_0^{\pi} \sin kx = -\frac{2}{\pi k} \left[\cos kx\right]_0^{\pi}$$

$$= -\frac{2}{\pi k} \left((-1)^k - 1\right) = \frac{2}{\pi k} \left(1 - (-1)^k\right)$$

$$\Rightarrow F_{\operatorname{sgn} x} = \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{k}\left(1-(-1)^k\right)\sin kx = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left((2k+1)x\right)}{(2k+1)}$$

<u>Parseval</u>: $2\pi = \int_{-\pi}^{\pi} (\operatorname{sign} x)^2 = \sum_{k=0}^{\infty}\left(\frac{4\sqrt{\pi}}{\pi(2k+1)}\right)^2 = \frac{16}{\pi}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

KONVERGENCE BODOVÁ (STEJNOMĚRNÁ LOKÁLNÍ) na $(0,\pi)$ i na $(-\pi,0)$

$$\Rightarrow x=\frac{\pi}{2} \Rightarrow 1 = \operatorname{sign}\frac{\pi}{2} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin\left((2k+1)\frac{\pi}{2}\right)}{(2k+1)} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)}$$

$$\Rightarrow \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)} = \frac{\pi}{4}$$

(1) $f = x^4$ na $[-\pi, \pi]$ (f je sudá, členy o $\sin nx$ jsou nulové)

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 = \left[\frac{x^5}{5\pi}\right]_{-\pi}^{\pi} = \frac{2\pi^4}{5}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 \cos kx = \frac{2}{\pi}\int_{0}^{\pi} x^4 \cos kx = \left[\frac{2x^4 \sin kx}{k\pi}\right]_0^{\pi}$$

$$- \int_0^{\pi} \frac{8x^3}{k\pi}\sin kx = \left[\frac{8x^3}{k^2\pi}\cos kx\right]_0^{\pi} - \int_0^{\pi}\frac{24x^2}{k^2\pi}\cos kx$$

$$= \frac{(-1)^k 8\pi^2}{k^2} - \left[\frac{24x^2}{k^3\pi}\sin kx\right]_0^{\pi} + \int_0^{\pi}\frac{48x}{k^3\pi}\sin kx$$

$$= \frac{(-1)^k 8\pi^2}{k^2} + \left[-\frac{48x}{k^4\pi}\cos kx\right]_0^{\pi} + \int_0^{\pi}\frac{48}{k^4\pi}\cos kx$$

$$= \frac{(-1)^k 8\pi^2}{k^2} - \frac{(-1)^k 48}{k^4}$$

$$\Rightarrow F_{x^4} = \frac{\pi^4}{5} + \sum_{k=1}^{\infty} \cos kx \left[\frac{(-1)^k 8\pi^2}{k^2} - \frac{48(-1)^k}{k^4}\right]$$

$f = x^4$ dodefinujme periodicky mimo interval $[-\pi, \pi]$

f je $\underline{\underline{\text{SPOJITÁ}}}$, po částech $C^1 \Rightarrow F_{x^4} \rightrightarrows f$ na $[-\pi, \pi]$

$f' = 4x^3$ – po periodickém rozšíření není spojitá na $\mathbb{R}$!

konvergence lokální $F_{x^3} \overset{loc}{\rightrightarrows} f$ na $(-\pi, \pi)$

a $F_x(\pm\pi) \to 0 = \dfrac{f'(\pi_+) + f'(\pi_-)}{2}$ (po dodefinování f periodicky)

(2) $f \in L^1(-\pi,\pi)$, $f(-x) = f(x)$ a $f(x+\pi) = -f(x)$

- f je lichá => obsahuje pouze $\sin nx$

- $$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx = \frac{1}{\pi}\int_0^{\pi} \sin kx\, f(x)$$

$$+ \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin kx = \frac{1}{\pi}\int_0^{\pi} \sin kx\, f(x)$$

$$- \frac{1}{\pi}\int_{-\pi}^{0} f(x+\pi)\sin kx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin kx\, f(x) - \frac{1}{\pi}\int_{-\pi}^{0} f(x+\pi)\sin[k(x+\pi)-k\pi]$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,(\sin kx - \sin(kx-k\pi)) = \frac{1}{\pi}\int_0^{\pi} f(x)\sin kx\,(1-(-1)^k)$$

=> řada bude obsahovat pouze členy s lichým "k",

$$\text{Tedy } f \sim \sum_{k=0}^{\infty} b_{2k+1}\sin(2k+1)x$$

(3) 1) Fourierova řada obsahuje pouze $\cos kx$ => f musí být sudá!?

Dále chceme aby $0 = \int_{-\pi}^{\pi} f(x)\cos(2k+1)x$

$$= 2\int_0^{\pi} f(x)\cos(2k+1)x = 2\int_0^{\frac{\pi}{2}} f(x)\cos(2k+1)x + 2\int_{\frac{\pi}{2}}^{\pi} f(x)\cos(2k+1)x$$

$$= 2\int_0^{\frac{\pi}{2}} f\left(\tfrac{\pi}{2}-x\right)\cos(2k+1)\left(x-\tfrac{\pi}{2}\right) + f\left(x+\tfrac{\pi}{2}\right)\cos(2k+1)\left(x+\tfrac{\pi}{2}\right)$$

$$= 2\int_0^{\frac{\pi}{2}} \left(f\left(\tfrac{\pi}{2}-x\right) + f\left(\tfrac{\pi}{2}+x\right)\right)(-1)^{k+1}\sin(2k+1)x$$

=> požadujeme $f\left(\frac{\pi}{2}+x\right) = -f\left(\frac{\pi}{2}-x\right)$

(4) a) $f = \sin^4 x$ na $(0,\pi)$ — rozšiřit do 2π

$$f \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos 2kx + b_k \sin 2kx$$

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4} - \frac{\cos 2x}{2} + \frac{\cos^2 2x}{4}$$

$$= \frac{1}{4} - \frac{\cos 2x}{2} + \frac{1+\cos 4x}{8} = \underline{\underline{\frac{3}{8} - \frac{\cos 2x}{2} + \frac{\cos 4x}{8}}}$$

COŽ JE FOURIEROVA ŘADA

b) $f(x) = ax \quad x \in (-\pi, 0]$

$\qquad\quad bx \quad x \in (0, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f = \frac{1}{\pi}\int_{-\pi}^{0} ax + \frac{1}{\pi}\int_{0}^{\pi} bx = \pi(b-a)$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos kx = \frac{1}{\pi}\int_{-\pi}^{0} ax \cos kx + \frac{1}{\pi}\int_{0}^{\pi} bx \cos kx =$$

$$= \frac{1}{k\pi}\left(-\int_{-\pi}^{0} a \sin kx - \int_{0}^{\pi} b \sin kx\right) = \frac{1}{k^2\pi}\left(\left[a\cos kx\right]_{-\pi}^{0} + \left[b \cos kx\right]_{0}^{\pi}\right)$$

$$= \frac{1}{k^2\pi}\left((a-b)(1-(-1)^k)\right)$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin kx = \frac{1}{\pi}\int_{-\pi}^{0} ax \sin kx + \frac{1}{\pi}\int_{0}^{\pi} bx \sin kx$$

$$= -\frac{1}{\pi k}\left(\left[ax\cos kx\right]_{-\pi}^{0} + \left[bx \cos kx\right]_{0}^{\pi}\right) + \frac{1}{\pi k}\left(\int_{-\pi}^{0} a \cos kx + \int_{0}^{\pi} b \cos kx\right)$$

$$= -\frac{1}{\pi k}\left(a\pi(-1)^k + b\pi(-1)^k\right) = \frac{(-1)^{k+1}}{k}(a+b)$$

$$f = \frac{\pi}{2}(b-a) + \sum_{k=1}^{\infty} \frac{1}{k^2\pi}(a-b)(1-(-1)^k)\cos kx + \frac{(-1)^{k+1}}{k}(a+b)\sin kx$$

řada konverguje lokálně stejnoměrně na $(-\pi,\pi)$, v bodě $x = \pm\pi$ konverguje k $\frac{\pi(b-a)}{2}$, pokud $a=-b$ pak konverguje stejnoměrně

(7) c) $|\sin x|$ na délce periody $(0,\pi)$

Tvar do tvaru $\frac{a_0}{2} + \sum a_k \cos 2kx + b_k \sin 2kx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} |\sin x| = \frac{4}{\pi}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} \sin x \cos 2kx = \frac{2}{\pi}\left[-\cos x \cos 2kx\right]_0^{\pi} - \frac{4k}{\pi}\int_0^{\pi} \cos x \sin 2kx$$

$$= \frac{4}{\pi} - \frac{4k}{\pi}\left[\sin x \sin 2kx\right]_0^{\pi} + \frac{8k^2}{\pi}\int_0^{\pi} \sin x \cos 2kx$$

$$= \frac{4}{\pi} + 4k^2 a_k \Rightarrow a_k = \frac{-\frac{4}{\pi}}{4k^2-1}$$

$$b_k = \frac{2}{\pi}\int_0^{\pi} \sin x \sin 2kx = 0$$

$$\Rightarrow |\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{\cos 2kx}{4k^2-1}$$

$|\sin x|$ je periodická spojitá, po částech C^1 $\Rightarrow$ konverguje stejnoměrně.

d) $f = \max(0,x) = \begin{cases} x & x \geq 0 \\ 0 & x \leq 0 \end{cases}$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x = \frac{\pi}{2}$$

$$a_k = \frac{1}{\pi}\int_0^{\pi} x \cos kx = -\frac{1}{\pi k}\int_0^{\pi} \sin kx = \left[\frac{\cos kx}{\pi k^2}\right]_0^{\pi} = \frac{(-1)^k - 1}{\pi k^2}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} x \sin kx = \left[-\frac{x \cos kx}{k\pi}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos kx}{k\pi} = \frac{(-1)^{k+1}}{k}$$

$$\max(0,x) = \frac{\pi}{4} + \sum_{k=1}^{\infty} \frac{(-1)^k - 1}{\pi k^2}\cos kx + \frac{(-1)^{k+1}}{k}\sin kx$$

konverguje lokálně stejnoměrně

pro $x = \pi$ konverguje k $\frac{\pi + 0}{2} = \frac{\pi}{2}$!

e) e^{ax} na $(-1,1)$

per partes....

(5) Rozklad do sinové řady $\Leftrightarrow$ funkci dodefinujeme liše

$$\Rightarrow f(x) = \begin{cases} x^2 & \text{na } (0,\pi) \\ -x^2 & \text{na } (-\pi,0) \end{cases}$$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin kx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin kx = \left[-\frac{2x^2\cos kx}{k\pi}\right]_0^{\pi}$$

$$+ \int_0^{\pi} \frac{4x}{k\pi}\cos kx = \frac{(-1)^{k+1}\cdot 2\pi}{k} + \left[\frac{4x\sin kx}{k^2\pi}\right]_0^{\pi} - \int_0^{\pi}\frac{4\sin kx}{k^2\pi}$$

$$= \frac{(-1)^{k+1}2\pi}{k} + \left[\frac{4\cos kx}{k^3\pi}\right]_0^{\pi} = \frac{(-1)^{k+1}2\pi}{k} + \frac{4((-1)^k-1)}{k^3\pi}$$

$$x^2 = \sum_{k=1}^{\infty}\left(\frac{(-1)^{k+1}2\pi}{k} + \frac{4((-1)^k-1)}{k^3\pi}\right)\sin kx \quad \text{na } (0,\pi)$$

konverguje lokálně stejnoměrně. v π konverguje k 0 !

b) cosinová řada $\Leftrightarrow$ FCI dodefinuji sudě

$$f = x^2$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 = \frac{2\pi^2}{3}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} x^2\cos kx = -\frac{4}{\pi k}\int_0^{\pi} x\sin kx$$

$$= \left[\frac{4x\cos kx}{\pi k^2}\right]_0^{\pi} - \int_0^{\pi}\frac{4\cos kx}{\pi k^2} = \frac{4(-1)^k}{k^2}$$

$$\Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{k^2}\cos kx$$ konverguje stejnoměrně na $[-\pi,\pi]$

(6) c) a d) už známe – rozvoj signx a x

a) → rozvoj $-\ln\left(2\sin\frac{x}{2}\right)$

b) $\left(\sum_{n=1}^{\infty}\left(\frac{\sin nx}{n^2}\right)\right)' = \sum \frac{\cos nx}{n} \overset{a)}{=} -\ln\left(2\sin\frac{x}{2}\right)$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{\sin nx}{n^2} = \int_0^x -\ln\left(2\sin\frac{t}{2}\right)dt$$

(7) a) $\sum_{n=1}^{\infty}\frac{1}{n^4}$ a d) $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$

Použijeme príklad 5)

$$x^2 = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4(-1)^k}{k^2}\cos kx \quad \text{(stejnoměrně)}$$

d) Volba $x=0 \Rightarrow 0 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^k}{k^2} \Rightarrow \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = -\frac{\pi^2}{12}$

a) Parsevalova rovnost v $L^2(-\pi,\pi)$

$$x^2 = \frac{\sqrt{2\pi}\,\pi^2}{3}\frac{1}{\sqrt{2\pi}} + \sum_{k=1}^{\infty}\frac{\sqrt{\pi}\,4(-1)^k}{k^2}\frac{\cos kx}{\sqrt{\pi}}$$

$$\Rightarrow \int_{-\pi}^{\pi}(x^2)^2 = \left(\frac{\sqrt{2\pi}\,\pi^2}{3}\right)^2 + \sum_{k=1}^{\infty}\left(\frac{\sqrt{\pi}\,4(-1)^k}{k^2}\right)^2$$

$$\frac{2\pi^5}{5} = \frac{2\pi^5}{9} + 16\pi\sum_{k=1}^{\infty}\frac{1}{k^4}$$

$$\Rightarrow \sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{8}\left(\frac{1}{5}-\frac{1}{9}\right) = \frac{\pi^4}{90}$$

7c) Fourierova řada pro x na $(-\pi,\pi)$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin kx = \left[-\frac{x\cos kx}{k\pi}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos kx}{k\pi}$$
$$= -\frac{2(-1)^k}{k}$$

$$x = -2\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\sin kx$$

integrace

$$\frac{x^2}{2} = \frac{a_0}{2} + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx$$

spočítáme a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{x^2}{2} = \frac{\pi^2}{3}$$

$$\Rightarrow \frac{x^2}{2} = \frac{\pi^2}{3} + 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos kx$$

integrace:

$$C + \frac{x^3}{6} - \frac{x\pi^2}{3} = 2\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3}\sin kx$$

C musí být zvoleno aby

$$\int_{-\pi}^{\pi} C + \frac{x^3}{6} - \frac{x}{3}\pi^2 = 0$$

$\Rightarrow C = 0$

$$\Rightarrow \frac{x^3}{6} - \frac{x\pi^2}{3} = \sum_{k=1}^{\infty}\frac{2(-1)^k\sqrt{\pi}}{k^3}\,\frac{\sin kx}{\sqrt{\pi}}$$

$$\Rightarrow \int_{-\pi}^{\pi}\left(\frac{x^3}{6} - \frac{x\pi^2}{3}\right)^2 = \sum_{k=1}^{\infty}\left(\frac{2(-1)^k\sqrt{\pi}}{k^3}\right)^2 = 4\pi\sum_{k=1}^{\infty}\frac{1}{k^6}$$

$$\parallel$$

$$\int_{-\pi}^{\pi}\frac{x^6}{36} + \frac{x^2\pi^4}{9} - \frac{\pi^2x^4}{9} = \left[\frac{x^7}{7\cdot 36} + \frac{x^3\pi^4}{3\cdot 9} - \frac{\pi^2x^5}{45}\right]_{-\pi}^{\pi}$$

Řešení parciálních diferenciálních rovnic

Fourierova metoda: uvažujme $u(t,x) := \sum_k \alpha_k(t) w_k(x)$

kde $w_k''(x) = -\lambda_k w_k(x)$

a $\{w_k\}_{k=1}^{\infty}$ tvoří ortonormální systém, který je úplný – musí splňovat okrajové podmínky.

pro rovnici $\partial_t u - u'' = 0$ chceme

$$0 = \sum \alpha_k' w_k - \alpha_k w_k'' = \sum (\alpha_k' + \lambda_k \alpha_k) w_k$$

$$\Rightarrow \quad \alpha_k(t) = \alpha_k(0) e^{-\lambda_k t}$$

$\Rightarrow u(t,x) = \sum \alpha_k(0) e^{-\lambda_k t} w_k(x)$ – zbývá určit $\alpha_k(0)$.

Tak aby $u(0) = \underbrace{\sum \alpha_k(0) w_k(0)}_{\text{Fourierova řada}}$

Pro rovnici $0 = \partial_{tt}^2 u - \partial_{xx} u = \sum (\alpha_k''(t) + \lambda_k \alpha_k) w_k$

$$\Rightarrow \quad \alpha_k = C_k \cos \lambda_k t + D_k \sin \lambda_k t$$

a C_k a D_k jsou určeny podle $u(0)$ a $\partial_t u(0)$

8) předepisujeme DIRICHLETOVY OKRAJOVÉ PODMÍNKY

$$u(t,0) = u(t,l) = 0$$

Uvažujme sinovou řadu u na $(0,l)$ v bázi $\{\sin\frac{\pi k x}{l}\}_{k=1}^{\infty}$ na $(0,l)$. Báze splňuje okrajové podmínky.

Dostaneme $u(t,x) = \sum b_k(t) \sin\frac{kx}{l}\pi$ kde $b_k(t) = \frac{2}{l}\int_0^l u(t,x)\sin\frac{kx}{l}\pi$

$$0 = \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \sum \sin\frac{kx}{l}\pi\left(b_k''(t) + c^2\left(\frac{k\pi}{l}\right)^2 b_k(t)\right)$$

$$\Rightarrow \quad b_k = A_k\cos\frac{k\pi c}{l}t + B_k\sin\frac{k\pi c}{l}t$$

a musí platit

$$u(0,x) = \sum_{k=1}^{\infty} A_k \sin\frac{k\pi}{l}x \quad !!$$

$$\partial_t u(0,x) = \sum_{k=1}^{\infty}\frac{k\pi c}{l}B_k\cos\frac{k\pi c}{l}t \quad !!$$

$$0 = \partial_t u(0,x) \Rightarrow B_k \equiv 0$$

$$x(l-x) = \sum A_k \sin\frac{k\pi}{l}x \Rightarrow A_k = \frac{2}{l}\int_0^l x(l-x)\sin\frac{k\pi x}{l}$$

$$= \frac{2}{l}\left[\frac{x(x-l)\cos\frac{k\pi x}{l}}{\frac{k\pi}{l}}\right]_0^l - \frac{2}{k\pi}\int_0^l (2x-l)\cos\frac{k\pi x}{l}$$

$$= \left[l\frac{2(l-2x)}{(k\pi)^2}\sin\frac{k\pi x}{l}\right]_0^l + \int_0^l \frac{4l}{(k\pi)^2}\sin\frac{k\pi x}{l}$$

$$= \left[-\frac{4l^2}{(k\pi)^3}\frac{\cos k\pi x}{l}\right]_0^l = \frac{4l^2}{k\pi^3}\left((-1)^{k+1} - 1\right)$$

$$\Rightarrow u(t,x) = \sum_{k=1}^{\infty}\frac{4l^2}{(k\pi)^3}\cos\frac{k\pi c}{l}t\,\sin\frac{k\pi x}{l}\left[(-1)^{k+1}-1\right]$$

9) Jiné okrajové podmínky NEUMANNOVY – použijeme cosinové řady

tj $u = \frac{a_0(t)}{2} + \sum a_k(t) \cos \frac{k\pi x}{\ell}$

kde $a_0(t) = \frac{2}{\ell} \int_0^\ell u(t,x)$

$a_k(t) = \frac{2}{\ell} \int_0^\ell \cos \frac{k\pi x}{\ell} u(t,x)$

jinak stejné jako 8)

10) Rozvineme do sinové řady na $(0, 2\ell)$

musíme funkci dodefinovat na $(\ell, 2\ell)$ sudě, tj.

$u(x+\ell) = u(\ell - x)$

$u(t,x) = \sum a_k(t) \sin (2k+1) \frac{x\pi}{2\ell}$

zbytek stejně

11) napíšeme u v polárních souřadnicích $u(r,\varphi)$

a) $0 = \Delta u(r,\varphi) = \boxed{\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}}$

b) u rozvineme do F řady vzhledem k φ

$u(r,\varphi) = \frac{a_0(r)}{2} + \sum_{k=1}^{\infty} a_k(r) \cos k\varphi + b_k(r) \sin k\varphi$

$f_1(\varphi) = u(1,\varphi) = \frac{a_0(1)}{2} + \sum a_k(1) \cos k\varphi + b_k(1) \sin k\varphi$

$f_2(\varphi) = u(2,\varphi) = \frac{a_0(2)}{2} + \sum a_k(2) \cos k\varphi + b_k(2) \sin k\varphi$

$\Delta u = \frac{a_0''(r)}{2} + \frac{1}{r}\frac{a_0'(r)}{2} + \sum_{k=1} \left(a_k''(r) + \frac{a_k'(r)}{r}\right) \cos k\varphi + \left(b_k''(r) + \frac{b_k'(r)}{r}\right) \sin k\varphi$

$- \sum_{k=1} \frac{a_k(r)}{r^2} k^2 \cos k\varphi + \frac{b_k(r)}{r^2} k^2 \sin k\varphi$

$\Rightarrow \begin{cases} r a_0'' + a_0' = 0 \\ r^2 a_k'' + r a_k' - k^2 a_k = 0 \quad r^2 b_k'' + r b_k' - k^2 b_k = 0 \end{cases}$

$$r a_0'' + a_0' = 0$$
$$r^2 a_k'' + r a_k' - k^2 a_k = 0$$
$$r^2 b_k'' + r b_k' - k^2 a_k = 0$$

$$\Rightarrow \quad a_0 = C \ln r + D \quad \Rightarrow \quad a_0 = a_0(1) + \frac{\ln r}{\ln 2}\left(a_0(2) - a_0(1)\right)$$

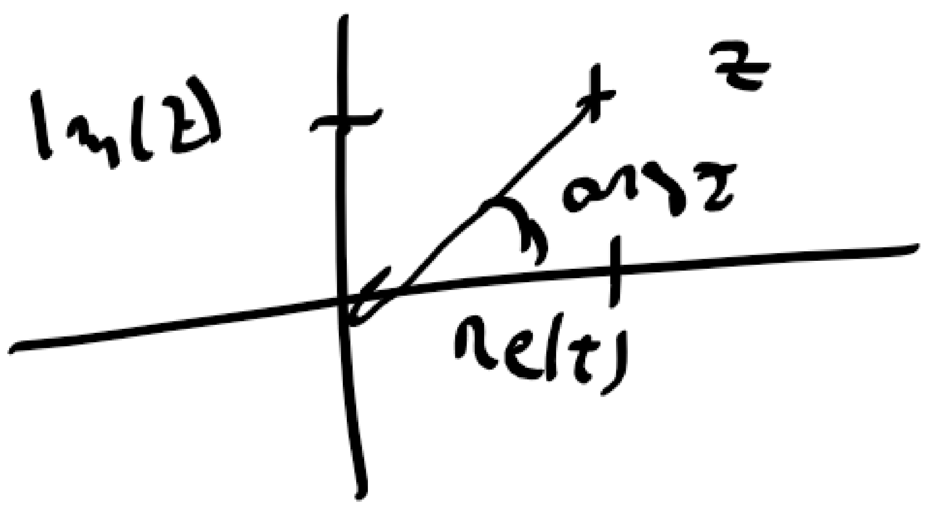
$$a_k = C r^k + D r^{-k} \quad \Rightarrow \quad \text{etc}$$

$$b_k = C r^{-k} + D r^{-k}$$

KOMPLEXNÍ ČÍSLA, FCE KOMPLEXNÍ PROMĚNNÉ

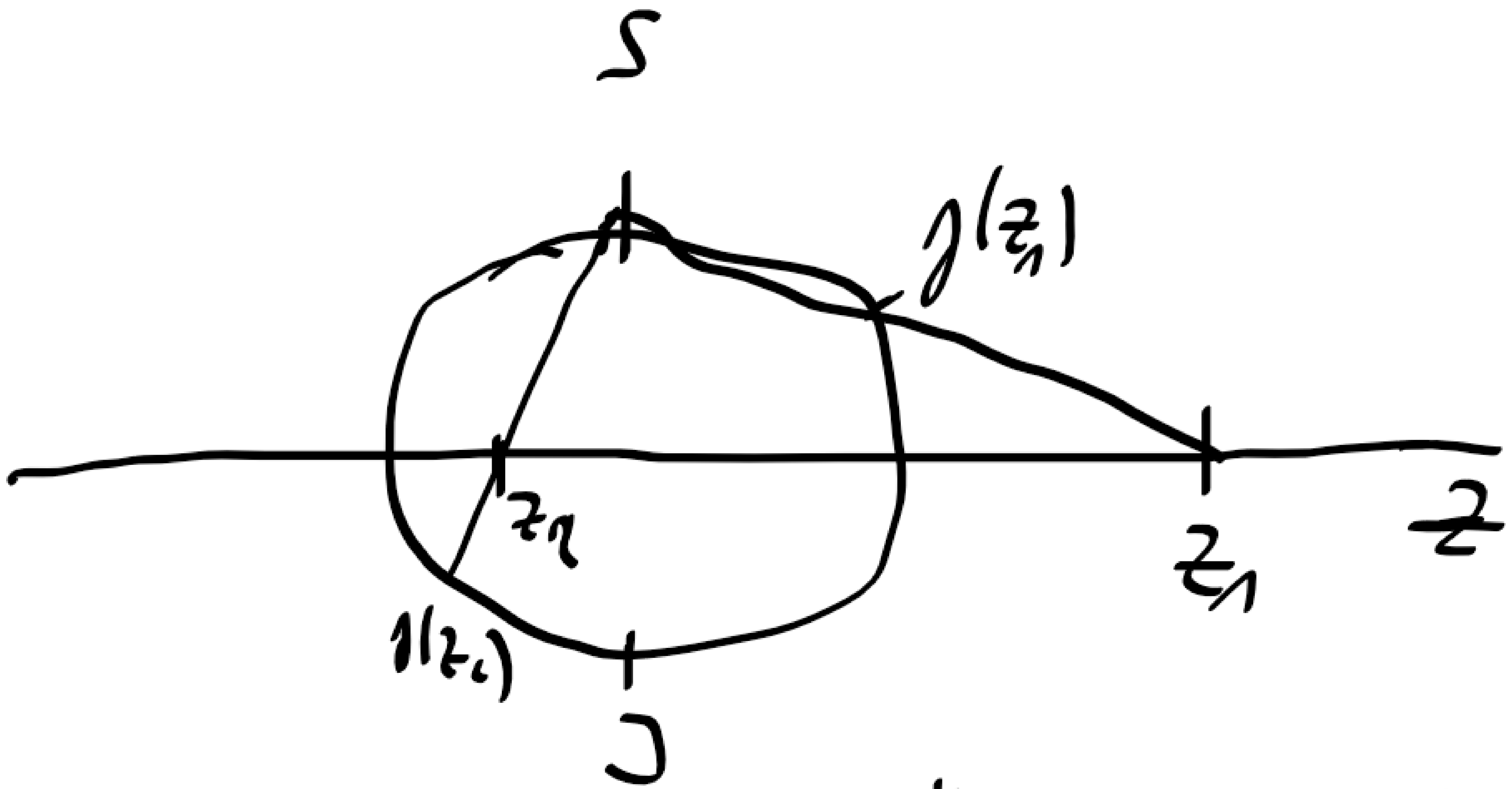
$$z = \mathrm{Re}(z) + i\,\mathrm{Im}(z) = |z|\,(\cos \arg z + i \sin \arg z)$$

reálná část → imaginární část



Zavedení "∞" a vztah s nekonečnem

Bijekce mezi sférou a komplexní rovinou



každý bod se zobrazí na "S" - identifikujeme jako "∞"

aritmetika: $\forall z \in \mathbb{C}$

$$z \pm \infty = \infty$$

$$z \cdot \infty = \infty \quad (z \neq 0)$$

$$\frac{z}{0} = \infty \quad (z \neq 0)$$

NEDEFINUJEME

$\infty \pm \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$, $0 \cdot \infty$

$f(z): \mathbb{C} \to \mathbb{C}$

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

pokud na definiční $\Omega \subseteq \mathbb{C}$ $\exists$ existuje $f'(z)$, říkáme

že $f(z)$ je na Ω holomorfní

Pro $f(z) = u(x,y) + i\,v(x,y)$ | kde $z = x + iy$

platí: 1) f je holomorfní na $\Omega \Rightarrow$

$$\underbrace{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}_{\text{Cauchy-Riemann podmínky}}$$

2) Pokud v, u mají spojité derivace a splňují C-R podm.

f je holomorfní

Elementární fce

z^k - polynom, $(z^k)' = k z^{k-1}$ $k \in \mathbb{N}$

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\cos z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

$$\sin z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Řady konvergují stejnoměrně

poloměr konvergence je ∞

MŮŽU DERIVOVAT

A KT. ČLEN PO ČLENU

VZTAHY $e^z, \sin z, \cos z$

$$\frac{e^{iz}+e^{-iz}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} + \frac{(-iz)^n}{n!} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(iz)^{2k} + (iz)^{2k}}{(2k)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} = \cos z$$

$$\frac{e^{iz}-e^{-iz}}{2i} = \sin z$$

Pro $z = x+iy$, $x, y \in \mathbb{R}$ $\quad e^{x+iy} = e^x(\cos y + i \sin y)$

(1)

a) $\cos(2+i) = \dfrac{e^{i(2+i)} + e^{-i(2+i)}}{2} = \dfrac{e^{-1+2i} + e^{1-2i}}{2}$

$= \dfrac{e^{-1}(\cos 2 + i\sin 2) + e(\cos 2 - i\sin 2)}{2}$

$= \underbrace{\cos 2 \cosh 1}_{Re} - i \underbrace{\sin 2 \sinh 2}_{Im}$

b) $\sin(2i) = \dfrac{e^{i(2i)} - e^{-i(2i)}}{2i} = -\dfrac{e^{-2} - e^{2}}{2} i = i \sinh 2$

c) $tg(2-i) = \dfrac{\sin(2-i)}{\cos(2-i)} = \dfrac{1}{i} \dfrac{e^{i(2-i)} - e^{-i(2-i)}}{e^{i(2-i)} + e^{-i(2-i)}}$

$= -i \dfrac{e^{1+2i} - e^{-1-2i}}{e^{1+2i} + e^{-1-2i}} = -i \dfrac{e^{2+4i} - 1}{e^{2+4i} + 1} \cdot \dfrac{e^{2-4i} + 1}{e^{2-4i} + 1}$

$= -i \dfrac{(e^2 - 1 + 2e^2 i \sin 4)}{e^2 + 1 + 2e^2 \cos 4} = \dfrac{2e^2 \sin 4}{e^2 + 1 + 2e^2 \cos 4} + i \dfrac{(1-e^2)}{e^2 + 1 + 2e^2 \cos 4}$

(2) $e^{z_1} \cdot e^{z_2} = \left(\sum_{k=0}^{\infty} \frac{z_1^k}{k!} \right) \cdot \left(\sum_{m=0}^{\infty} \frac{z_2^m}{m!} \right)$

$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{k!} \frac{1}{m!} z_1^k z_2^m$

$n = k + m$

$\sum_{n=0}^{\infty} \sum_{m=0}^{n} z_1^{n-m} z_2^m \frac{1}{(n-m)!\, m!} =$

$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m=0}^{n} z_1^{n-m} z_2^m \binom{n}{m} = \sum_{n=0}^{\infty} \frac{(z_1+z_2)^n}{n!}$ $\square$

(3) a) $\sin(z_1+z_2) = \frac{e^{i(z_1+z_2)} - e^{-i(z_1+z_2)}}{2i} =$

$= \frac{(e^{iz_1} - e^{-iz_1})}{2i} \cdot \frac{(e^{iz_2} + e^{-iz_2})}{2} + \frac{(e^{iz_1} + e^{-iz_1})}{2} \cdot \frac{(e^{iz_2} - e^{-iz_2})}{2i}$

$= \sin z_1 \cos z_2 + \cos z_1 \sin z_2$

b) – STEDLE

c) $\sin^2 t + \cos^2 t = \left(\frac{e^{it} - e^{-it}}{2i} \right)^2 + \left(\frac{e^{it} + e^{-it}}{2} \right)^2$

$= \frac{e^{2it} + e^{-2it} - 2e^{it}e^{-it}}{-4} + \frac{e^{2it} + e^{-2it} + 2e^{it} \cdot e^{-it}}{4} = e^0 = 1$

③ a) $\sin(iz) = \dfrac{e^{i(iz)} - e^{-i(iz)}}{2i}$

$$= -i\,\frac{e^{-z} - e^{z}}{2} = i \sinh z$$

c) oldalra ..

④ a) $\sin z + \cos z = 2$

$$2 = \frac{e^{iz} - e^{-iz}}{2i} + \frac{e^{iz} + e^{-iz}}{2} \qquad \text{Legyen } w := e^{iz} \in \mathbb{C}$$

$$4 = i\left(\frac{1}{w} - w\right) + w + \frac{1}{w}$$

$$4w = i - iw^2 + w^2 + 1$$

$$w^2 - \frac{4}{1-i}\,w + \frac{1+i}{1-i} = 0 \qquad z = \frac{\pi}{4} + 2k\pi - i\sqrt{2}$$

$$w^2 - 2(1+i)\,w + (1+i)^2 = 0$$

$$\Longleftrightarrow$$

$$\big(w - (1+i)\big)^2 = 0 \Longleftrightarrow w = 1+i$$

$$\Rightarrow e^{iz} = 1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4} + 2k\pi\right) + i \sin\left(\frac{\pi}{4} + 2k\pi\right)\right)$$

$$= e^{\sqrt{2} + i\left(\frac{\pi}{4} + 2k\pi\right)} = e^{i\left(\frac{\pi}{4} + 2k\pi - i\sqrt{2}\right)}$$

b) $\sinh z - \cosh z = 2i$

$$\frac{e^{z}-e^{-z}}{2} - \frac{e^{z}+e^{-z}}{2} = 2i$$

$$-e^{-z} = 2i$$

$$\Rightarrow e^{-z} = -2i = 2\left(\cos\left(\frac{3\pi}{2}+2k\pi\right) + i\sin\left(\frac{3\pi}{2}+2k\pi\right)\right)$$

$$= e^{2 + i\left(\frac{3\pi}{2}+2k\pi\right)} \Rightarrow z = -2 - i\left(\frac{3\pi}{2}+2k\pi\right)$$

$$\sum_{k=0}^{n} \cos(\alpha+k\beta) = \sum_{k=0}^{n} \frac{e^{i(\alpha+k\beta)} + e^{-i(\alpha+k\beta)}}{2}$$

$$= \frac{e^{i\alpha}}{2} \sum_{k=0}^{n} e^{ik\beta} + \frac{e^{-i\alpha}}{2} \sum_{k=0} e^{-ik\beta}$$

$$= \frac{e^{i\alpha}}{2} \underbrace{\sum_{k=0}^{n} \left(e^{i\beta}\right)^{k}}_{\text{Geom}} + \frac{e^{-i\alpha}}{2} \underbrace{\sum_{k=0}^{n} \left(e^{-i\beta}\right)^{k}}_{\text{Geom}}$$

$$= \frac{e^{i\alpha}}{2} \frac{e^{ik\beta}-1}{e^{i\beta}-1} + \frac{e^{-i\alpha}}{2} \frac{e^{-ik\beta}-1}{e^{-i\beta}-1}$$

$$= \frac{e^{i\alpha}}{2} \frac{e^{ik\beta}-1}{e^{i\beta}-1} + \frac{e^{-i\alpha}}{2} \frac{e^{-ik\beta}-1}{e^{-i\beta}-1}$$

$$= \frac{\cos\alpha + i\sin\alpha}{2} \cdot \frac{\cos k\beta + i\sin k\beta - 1}{\cos\beta + i\sin\beta - 1}$$

$$+ \frac{\cos\alpha - i\sin\alpha}{2} \cdot \frac{\cos k\beta - i\sin k\beta - 1}{\cos\beta - i\sin\beta - 1}$$

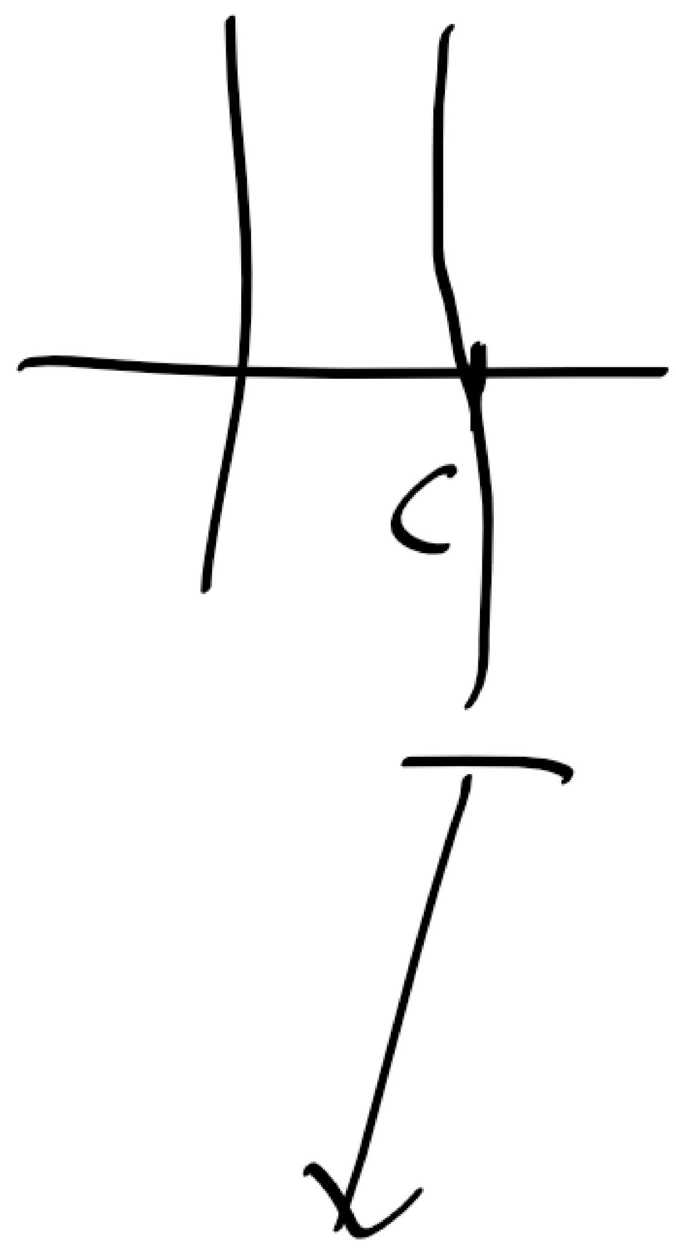
$$= \frac{\cos\alpha + i\sin\alpha}{2} \quad \frac{\cos k\beta + i\sin k\beta - 1}{(\cos\beta - 1)^2 + \sin^2\beta} \left(\cos\beta - 1 - i\sin\beta\right)$$

$$+ \frac{\cos\alpha - i\sin\alpha}{2} \quad \frac{\cos k\beta - i\sin k\beta - 1}{(\cos\beta - 1)^2 + \sin^2\beta} \left(\cos\beta - 1 + i\sin\beta\right)$$

$= \mathrm{dld} ..$

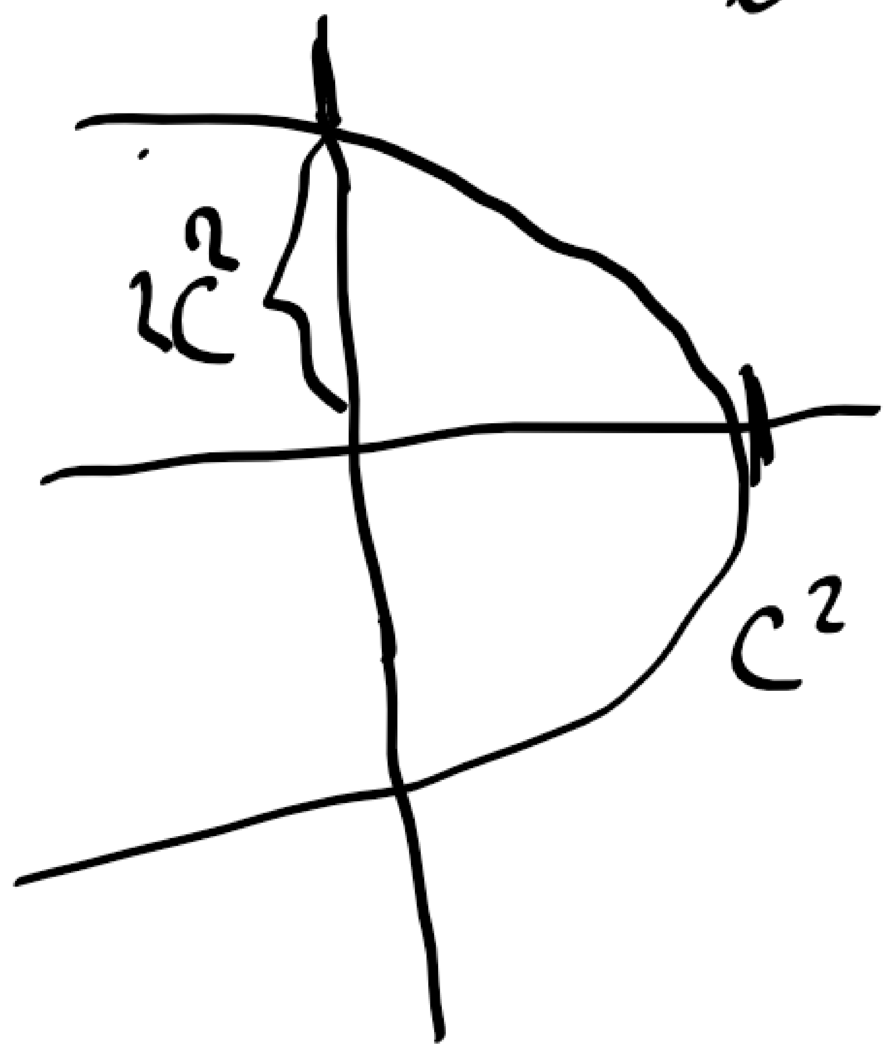
(6) zobrazení $w = z^2$ $c \in \mathbb{R}$

a) obraz přímky $z = c + iy$



$$u = z^2 = (c+iy)^2 = \underbrace{c^2 - y^2}_{u} + i\,\underbrace{2cy}_{v}$$

$$c^2 - \frac{v^2}{4c^2} = u \quad \xRightarrow{\text{OBRAZ}}$$



(b) obraz kružnice $|z| = R$ $z = Re^{i\varphi}$

$\varphi \in [0, 2\pi)$

$z^2 = R^2 e^{i2\varphi}$ = 2x oběhlá kružnice o poloměru R^2

$\Rightarrow$ obraz $|z| = R$ je $|w| = R^2$

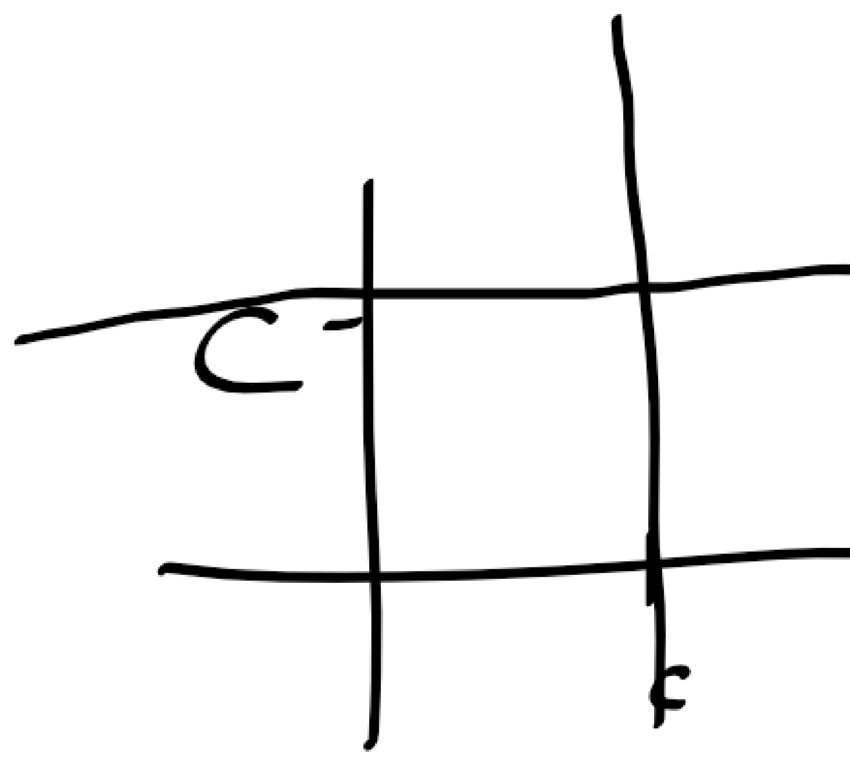
c) Iščemo izvor premice $u=c$

$$(x+iy)^2 = c+iv$$

$$\iff$$

$$x^2-y^2+2ixy = c+iv \iff \begin{array}{l} 2xy = v \\ x^2-y^2 = c \end{array}$$

Izvor je HIPERBOLA

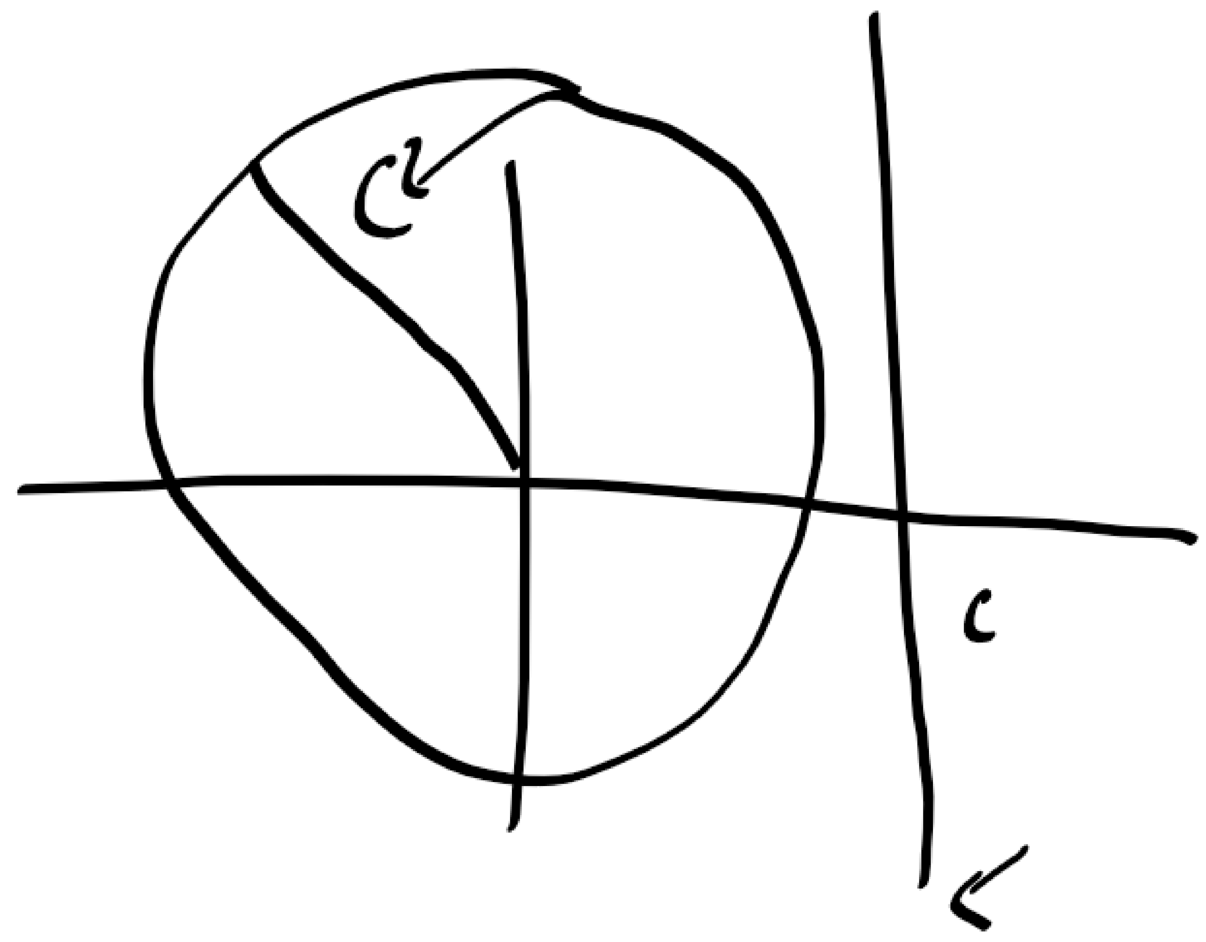


(7) OBRAZ $c+iy$ in $x+ic$

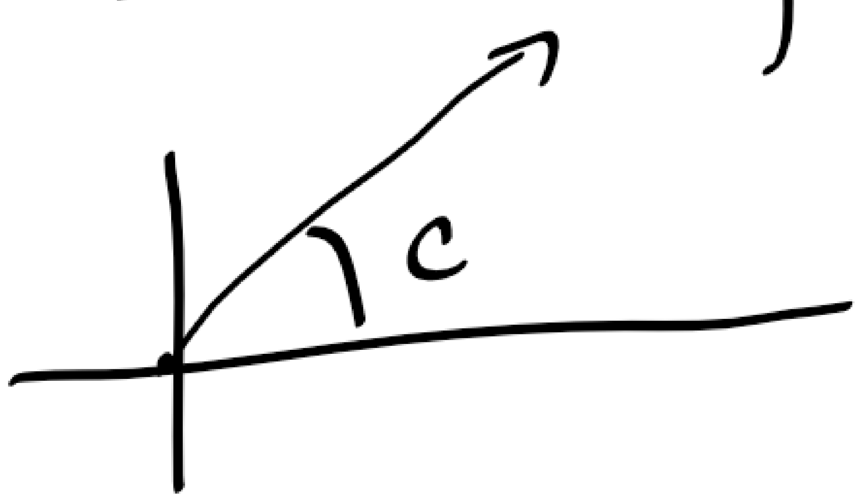
po zobrazenju $w=e^z$

a) $c+iy$ $\quad e^{c+iy} = e^c e^{iy}$

$\qquad\qquad\qquad\qquad e^c \to$ polmer, $\quad e^{iy} \to$ krožnica



b) $x+ic$ $\quad e^{x+ic} = e^x(\cos c + i\sin c)$ polpremica



8) obraz $\operatorname{Im} z = 1$ při $w = \frac{z-1}{z+1}$

$z = x + i$

$$\frac{z-1}{z+1} = 1 - \frac{2}{z+1} = 1 - \frac{2}{x+1+i}$$

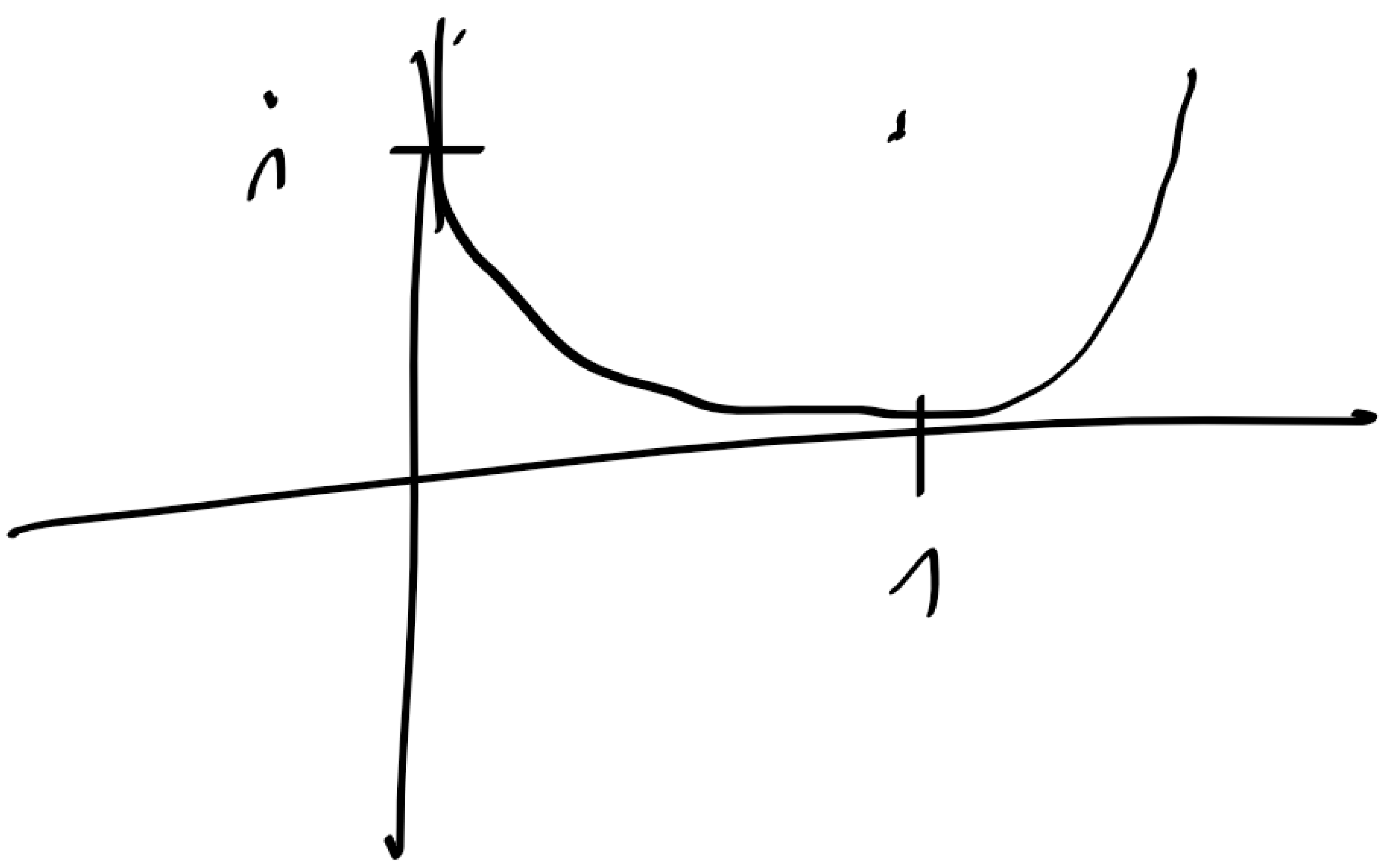
$$= 1 - \frac{2(x+1)-2i}{(x+1)^2+1} = \underbrace{1 - \frac{2(x+1)}{(x+1)^2+1}}_{u} + i\underbrace{\frac{2}{(x+1)^2+1}}_{v}$$

$$\Rightarrow (u-1)^2 + v^2 = \frac{4(x+1)^2}{\left((x+1)^2+1\right)^2} + \frac{4}{\left[(x+1)^2+1\right]^2}$$

$$= \frac{4}{(x+1)^2+1} = 2v$$

$$\Rightarrow (u-1)^2 + (v-1)^2 = 1$$

obraz je kružnice se středem $1+i$ a poloměrem 1



(9) OBRAZ $|z+1| = 1$ při $u = \frac{1}{z}$

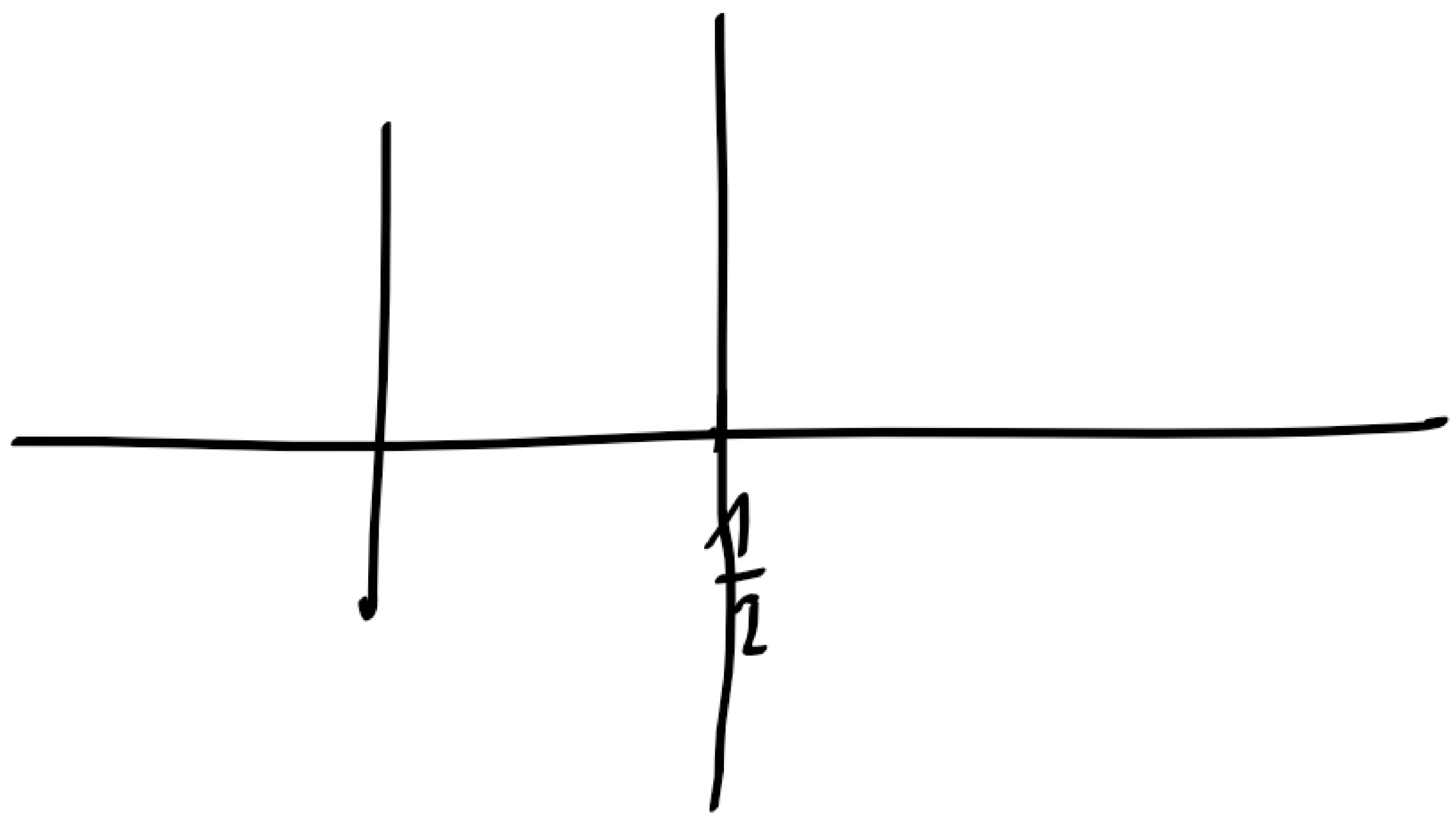
$$z + 1 = e^{i\varphi} \qquad \varphi \in \mathbb{R}$$

$$z = 1 - e^{i\varphi} \qquad u = \frac{1}{1 - e^{i\varphi}} = \frac{1}{1 - \cos\varphi - i\sin\varphi}$$

$$= \frac{1 - \cos\varphi + i\sin\varphi}{(1-\cos\varphi)^2 + \sin^2\varphi} = \frac{1 - \cos\varphi + i\sin\varphi}{2(1-\cos\varphi)}$$

$$= \frac{1}{2} + i \underbrace{\frac{\sin\varphi}{1-\cos\varphi}}_{\text{nabývá všech hodnot}}$$

OBRAZ



(10) obraz $\underbrace{|z| = 2}$ při $u = \frac{1}{2}(z + z^{-1})$

$$z = 2e^{i\varphi} \qquad \varphi \in (-\pi, \pi]$$

$$u = \frac{1}{2}\left(2e^{i\varphi} + \frac{e^{-i\varphi}}{2}\right) = \cos\varphi + i\sin\varphi + \frac{1}{4}\cos\varphi - \frac{1}{4} i\sin\varphi$$

$$= \underbrace{\frac{5}{4}\cos\varphi}_{\mu} + i\,\underbrace{\frac{3}{4}\sin\varphi}_{\nu} \qquad \left(\frac{4}{5}\mu\right)^2 + \left(\frac{4}{3}\nu\right)^2 = 1$$

$$\left(\frac{4}{5}u\right)^2+\left(\frac{4}{3}v\right)^2=1 \iff \frac{u^2}{\left(\frac{5}{4}\right)^2}+\frac{v^2}{\left(\frac{3}{4}\right)^2}=1$$

Elipsa se středem $v(0,0)$ a poloosy $\frac{5}{4}$ i $\frac{3}{4}$

(11) $f(z)=|z|$ $\quad u(x,y)=\sqrt{x^2+y^2}$ $\quad v(x,y)=0$

$f(z)=u(x,y)+i\,v(x,y)$ $\quad z=x+iy$

$$0=\frac{\partial v}{\partial y}\neq\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y}}$$ pouze pro $x=0$ a $y\neq0$

$$0=\frac{\partial v}{\partial x}\neq-\frac{\partial u}{\partial y}=-\frac{y}{\sqrt{x^2+y}}$$ pouze pro $y=0$ a $x\neq0$

není nikde holomorfní

(12) a) $f(z)=\underbrace{x+ay}_{u}+i\underbrace{(bx+cy)}_{v}$

Cauchy-Riemann $\quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \iff 1=c$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x} \iff a=-b$$

parciální derivace jsou spojité a tedy C-R jsou i postačující

$f(z)=x+ay+i(-ax+y)$

b) $f(z)=\cos x(\cosh y+a\sinh y)+i\sin x(\cosh y+b\sinh y)$

$$\frac{\partial u}{\partial x}=-\sin x(\cosh y+a\sinh y) \qquad \frac{\partial v}{\partial y}=\sin x(\sinh y+b\cosh y)$$

$\Rightarrow a=b=-1$ D.

(13) $f(z) = \sqrt{|xy|}$ $u = \sqrt{|xy|}$ $v = 0$

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h} = 0$$

$$\frac{\partial u}{\partial y}(0,0) = \dots = 0$$

C-R središ

de $\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \overset{z=0}{=} \lim_{(h_1,h_2)\to 0} \frac{\sqrt{|h_1||h_2|}}{\sqrt{h_1^2+h_2^2}}$

neeksistira!

(14) vse stejne deriviraj red člen po členu

npr. $(\sin z)' = \left(\sum_{n=0} \frac{(-1)^n z^{2n+1}}{(2n+1)!}\right)' \overset{\text{POLOMER} = \infty}{=}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (z^{2n+1})'}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{2n!}$$

$= \cos z$

5) $f(z) = u(x,y) + i\,v(x,y)$ f je holomorfní

a) $u(x,y) = x^2 - y^2 + e^x (x\cos y - y \sin y)$

první splnění C-R

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x + e^x(x\cos y - y\sin y) + e^x \cos y$$

$$\Rightarrow v(x,y) = C(x) + \int 2x + e^x x\cos y - e^x y \sin y + e^x \cos y \, dy$$

$$= C(x) + 2xy + e^x x \sin y + e^x y \cos y$$

druhá podmínka

$$-2y + e^x(-x\sin y - \sin y - y\cos y) = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} =$$

$$= -C'(x) - 2y + e^x x \sin y \cancel{+} e^x \sin y - e^x y \cos y$$

$\Rightarrow \quad C'(x) = 0 \quad$ a tedy $C = \text{konst.}$

b) $u(x,y) = x^2 - y^2 + 5x + y - \dfrac{y}{x^2+y^2}$ $(x,y) \neq 0$

$$\frac{\partial u}{\partial x} = 2x + 5 + \frac{2xy}{(x^2+y^2)^2} \quad , \quad \frac{\partial^2 u}{\partial x^2} = 2 + \frac{2y}{(x^2+y^2)^2} - \frac{8x^2y}{(x^2+y^2)^3}$$

$$\frac{\partial u}{\partial y} = -2y + 1 - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2} \; ; \quad \frac{\partial^2 u}{\partial y^2} = -2 + \frac{2y}{(x^2+y^2)^2} + \frac{4y}{(x^2+y^2)^2} - \frac{8y^3}{(x^2+y^2)^3}$$

$$\Delta u = \frac{2yx^2 + 2y^3 - 8x^2y + 6yx^2 + 6y^3 - 8y^3}{(x^2+y^2)^3} = 0 \quad \checkmark$$

$$v = C(x) + \int 2x+5 + \frac{2xy}{(x^2+y^2)^2}\, dy = C(x) + (2x+5)y - \frac{x}{x^2+y^2}$$

$$-\frac{\partial v}{\partial x} = -C'(x) - 2y + \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{\partial u}{\partial y} = -2y + 1 - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}$$

$$\Rightarrow C'(x) = \frac{\cancel{x^2+y^2 - 2x^2 + x^2+y^2 - 2y^2}}{(x^2+y^2)^2} - 1 = -1 \Rightarrow \boxed{C = -x}$$

c) $u(x,y) = \ln(x^2+y^2) + x - 2y$ $\qquad (x,y) \neq 0 \quad z \neq 0$

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} + 1 \Rightarrow v(x,y) = C(x) + \int\left(\frac{2x}{x^2+y^2} + 1\right)dy$$

$$\frac{\partial u}{\partial y} = \frac{2y}{(x^2+y^2)} - 2 \qquad = C(x) + x + 2\arctan\frac{y}{x}$$

$$\frac{\partial v}{\partial x} = c'(x) + 1 - \frac{2}{1+\frac{y^2}{x^2}}\frac{y}{x^2}$$

$$\Rightarrow -\frac{2y}{x^2+y^2} + 2 = c'(x) + 1 - \frac{2y}{x^2+y^2} \Rightarrow c' = 1$$

$$\Rightarrow v = 2x + 2\arctan\frac{y}{x} + c$$

KŘIVKOVÝ INTEGRÁL

$\varphi : [a,b] \to \mathbb{C}$ – křivka, uvažujeme později po částech C^1 křivky

φ je regulární pokud $\varphi'(t) \neq 0$

φ je jednoduchá, pokud je prostá na $[a,b)$.

φ je uzavřená pokud $\varphi(a) = \varphi(b)$

φ je Jordanova, pokud je jednoduchá a uzavřená

$$\boxed{\int_\varphi f(z)\,dz := \int_a^b f(\varphi(t))\,\varphi'(t)\,dt}$$ křivkový int. v $\mathbb{C}$

pokud $F'(z) = f(z)$ v Ω a $\langle \varphi \rangle \subseteq \Omega$

pak $\int_\varphi f(z)\,dz = F(\varphi(b)) - F(\varphi(a))$ – integrál nezávisí na cestě

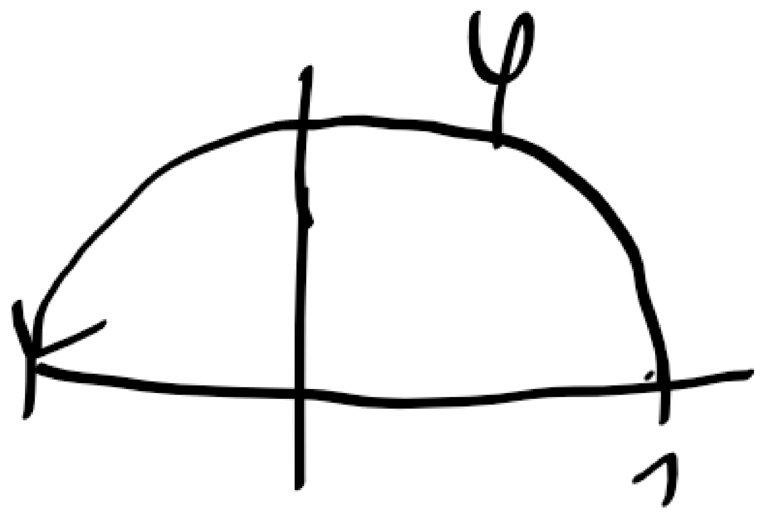
Cauchyova věta: Buď f holomorfní v Ω a

$\langle \underbrace{\mathrm{int}}_{\text{vnitřek křivky}}\, \varphi \rangle \cup \langle \varphi \rangle \subseteq \Omega$ $\quad$ φ je Jordanova

$$\Rightarrow \boxed{\int_\varphi f(z)\,dz = 0}$$

Křivkový integrál

(1) $\int_\varphi z\,dz$

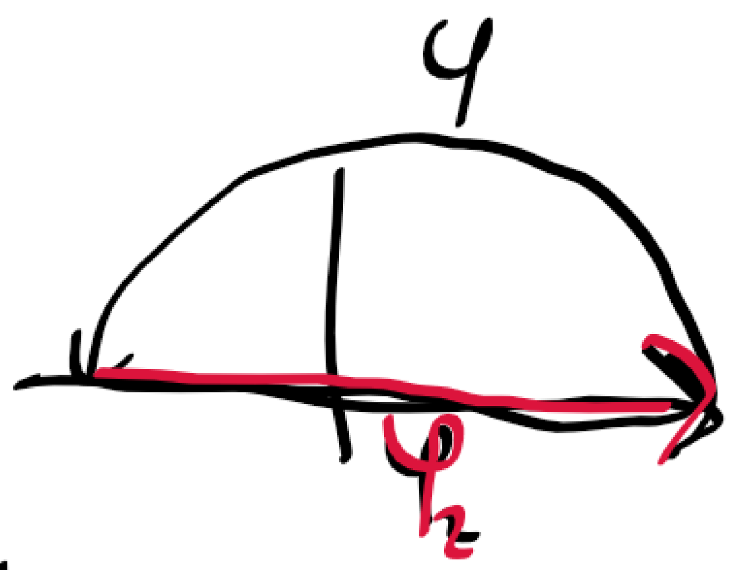


$\varphi = e^{it}$ $\quad t \in [0, \pi)$ $\quad \varphi'(t) = ie^{it}$

$\Rightarrow$ a) $\int_\varphi z\,dz \overset{def}{=} \int_0^\pi e^{it}\, i\, e^{it}\,dt = i\int_0^\pi e^{i2t}\,dt = \left[\frac{e^{i2t}}{2}\right]_0^\pi$

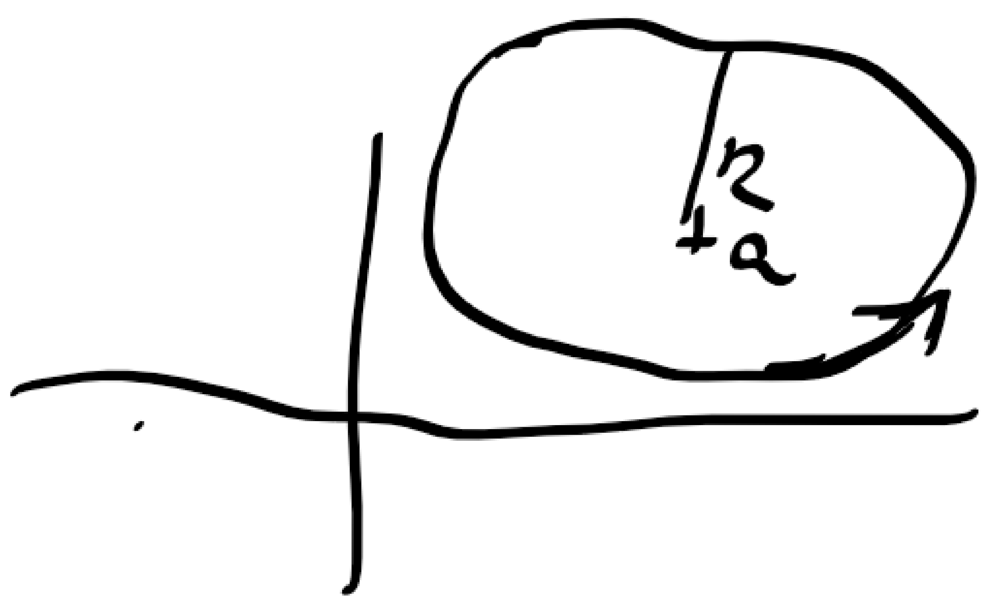
$= \frac{1}{2}\left(e^{2\pi i} - 1\right) = 0$

b) Pomocí Cauchyho věty



z je holomorfní $\Rightarrow \int_{\varphi \cup \varphi_2} z\,dz = 0 \Rightarrow \int_\varphi z\,dz = -\int_{\varphi_2} z\,dz = -\int_{-1}^{1} x\,dx = 0$

(2) $\int_\varphi (z-a)^n\,dz \quad n \in \mathbb{Z}$



(Poznámka pro $n \geq 0$ je integrál nulový přímo z Cauchyho věty...)

$\varphi(t) = Re^{it} + a \qquad \varphi'(t) = iRe^{it} \qquad t \in [-\pi, \pi)$

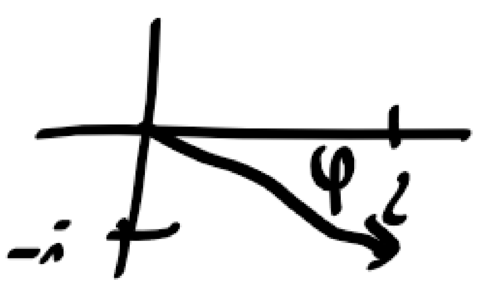
$\int_\varphi (z-a)^n\,dz = \int_{-\pi}^{\pi} \left(Re^{it}\right)^n iRe^{it}\,dt =$

$= R^{n+1}\, i \int_{-\pi}^{\pi} e^{it(n+1)}\,dt = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = 1 \end{cases}$

Důležité!

Nezávisí na R, a

(3) $\int_\varphi |z|\,dz$

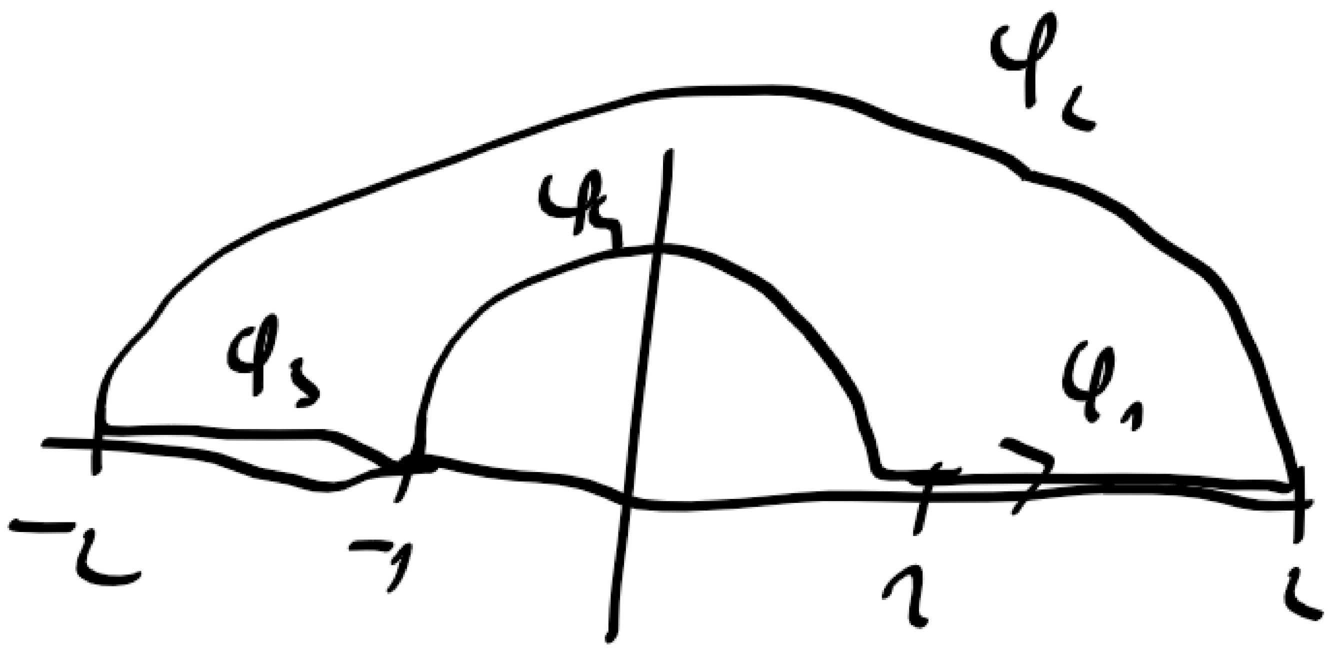


$\varphi: (2-i)t \qquad t \in [0, 1]$

$\varphi' = (2-i)$

$\int_\varphi |z|\,dz = \int_0^1 t\,|2-i|\,(2-i)\,dt = \sqrt{5}\,(2-i)$

(4) $\int_\varphi \frac{z}{\bar z}\,dz$



$\int_\varphi \frac{z}{\bar z}dz = \int_\varphi \frac{z^2}{|z|^2}\,dz$

$\varphi_1(t) = 1+t \quad (t\in(0,1)) \qquad \varphi_1' = 1$

$\varphi_2(t) = 2e^{it} \quad t\in(0,\pi) \qquad \varphi_2' = i2e^{it}$

$\varphi_3(t) = -2+t \quad t\in(0,1) \qquad \varphi_3' = 1$

$-\varphi_4 = e^{it} \quad t\in(0,\pi) \qquad -\varphi_4' = ie^{it}$

$$= \int_0^1 \frac{(1+t)^2}{(1+t)^2}\,1\,dt \quad (\varphi_1)$$

$$+ \int_0^\pi \frac{(2e^{it})^2}{|2e^{it}|^2}\,2ie^{it}\,dt \quad (\varphi_2)$$

$$+ \int_0^1 \frac{(t-2)^2}{(t-2)^2}\,dt \quad (\varphi_3)$$

$$- \int_0^\pi \frac{(e^{it})^2}{|e^{it}|^2}\,ie^{it}\,dt \quad (-\varphi_4)$$

$$= 2\int_0^1 dt + i\int_0^\pi e^{3it}\,dt = 2 + \left[\frac{e^{3it}}{3}\right]_0^\pi = 2 - \frac{2}{3}$$

(5) $\int_\varphi \frac{dz}{z^2+9}$

φ je uzavřená křivka, která neprotíná body $\pm 3i$

$$\frac{6i}{z^2+9} = -\frac{1}{z+3i} + \frac{1}{z-3i} \Rightarrow \frac{1}{z^2+9} \text{ je holomorfní kromě bodů } \pm 3i$$

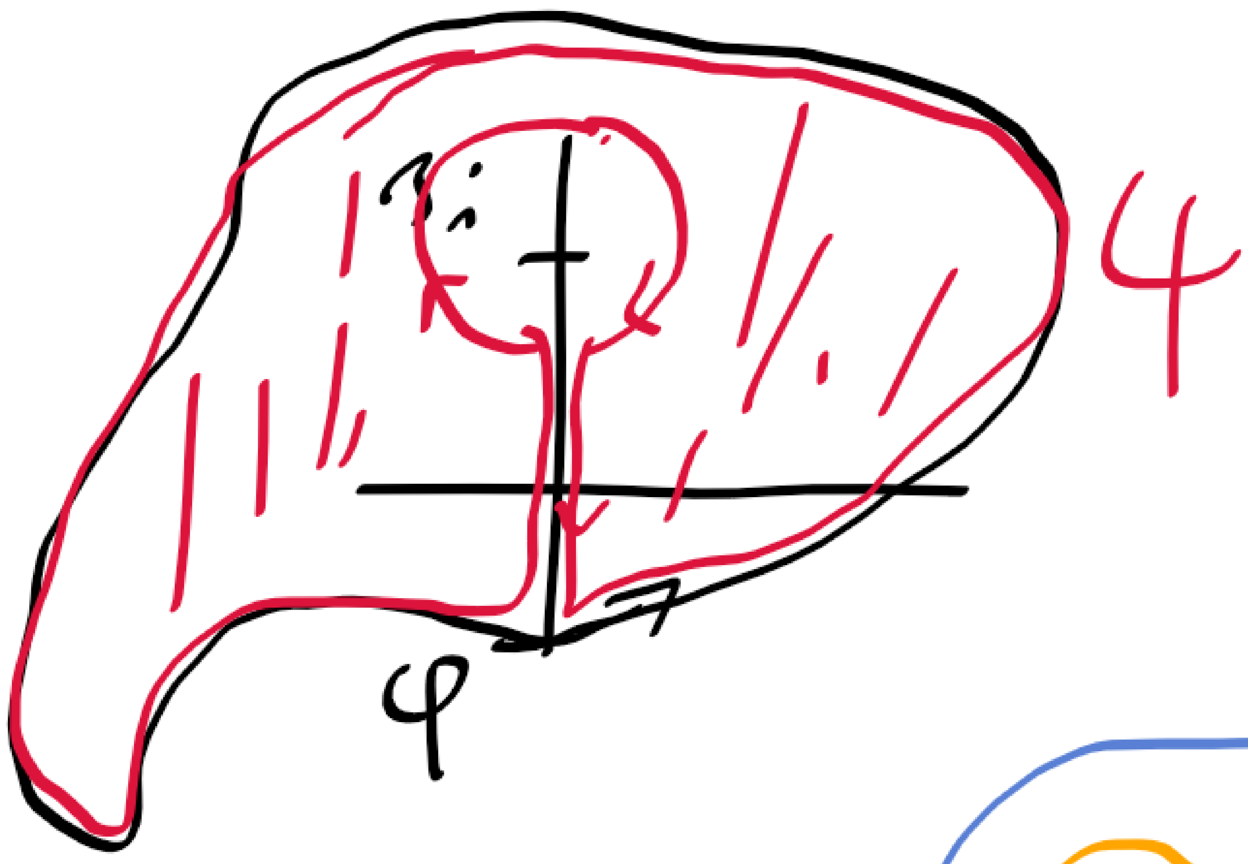
a) $\pm 3i \notin \operatorname{int}\varphi$

$\Rightarrow f$ je holomorfní na $\operatorname{int}\varphi \Rightarrow$ Cauchyho věta $\Rightarrow \int_\varphi \frac{1}{z^2+9}dz = 0$

b) $3i \in \operatorname{int}\varphi$ a $-3i \notin \operatorname{int}\varphi$

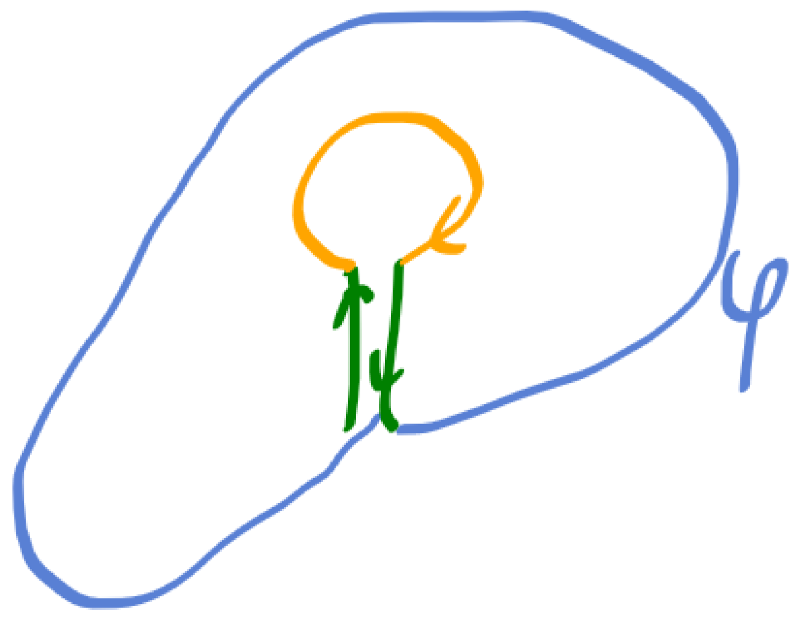
$$\Rightarrow \int_\varphi \frac{1}{z^2+9}dz = \frac{1}{6i}\int_\varphi \frac{1}{z-3i} - \underbrace{\frac{1}{z+3i}}_{\text{holomorfní na int}\,\varphi} = \frac{1}{6i}\int_\varphi \frac{1}{z-3i}$$

Stačí: $\int_\varphi \frac{1}{z-3i}$



$\frac{1}{z-3i}$ je holomorfní

na int ψ $\Rightarrow \int_\psi \frac{dz}{z-3i} = 0$



$0 = \int_\psi = \int_\varphi + \underbrace{\int_\uparrow + \int_\downarrow}_{=0} + \int_{\circlearrowleft} \Rightarrow \int_\varphi \frac{1}{z-3i} = \int_{\circlearrowleft} \frac{1}{z-3i}$ $\forall \varphi$

$\circlearrowleft$ je kružnice se středem v $3i$ a kladným poloměrem

z příkladu (2) víme že $\int_{\circlearrowleft} \frac{1}{z-3i} = 2\pi i$

a tedy $\int_\varphi \frac{1}{z^2+9}\,dz = \frac{1}{6i}\int_{\circlearrowleft} \frac{1}{z-3i} = \frac{\pi}{3}$

c) $3i \notin \operatorname{int}\varphi$ a $-3i$ je v int φ

$\Rightarrow \int_\varphi \frac{1}{z^2+9}\,dz = \frac{1}{6i}\int_\varphi \frac{1}{z-3i} - \frac{1}{z+3i} = -\frac{1}{6i}\int_\varphi \frac{1}{z+3i}\,dz$

$= -\frac{\pi}{3}$

d) $\pm 3i \in \operatorname{int}\varphi$

$\Rightarrow \int_\varphi = \frac{1}{6i}\int_\varphi \frac{1}{z-3i} - \frac{1}{z+3i} = \frac{1}{6i}(2\pi i - 2\pi i) = 0$

(6) $\displaystyle\int_\varphi \frac{1}{z(z^2-1)}\,dz$ $\qquad$ φ je kružnice s poloměrem $\frac{1}{2}$ a středem

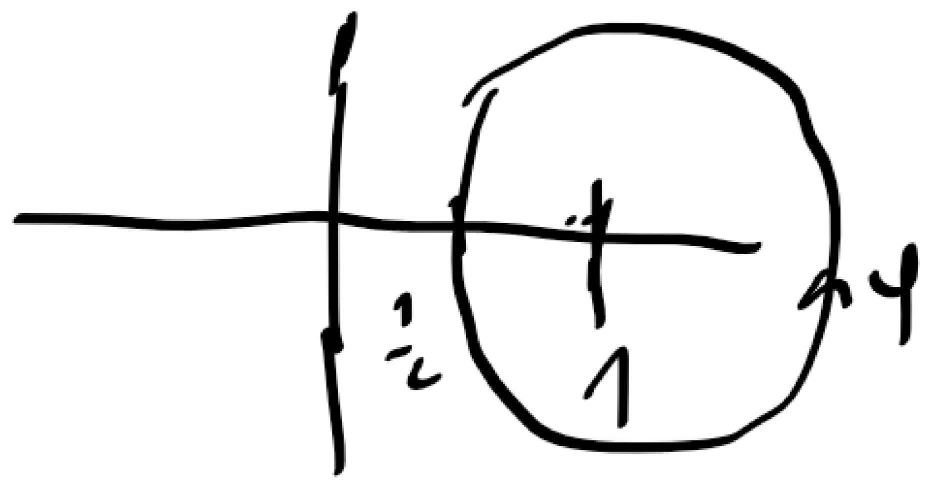
a) 1 ; b) 0 ; c) -1

$$\frac{1}{z(z^2-1)} = \frac{1-z}{z(z^2-1)} + \frac{z}{z(z^2-1)} = -\frac{1}{z(z+1)} + \frac{1}{z^2-1}$$

$$= -\frac{1}{z} + \frac{1}{z+1} + \frac{\frac{1}{2}}{z-1} - \frac{\frac{1}{2}}{z+1}$$

$$= -\frac{1}{z} + \frac{1}{2}\left(\frac{1}{z-1} + \frac{1}{z+1}\right)$$

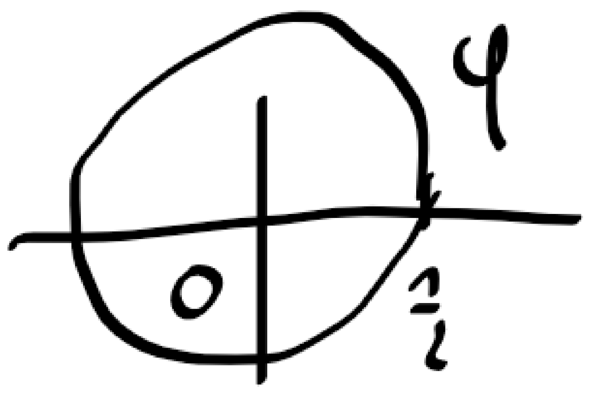
a)



$\frac{1}{z}$ a $\frac{1}{z+1}$ je holomorfní na int φ $\Rightarrow$

$$\int_\varphi \frac{1}{z(z^2-1)}\,dz = \frac{1}{2}\int_\varphi \frac{1}{z-1}\,dz = \pi i$$

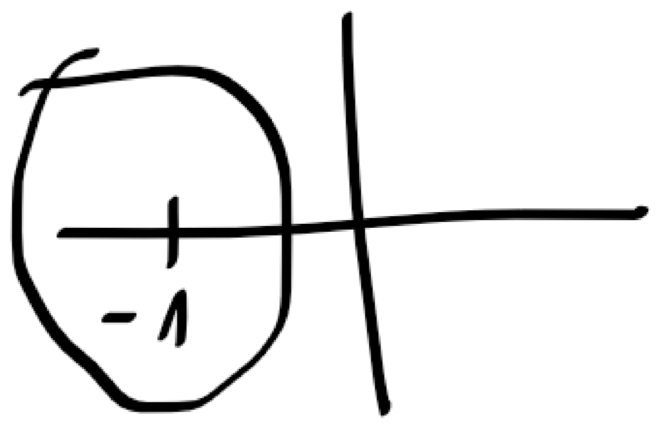
b)



$\frac{1}{z\pm1}$ holomorfní na int φ

$$\Rightarrow \int_\varphi \frac{1}{z(z^2-1)} = \int_\varphi -\frac{1}{z}\,dz = -2\pi i$$

c)



$$\int_\varphi \frac{1}{z(z^2-1)} = \frac{1}{2}\int_\varphi \frac{1}{z+1} = \pi i$$

(7) $\displaystyle\int_\varphi \frac{e^z\,dz}{z(z-1)^3}$ $\qquad$ φ je kladně orientovaná kružnice o poloměru $\frac{3}{2}$ a středu a) -1; b) 2 ; c) $\frac{1}{2}$

$$\frac{1}{z(z-1)^3} = \frac{1-z+z}{z(z-1)^3} = -\frac{1}{z(z-1)^2} + \frac{1}{(z-1)^3} = \frac{1}{z(z-1)} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}$$

$$= -\frac{1}{z} + \frac{1}{z-1} - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3}$$

$$\Rightarrow \int_\varphi \frac{e^z}{z(z-1)^3}\,dz = \int_\varphi -\frac{e^z}{z} + \frac{e^z}{z-1} - \frac{e^z}{(z-1)^2} + \frac{e^z}{(z-1)^3}$$

$$\int_\varphi -\frac{e^z}{z} + \frac{e^z}{z-1} - \frac{e^z}{(z-1)^2} + \frac{e^z}{(z-1)^3}$$

$$= \int_\varphi \underbrace{-\frac{e^z-1}{z}} + e^{-1}\underbrace{\frac{e^{z-1}-1}{z-1}} - e^{-1}\underbrace{\frac{e^{z-1}-1-(z-1)}{(z-1)^2}} + e^{-1}\underbrace{\frac{e^{z-1}-1-(z-1)-\frac{(z-1)^2}{2}}{(z-1)^3}}$$

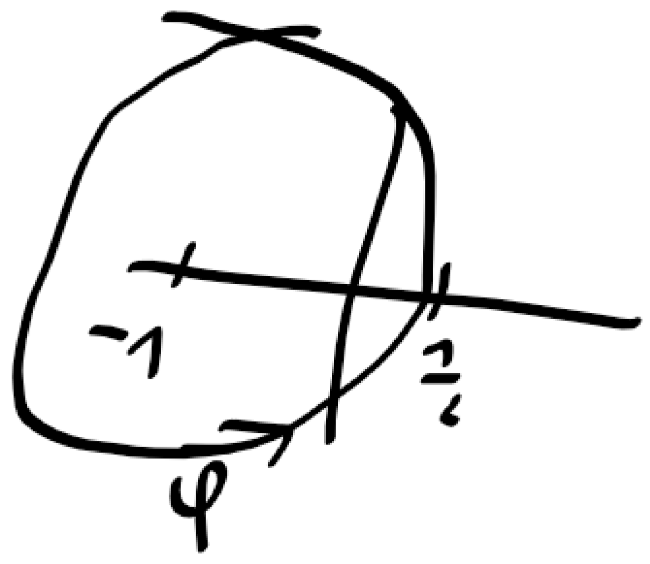
$$+\int_\varphi -\frac{1}{z} + \frac{e^{-1}}{z-1} - e^{-1}\frac{1+z-1}{(z-1)^2} + e^{-1}\frac{1+(z-1)+\frac{(z-1)^2}{2}}{(z-1)^3}$$

Všechny ty $\underbrace{\quad}$ jsou holomorfní a tedy integrál z nich je nulový

$\Rightarrow$
$$= \int_\varphi -\frac{1}{z} + \frac{e^{-1}}{z-1} - \frac{e^{-1}}{(z-1)^2} - \frac{e^{-1}}{(z-1)} + \frac{e^{-1}}{(z-1)^3} + \frac{e^{-1}}{(z-1)^2} + \frac{e^{-1}}{2(z-1)}$$

$$= \int_\varphi -\frac{1}{z} + \frac{e^{-1}}{2(z-1)} + \frac{e^{-1}}{(z-1)^3}$$

a)

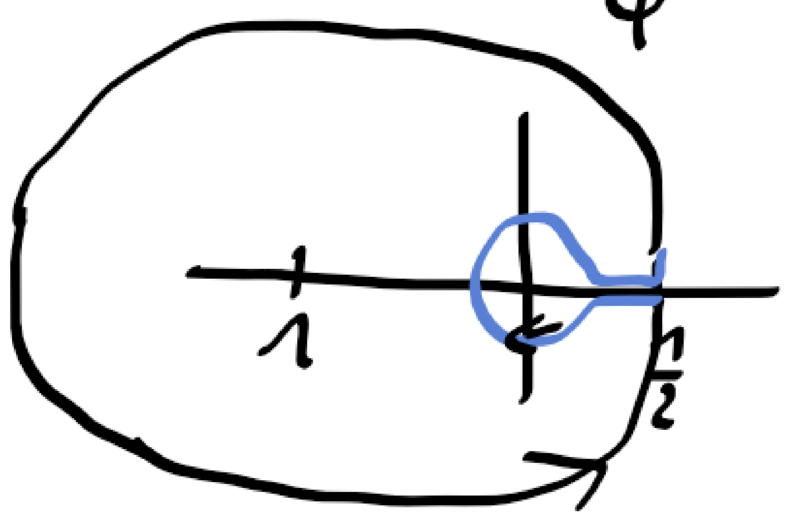


z přednášky víme že

$\int_\varphi \frac{1}{z-1} = 2\pi i$ a $\int_\varphi \frac{1}{(z-1)^3} = 0$

zbývá $\int_\varphi \frac{1}{z} \to$ problém v '0'

φ – nakreslíme



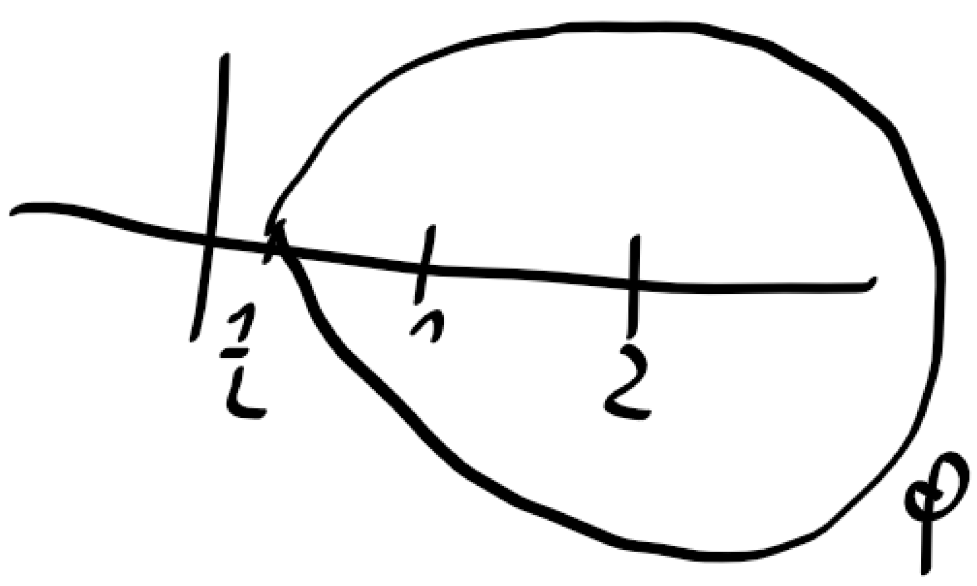
$\frac{1}{z}$ je holomorfní na místech

$\Rightarrow \int_\varphi \frac{1}{z} = \int_\psi \frac{1}{z}$ kde ψ je kružnice se středem 0

$\Rightarrow \int_\varphi \frac{1}{z} = 2\pi i$

$$\to \int_\varphi -\frac{1}{z} + \frac{e^{-1}}{2(z-1)} + \frac{e^{-1}}{(z-1)^3} = -2\pi i + e^{-1}\pi i = \pi i\,(e^{-1}-2)$$

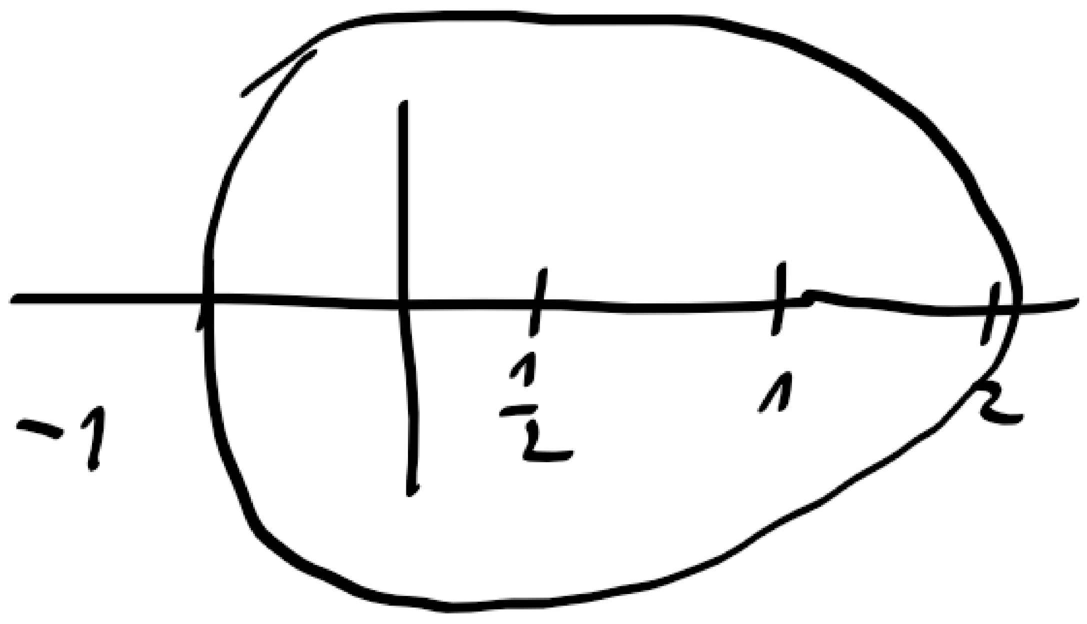
b)



odtud

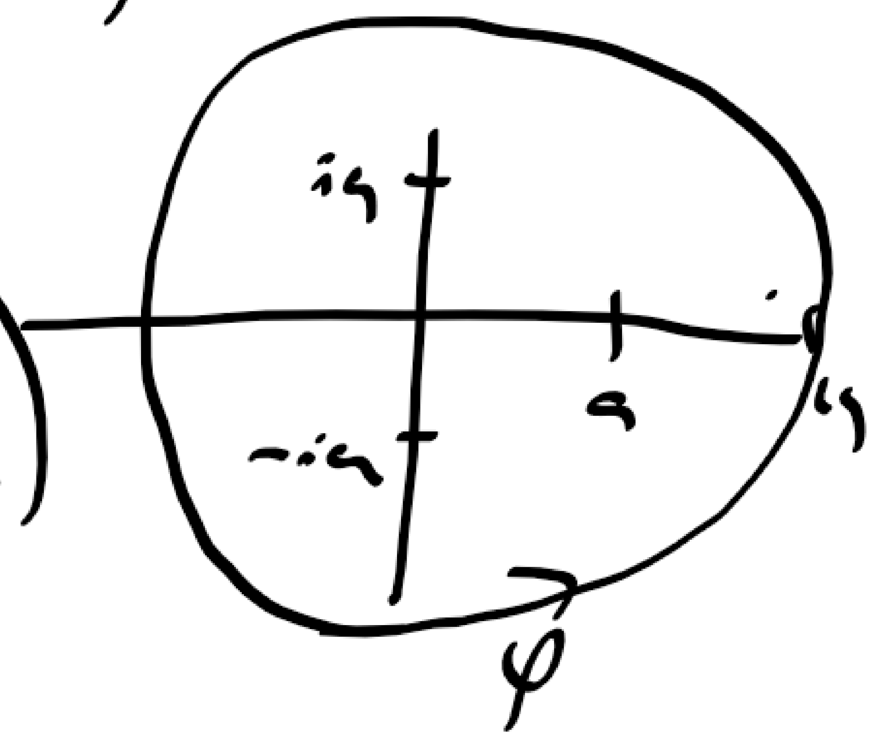
$$\int_\varphi -\frac{1}{z} + \underbrace{\frac{e^{-1}}{2(z-1)}}_{\text{zbylý}} + \frac{e^{-1}}{(z-1)^3} = \pi i\, e^{-1}$$

c)



$$\int_\varphi -\frac{1}{z} + \frac{e^{-1}}{2(z-1)} + \frac{e^{-1}}{(z-1)^3} = -2\pi i + \pi i e^{-1} = \pi i (e^{-1} - 2)$$

(8) $\frac{1}{2\pi i} \int_\varphi \frac{e^z}{z^2+a^2}\, dz \qquad \varphi := 2ae^{it} \qquad t \in [-\pi, \pi) \qquad a > 0$



$$\frac{1}{z^2+a^2} = \frac{1}{(z+ia)(z-ia)} = \frac{1}{2ia}\left(\frac{1}{z+ia} - \frac{1}{z-ia}\right)$$

$$\Rightarrow \frac{1}{2\pi i} \int_\varphi \frac{e^z}{z^2+a^2} = \frac{1}{4\pi a} \int_\varphi \frac{e^z}{z+ia} - \frac{e^z}{z-ia}$$

$$= \frac{1}{4\pi} \int_\varphi \underbrace{e^{-ia} \frac{e^{z+ia} - 1}{z+ia}}_{\text{Holom}} - \underbrace{e^{ia} \frac{e^{z-ia} - 1}{z-ia}}_{\text{Holom}} + \frac{e^{-ia}}{z+ia} - \frac{e^{ia}}{z-ia}$$

$$= \frac{1}{4\pi} \int_\varphi \frac{e^{-ia}}{z+ia} - \frac{e^{ia}}{z-ia} =$$

body $\pm ia$ jsou v int φ

$$\Rightarrow = -\frac{1}{4\pi} 2\pi i (e^{ia} - e^{-ia})$$

$$= \sin a$$

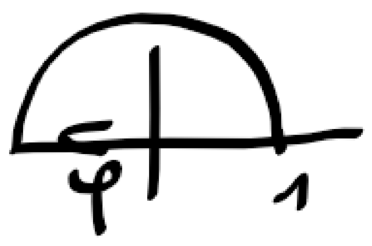
Cauchyova věta a Cauchyho vzorce

Buď f holomorfní v $\operatorname{int} \varphi$, kde φ je Jordanova křivka.

pak $\int_\varphi f(z)\,dz = 0$

$\forall z_0 \in \operatorname{int} \varphi \qquad f^{(k)}(z_0) = \frac{k!}{2\pi i} \int_\varphi \frac{f(z)}{(z-z_0)^{k+1}}$

① $\int_\varphi |z|\bar{z}\,dz$



$|z|\bar{z} = \frac{|z|^2}{z}$

$\int_\varphi |z|\bar{z}\,dz =$

$= \int_{\varphi_1} \frac{1}{z}\,dz + \int_{\varphi_2} x\,dx$

$\varphi = \varphi_1 + \varphi_3^\varepsilon + \dots$

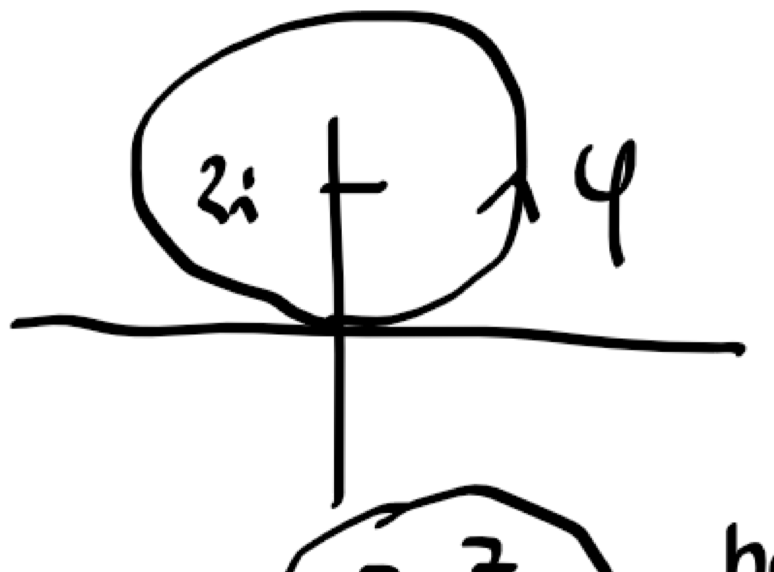
$\int_{\varphi_2} x\,dx = -\int_{-1}^{1} x\,dx = 0$

$\frac{1}{z}$ je holomorfní na $\operatorname{int} \varphi \Rightarrow \int_\varphi \frac{1}{z} = 0 \Rightarrow \int_{\varphi_1} \frac{1}{z}\,dz = \overbrace{\int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \int_{\varepsilon}^{1} \frac{1}{x}\,dx}^{=0} - \int_{\varphi_3} \frac{1\,dz}{z}$

$\varphi_3' = \varepsilon e^{it}$

$= -\int_0^\pi \frac{\varepsilon i e^{it}}{\varepsilon e^{it}}\,dt = -\pi i$

② $\int_\varphi \frac{z e^z}{z^2+4} =$



$= \int_\varphi \frac{z e^z}{(z+2i)(z-2i)} = \int_\varphi \frac{\frac{z e^z}{z+2i}}{z-2i}$ holomorfní na $\operatorname{int} \varphi$

$2i \in \operatorname{int} \varphi$

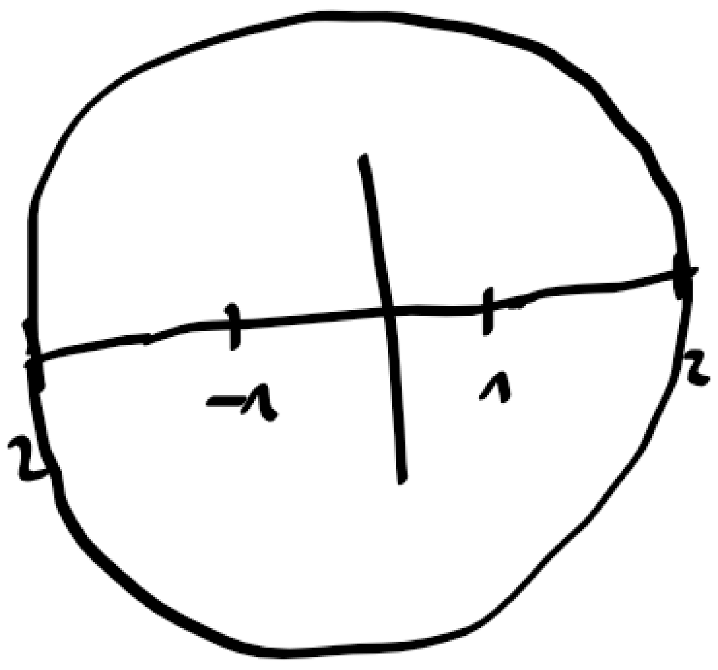
Cauchy $\Rightarrow \; = 2\pi i \left(\frac{2i e^{2i}}{2i+2i} \right) = \pi i e^{2i}$

3

a) $\int_{|z+i|=3} \frac{\sin z}{z+i}$ $\qquad$ $\sin z$ je holomorfní

$$= 2\pi i \sin(-i) = \pi(e^{-i\cdot i} - e^{i\cdot i}) = \pi(e - e^{-1})$$

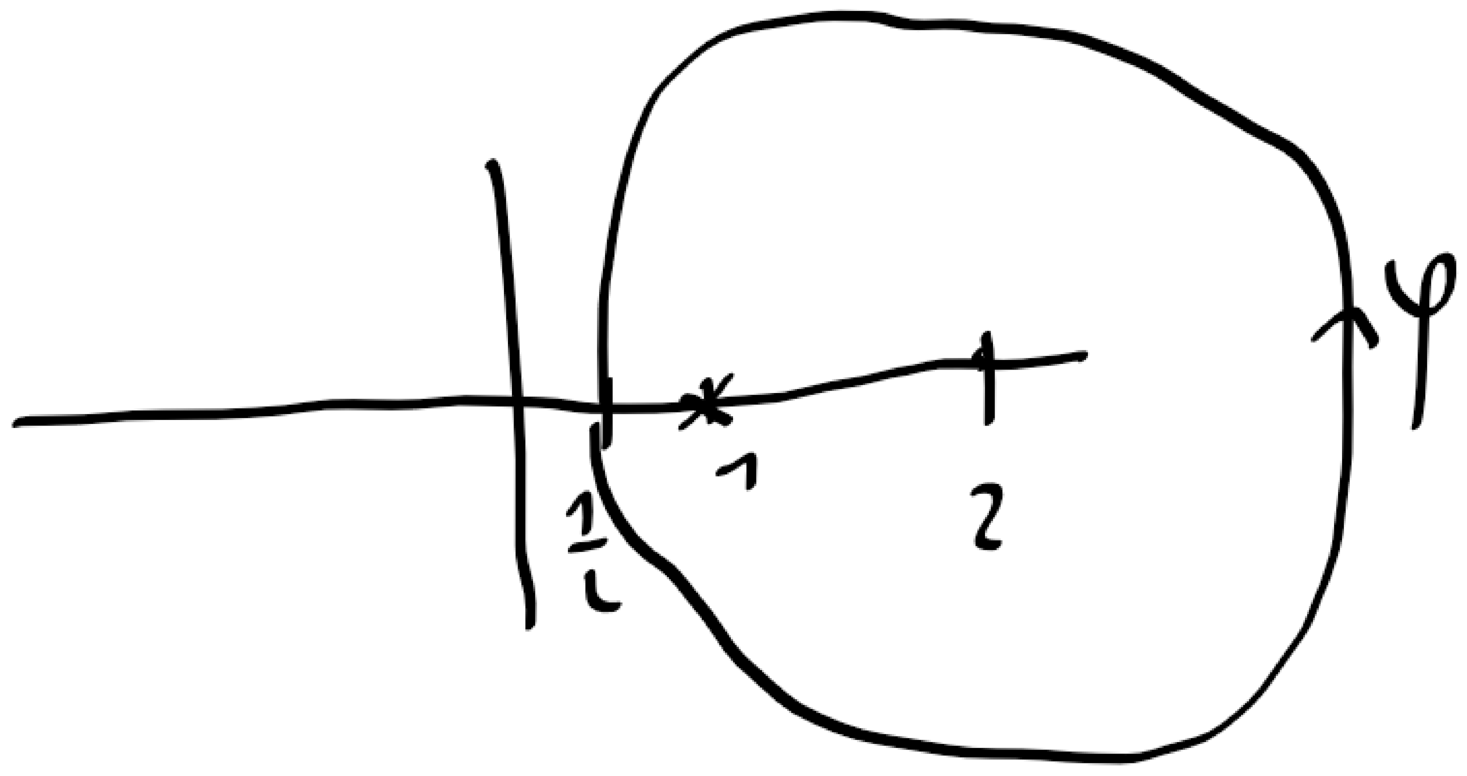
b) $\int_{|z|=2} \frac{e^z}{z^2-1}\, dz = \frac{1}{2}\int_{|z|} \frac{e^z}{z-1} - \frac{e^z}{z+1}$



$$= \frac{1}{2}\, 2\pi i\, 0!\, (e^0 - e^0) = 0$$

4

$\frac{1}{2\pi i}\int_\varphi \frac{e^z}{z(1-z)^3}$



$\frac{e^z}{z}$ je holomorfní na int φ

$$\Rightarrow \frac{1}{2\pi i}\int_\varphi \frac{e^z}{z(1-z)^3} = -\frac{1}{2\pi i}\int_\varphi \underbrace{\frac{e^z}{z}}_{= f(z)} \frac{1}{(z-1)^3} = \frac{f^{(2)}(1)}{2!}$$

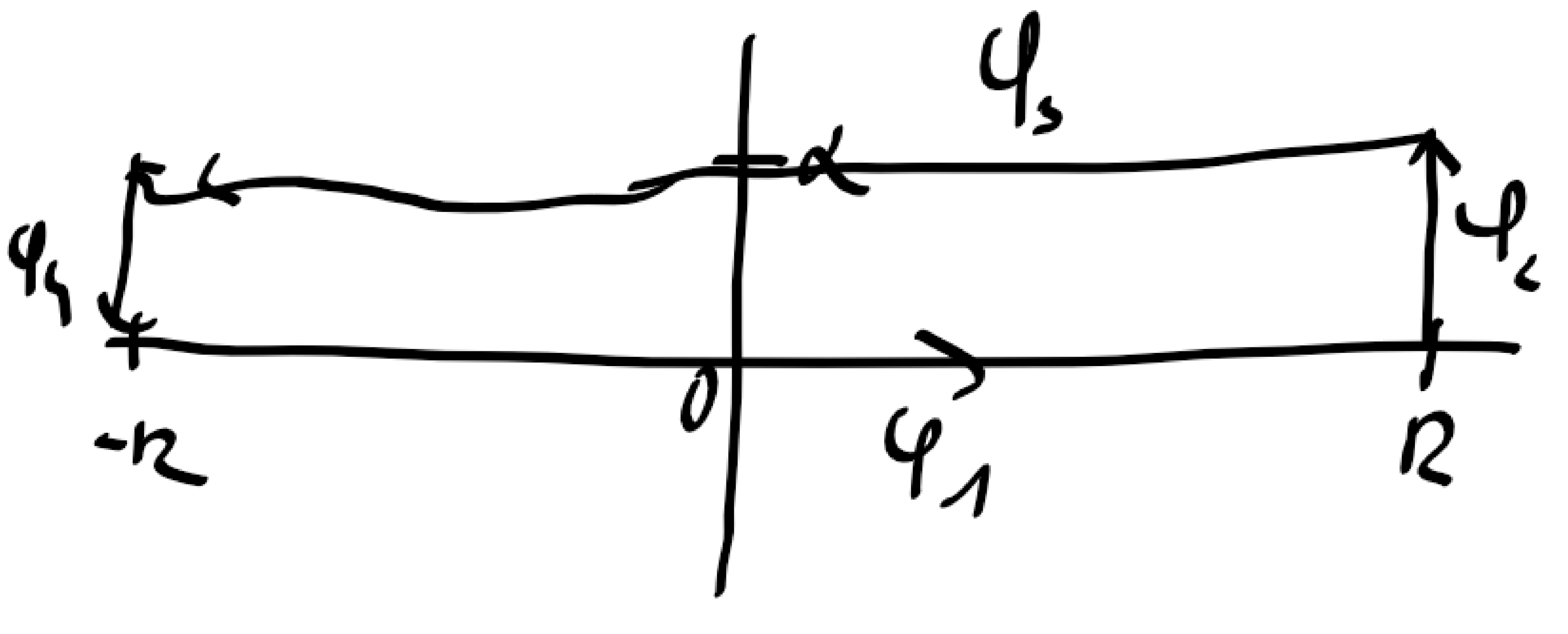
cauchy

$$f^{(2)} = \left(\frac{e^z}{z}\right)'' = \left(\frac{e^z}{z} - \frac{e^z}{z^2}\right)' = e^z\left(\frac{1}{z} - \frac{1}{z^2}\right) + e^z\left(-\frac{1}{z^2} + \frac{2}{z^3}\right)$$

$$\Rightarrow \frac{1}{2\pi i}\int_\varphi \frac{e^z}{z(1-z)^3} = \frac{e\left(\frac{1}{1} - \frac{1}{1^2}\right) + e^1\left(-\frac{1}{1^2} + \frac{2}{1^3}\right)}{2!} = \frac{e}{2}$$

5

Buď $R>0$ a $\varphi = \varphi_1 + \varphi_2 + \varphi_3 + \varphi_4$



Cauchy $\Rightarrow \int_\varphi f(z)\,dz = 0$

a mai me $\displaystyle\int_{ai-R}^{di+R} f(z)dz = -\int_{\varphi_3} f(z)dz = \int_{\varphi_1} f(z)dz + \int_{\varphi_2} f(z)dz + \int_{\varphi_4} f(z)dz$

ndme $\displaystyle\int_{\varphi_1} f(z)dz = \int_{-R}^{R} f(x)dx$ $\qquad \leq \left|\int f(\varphi_2(t))\varphi_2'(t)dt\right| + \left|\int f(\varphi_3(t))\varphi_3'(t)dt\right|$

$\leq \int | \quad " \quad | dt \dots$

a $\displaystyle\left|\int_{\varphi_2} f(z)dz + \int_{\varphi_3} f(z)dz\right| \leq 2 \sup_{y\in(-d,\alpha)} f(R+iy) \quad |d|$

$\displaystyle\Rightarrow \int_{ai-\infty}^{di+\infty} f(z)dz = \lim_{R\to\infty} \int_{ai-R}^{di+R} f(z)dz = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx + \lim_{R\to\infty} \int_{\varphi_2} + \int_{\varphi_3}$

$\displaystyle = \int_{-\infty}^{\infty} f(x)dx$

⑥ $\displaystyle\int_{C_r} f(z)\,dz = \int_{C_r} \underbrace{\frac{(z-a)f(z)-A}{z-a}}_{II} + \underbrace{\frac{A}{z-a}}_{I}\, dz$

$\displaystyle I = \int_{C_r} \frac{A}{z-a} = \int_0^b \frac{riAe^{it}}{re^{it}}\,dt = iAb$

$c_r = a + re^{it} \quad t\in(0,b)$

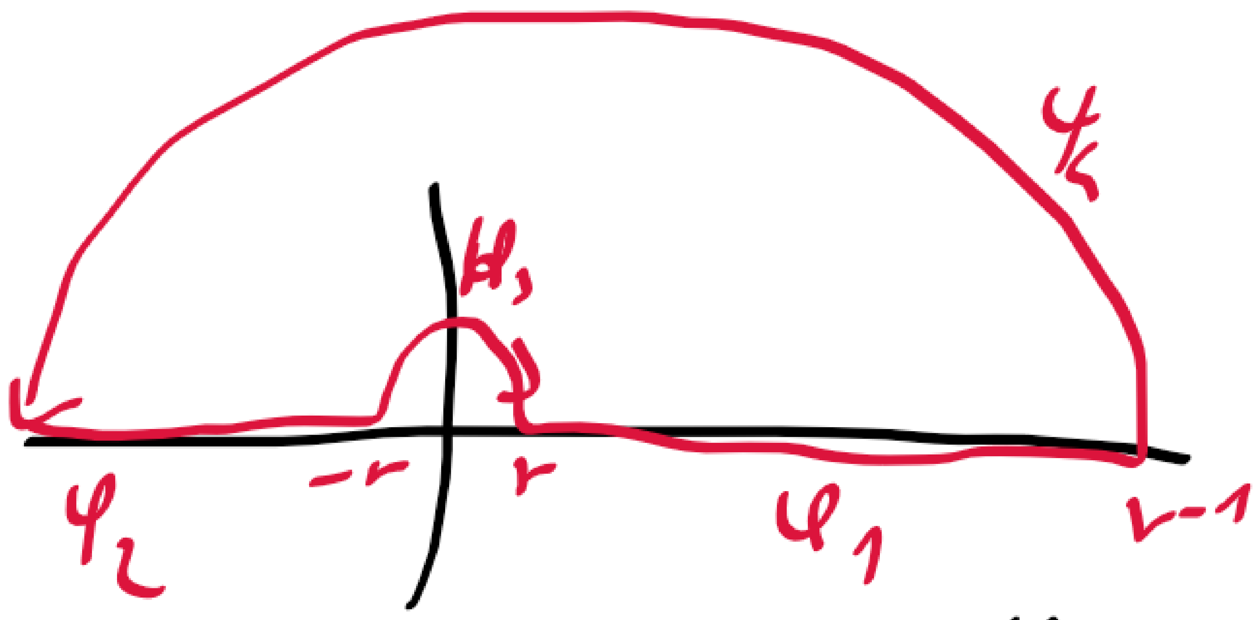
$c_r' = rie^{it}$

$\displaystyle |II| = \left|\int_{C_r} \frac{(z-a)f(z)-A}{z-a}dz\right| = \left|\int_0^b \frac{\left(rf(a+re^{it})-A\right)rie^{it}}{re^{it}}dt\right|$

$\displaystyle = \left|\int_0^b \left(rf(a+re^{it})-A\right)dt\right| \leq b \sup_{z\in C_r} |(z-a)f(z)-A| \xrightarrow{r\to 0} 0$

7

a) $\int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2}\lim_{r\to 0+} \underbrace{\int_{-r^{-1}}^{-r}\frac{\sin x}{x}dx}_{\int_{\varphi_2}} + \underbrace{\int_r^{r^{-1}}\frac{\sin x}{x}dx}_{\int_{\varphi_1}}$



$\int_\varphi \frac{e^{iz}}{z}dz = 0$ $\quad \int_{\varphi_4}\frac{e^{iz}}{z}dz = \int_0^\pi \frac{e^{ir^{-1}e^{it}}}{r^{-1}e^{it}}r^{-1}ie^{it}dt$

$= \int_0^\pi e^{r^{-1}i(\cos t + i\sin t)}dt = \int_0^\pi e^{-\frac{\sin t}{r}}e^{i\cos t\, r^{-1}}$

$\left|\int_{\varphi_4}\frac{e^{iz}}{z}dz\right| \leq \int_0^\pi \left|e^{-\frac{\sin t}{r}}e^{i(\cos t)r^{-1}}\right|dt < 2\int_0^{\frac{\pi}{2}} e^{-\frac{\sin t}{r}}dt$

$\frac{2}{\pi}t \leq \sin t \leq t$
$t\in[0,\frac{\pi}{2})$

$\leq 2\int_0^{\frac{\pi}{2}} e^{-\frac{2t}{\pi r}}dt = 2\frac{\pi r}{2}\left[-e^{-\frac{2t}{\pi r}}\right]_0^{\frac{\pi}{2}}$

$\leq \frac{2r}{\pi} \xrightarrow{r\to 0} 0$

$\Rightarrow \int_0^\infty \frac{\sin x}{x}dx = \frac{1}{2}\operatorname{Im}\Big(\underbrace{\lim \int \frac{e^{iz}}{z}dz}_{-\varphi_3}\Big) \overset{\text{Pr } 6}{=} \frac{1}{2}\operatorname{Im}(1\cdot\pi\cdot i)$

$\lim_{z\to 0}\left(z\cdot\frac{e^{iz}}{z}\right) = 1$

$= \frac{\pi}{2}$

$\int_0^\infty \frac{1-\cos x}{x^2} = \frac{1}{2}\lim_{r\to 0+} \int_{-r^{-1}}^{-r}\frac{1-\cos x}{x^2}dx + \int_r^{r^{-1}}\frac{1-\cos x}{x^2}dx$

$= \frac{1}{2}\operatorname{Re}\left(\lim_{r\to 0+}\int_{-r^{-1}}^{-r}\frac{1-e^{iz}}{z^2}dz + \int_r^{r^{-1}}\frac{1-e^{iz}}{z^2}dz\right)$

OBECNÝ LOGARITMUS

$\ln z := \ln|z| + i\,\mathrm{Arg}\, z$ (hlavní větev logaritmu)

$z \neq 0$

$$\boxed{e^{\ln z} = z}$$

jiná větev $\ln z := \ln|z| + i(\mathrm{Arg}\, z + 2k\pi)$ $k \in \mathbb{Z}$

$$e^{\ln|z| + i(\mathrm{Arg}\, z + 2k\pi)} = e^{\ln|z| + i\,\mathrm{Arg}\, z} \cdot \underbrace{e^{2k\pi i}}_{=1} = z$$

$z_1^{z_2} \hat{=} e^{z_2 \ln z_1}$ kde $\ln z_1$ je hlavní větev logaritmu
větvičkou

např. $\sqrt{4} <$ 2 $(2^2 = 4)$, -2 $((-2)^2 = 4)$

1 a) $\ln(-1) = \ln|-1| + i(\mathrm{Arg}(-1) + 2k\pi)$
$= i\,((2k-1)\pi)$

$k \in \mathbb{Z}$

b) $\ln i = \ln|i| + i(\mathrm{Arg}(i) + 2k\pi)$
$= i\left(\frac{\pi}{2} + 2k\pi\right)$

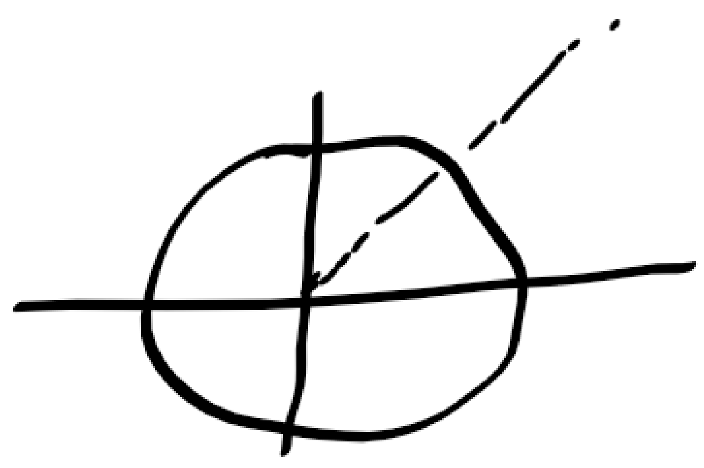
c) $\ln(-2+3i) = \ln|-2+3i| + i(\mathrm{Arg}(-2+3i) + 2k\pi)$
$= \ln\sqrt{13} + \quad + \quad$

$k \in \mathbb{Z}$

2 $1^{\sqrt{2}} = e^{\sqrt{2}\ln 1} = e^{\sqrt{2}(\ln|1| + i(\mathrm{Arg}(1) + 2k\pi))}$
$= e^{\sqrt{2}\, i\, 2k\pi} = (\cos\sqrt{2}\,2k\pi + i\sin\sqrt{2}\,2k\pi)$

(2) b) $2^i = e^{i\ln 2} = e^{i(\ln|2| + i(\mathrm{Arg}\,2 + 2k\pi))}$

$= e^{i\ln 2 - 2k\pi} = e^{-2k\pi}(\cos\ln 2 + i\sin\ln 2)$ $k\in\mathbb{Z}$



c) $(3+4i)^{1+i}$

$= e^{(1+i)\ln(3+4i)} = e^{(i+1)(\ln 5 + i(\mathrm{Arg}(3+4i)+2k\pi))}$

$= e^{\ln 5 - \mathrm{Arg}(3+4i) - 2k\pi} \cdot e^{i(\ln 5 + \mathrm{Arg}(3+4i) + 2k\pi)}$

V příkladech 3)-6) víme že $\mathrm{Im}(f(2)) = 0$

Zajímá nás $\lim_{t\to 2\pi} \mathrm{Arg}\, f(2e^{it})$.

3) $f(z) = (z-1)^{1/3}$ musíme zvolit "správný" logaritmus aby $\mathrm{Im}\, f(2)=0$

$f(z) = e^{\frac13 \ln(z-1)} = e^{\frac13 \ln|z-1| + i\frac13[\mathrm{Arg}(z-1)+2k\pi]}$

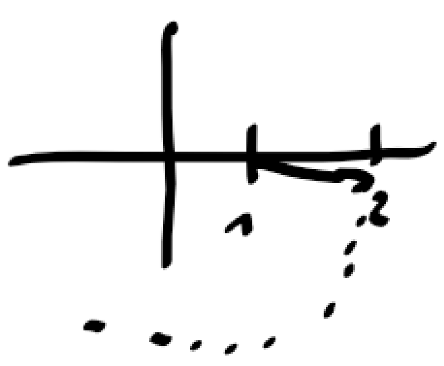
chceme aby $0 = \mathrm{Im}\, f(2) = \mathrm{Im}\left(e^{\frac13\ln|2-1| + i\frac13(\mathrm{Arg}(2-1)+2k\pi)}\right)$

$= e \sin\frac{2k\pi}{3} \Rightarrow k = 3m$ $m\in\mathbb{Z}$ jsou

přípustné hodnoty a navíc $\mathrm{Arg}\, f(2) = 0 \Rightarrow k=0$

$\lim_{t\to2\pi} \mathrm{Im}\, f(2e^{it}) = \lim_{t\to2\pi} \mathrm{Im}\left(e^{\frac13\ln|2e^{it}-1| + i\frac13(\mathrm{Arg}(z-1)+6m\pi)}\right)$

$= \sin\frac{2\pi}{3}$



a $\lim_{t\to2\pi}\mathrm{Arg}\, f(2e^{it}) = \mathrm{Arg}\left(e^{i\frac{2\pi}{3}}\right) = \frac{2\pi}{3}$

④ $f(z) = \sqrt{\frac{z-1}{z+1}} = e^{\frac{1}{2}\ln\left(\frac{z-1}{z+1}\right)} =$

$$e^{\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + i\underbrace{\frac{1}{2}\left(\operatorname{Arg}\frac{z-1}{z+1} + 2k\pi\right)}}$$

$$\frac{2e^{it}-1}{2e^{it}+1} = \frac{(2e^{it}-1)(2e^{-it}+1)}{(2e^{it}+1)(2e^{-it}+1)}$$

a chceme $\operatorname{Im} f(z) = 0$; $\operatorname{Arg} f(z) = 0$

pokud $\operatorname{Arg} f(z) = 0 \Rightarrow k = 0$

$$= \frac{3 + 4i\sin t}{5 + 4\cos t}$$

Pokud pouze $\operatorname{Im} f(z) = 0 \Rightarrow k \in \mathbb{Z}$

$$\lim_{t \to 2\pi} \operatorname{Im} f(2e^{it}) = e^{\frac{1}{2}\ln\frac{1}{3}} \lim_{t \to 2\pi} \sin\left(\frac{1}{2}\left[\operatorname{Arg}\left(\frac{3+4i\sin t}{5+4\cos t}\right) + 2k\pi\right)\right)$$

$= 0$

dodaži Arg ...

5) zadefinuji $\ln z := \ln|z| + i\operatorname{Arg} z$ $\qquad \operatorname{Arg} z \in [0, 2\pi)$

Pak Pak $\lim_{z} \quad \operatorname{Arg} 2\ln(z) \quad = 2\pi$

! kdybychom použili $\ln z = \ln|z| + i\operatorname{Arg} z \qquad z \in (-\pi, \pi)$

byla by se $f(z)$ nespojitá v bodě -2, což nechceme

⑥ $f(z) = \ln z + \ln(z+1) = \ln(|z||z+1|) + i\underbrace{\operatorname{Arg}(z(z+1))}_{[0, 2\pi)} + i2k\pi$

$\Rightarrow k = 0$

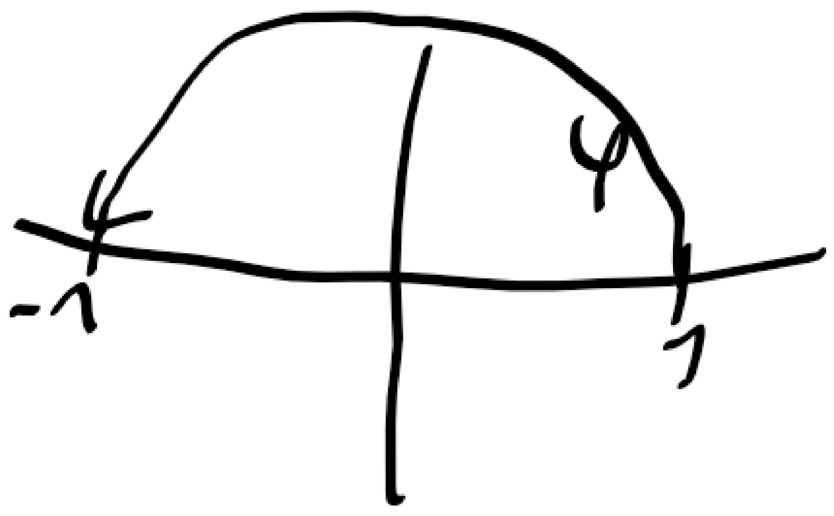
$2e^{it}(2e^{it}+1) = 4e^{2it} + 2e^{it} = 4\cos 2t + 2\sin 2t + i(4\sin 2t + 2\sin t)$

$2\sin t(4\cos t + 2)$

$\operatorname{Arg}(\ +,-\ ) \to 2\pi$

Arg nespojitý – skok v 2π

(7) $\int_\varphi \frac{dz}{\sqrt{z}}$



a $\sqrt{1} = 1$

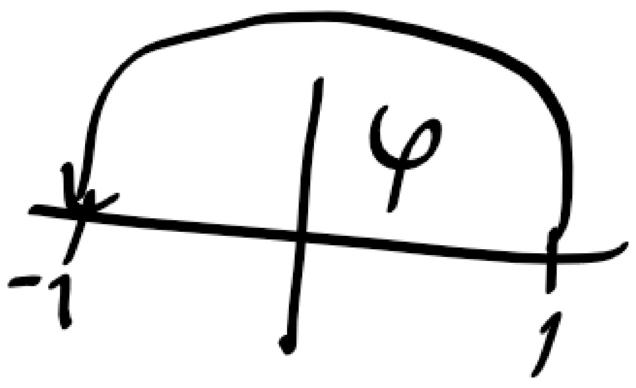
$\Rightarrow \sqrt{z} = e^{\frac{1}{2}\ln|z| + i\frac{1}{2}\operatorname{Arg} z}$ $\qquad \operatorname{Arg} z \in (-\pi, \pi]$

$\varphi = e^{it}, \ \varphi' = ie^{it}$

$$\int_\varphi \frac{dz}{\sqrt{z}} = \int_0^\pi \frac{ie^{it}\,dt}{e^{\frac{1}{2}\ln|e^{it}| + i\frac{1}{2}(\operatorname{Arg} e^{it})}} = \int_0^\pi \frac{ie^{it}\,dt}{e^{\frac{it}{2}}} = \int_0^\pi ie^{\frac{it}{2}}\,dt$$

$$= \left[2e^{\frac{it}{2}}\right]_0^\pi = 2e^{\frac{i\pi}{2}} - 2 = 2i - 2$$

(8) $\int_\varphi \frac{dz}{\sqrt{z}}$



ALE $\sqrt{1} = -1$!

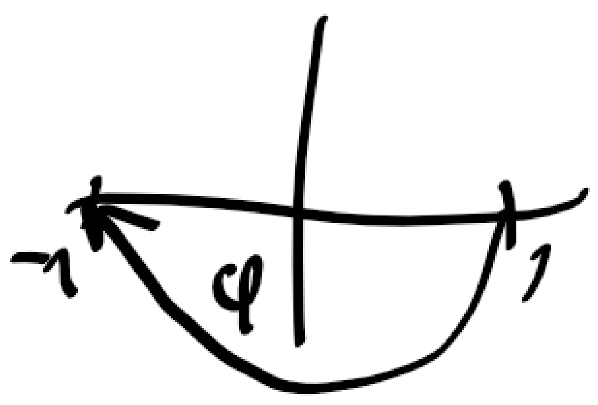
$\sqrt{z} = e^{\frac{1}{2}\ln|z| + i\frac{1}{2}(\operatorname{Arg} z + 2\pi)}$! $\qquad \operatorname{Arg} z \in (-\pi, \pi]$

Výběr správné větve

$$\int_\varphi \frac{dz}{\sqrt{z}} = \int_0^\pi \frac{ie^{it}\,dt}{e^{\frac{1}{2}\ln|e^{it}| + i\frac{1}{2}(\operatorname{Arg} e^{it} + 2\pi)}} = \int_0^\pi \frac{ie^{it}\,dt}{e^{i(\frac{t}{2}+\pi)}}$$

$$= e^{-i\pi} i \int_0^\pi e^{\frac{it}{2}}\,dt = 2e^{-i\pi}\left[e^{\frac{it}{2}}\right]_0^\pi = \underline{\underline{2-2i}}$$ (opačné znaménko než v 7)

(9) $\int_\varphi \frac{dz}{\sqrt{z}}$

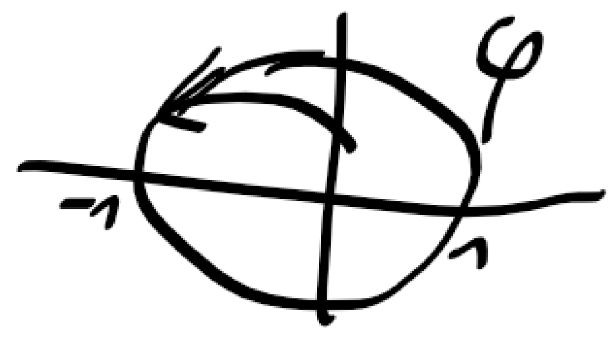


$\sqrt{1} = 1$

$$= -\int_{-\pi}^0 \frac{ie^{it}\,dz}{e^{\frac{1}{2}\ln|e^{it}| + \frac{i}{2}\operatorname{Arg}(e^{it})}} = -\int_{-\pi}^0 ie^{\frac{it}{2}} = -2\left[e^{\frac{it}{2}}\right]_{-\pi}^0 = -2\left(1 - e^{\frac{-i\pi}{2}}\right)$$

$$= -2(1+i)$$

(10) $\oint_{\varphi} \ln z \, dz$

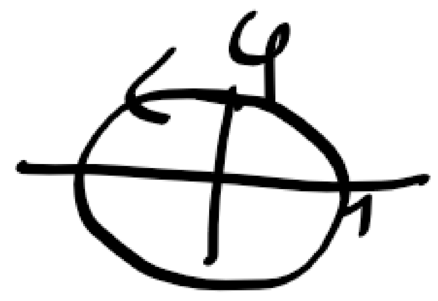


$\ln 1 = 0 \Rightarrow \ln z = \ln|z| + i \operatorname{Arg} z$ $\qquad$ $\operatorname{Arg} z \in (-\pi, \pi)$

$$= \int_{-\pi}^{\pi} \left[\ln|e^{it}| + i \operatorname{Arg}(e^{it})\right] i e^{it} dt = i \int_{-\pi}^{\pi} t e^{it} dt = \left[t e^{it} + i e^{it}\right]_{-\pi}^{\pi}$$

$$= \pi e^{i\pi} + \pi e^{-i\pi} + i\left(e^{i\pi} - e^{-i\pi}\right) = -2\pi$$

(11) $\oint_{\varphi} \ln z \, dz$



ale $\ln i = \frac{\pi i}{2}$ $\Rightarrow$ $\ln z := \ln|z| + i \operatorname{Arg} z$ $\qquad$ $\operatorname{Arg} z \in (-\pi, \pi]$

stejně jak (10)

(12) $\oint_{|z|=3} \ln z \, dz$ $\qquad$ a $\ln 1 = 2\pi i \Rightarrow \ln z := \ln|z| + i(\operatorname{Arg} z + 2\pi)$

$\operatorname{Arg} z \in (-\pi, \pi)$

$$\Rightarrow \; = \int_{\pi}^{3\pi} \left(\ln|z| + i \operatorname{Arg}(e^{it})\right) i e^{it} dt$$

nebo $\ln z := \ln|z| + i \operatorname{Arg} z$

a $\operatorname{Arg} z \in (\pi, 3\pi)$

$$= \int_{\pi}^{3\pi} (\ln 3 + it) i e^{it} = \left[\ln 3 \, e^{it} + t e^{it} + i e^{it}\right]_{\pi}^{3\pi}$$

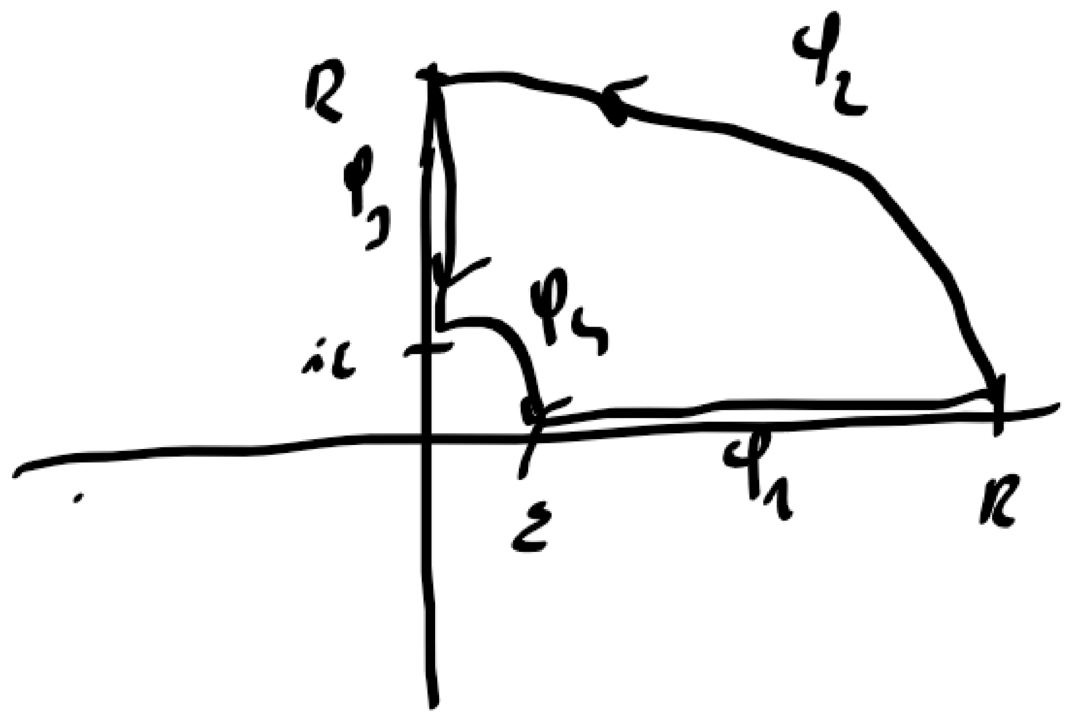
$$= \ln 3 \left(e^{3i\pi} - e^{i\pi}\right) + 3\pi e^{i3\pi} - \pi e^{i\pi} + i\left(e^{i3\pi} - i e^{i\pi}\right)$$

$$= -2\pi$$

(13) $\int_0^\infty x^{s-1} \sin x \, dx$ $\qquad\qquad$ $\int_0^\infty x^{s-1} \cos x \, dx$

$= \operatorname{Im} \int_0^\infty z^{s-1} e^{iz}$ $\qquad\qquad$ $= \operatorname{Re} \int_0^\infty z^{s-1} e^{iz} dz$

chceme teď $\int_0^\infty z^{s-1} e^{iz} dz$ v nějakém smyslu



$\varphi_1 = t \qquad t \in (\varepsilon, R]$

$\varphi_2 = Re^{it} \qquad t \in (0, \frac{\pi}{2})$

$-\varphi_3 = it \qquad t \in (\varepsilon, R)$

$-\varphi_4 = \varepsilon e^{it} \qquad t \in (0, \frac{\pi}{2})$

$z^{s-1} e^{iz} = e^{iz + (s-1)\ln|z| + (s-1) i \operatorname{Arg} z}$ $\qquad\qquad$ $\operatorname{Arg} z \in (-\pi, \pi)$

holomorfní fce na $\langle \operatorname{int} \varphi \rangle \Rightarrow$

$\int_\varphi z^{s-1} e^{iz} = 0$

$\int_{\varphi_1} z^{s-1} e^{iz} dz = \int_\varepsilon^R x^{s-1} e^{ix} dx$

$\int_{\varphi_2} z^{s-1} e^{iz} dz = \int_0^{\frac{\pi}{2}} iRe^{it} \, e^{iRe^{it} + (s-1)\ln(|Re^{it}|) + (s-1) i \operatorname{Arg}(Re^{it})} dt$

$= \int_0^{\frac{\pi}{2}} iR^s \, e^{ist + iR\cos t} \, e^{-R\sin t} \, dt$

$\sin t \geq \alpha t$ pro nějaké $\alpha > 0$

$\left| \int_{\varphi_2} \right| \leq \int_0^{\pi} R^s e^{-R\sin t} dt \leq \int_0^{\pi} R^s e^{-R\alpha t} dt = \left[-\frac{R^{s-1}}{\alpha} e^{-R\alpha t} \right]_0^{\pi}$

$\leq \frac{R^{s-1}}{\alpha} \xrightarrow{R \to \infty} 0 \quad (s < 1)$

$\int_{\varphi_3} z^{s-1} e^{iz} dz = -\int_\varepsilon^R i \, e^{i(it)} \, e^{(s-1)\ln|it| + i \overset{(s-1)}{\operatorname{Arg}(it)}} dt$

$= -ie^{i\frac{(s-1)\pi}{2}} \int_\varepsilon^R e^{-t} t^{s-1} dt \xrightarrow[\varepsilon \to 0]{R \to \infty} \left(\sin \frac{(s-1)\pi}{2} - i\cos\frac{(s-1)\pi}{2} \right) \underset{\text{Gamma fce}}{\Gamma(s)}$

$$\int_{\varphi_4} \doteq -\varepsilon i \int_0^{\frac{\pi}{2}} e^{it}\, e^{i(\varepsilon e^{it}) + (s-1)\ln(|\varepsilon e^{it}|) + i(s-1)\operatorname{Arg}(\varepsilon e^{it})}\, dt$$

$$= -\varepsilon^s i \int_0^{\frac{\pi}{2}} e^{its}\, e^{i\varepsilon\cos t}\, e^{-\varepsilon\sin t}\, dt$$

$$\left|\int_{\varphi_4}\right| \leq \varepsilon^s \frac{\pi}{2} \xrightarrow{\varepsilon\to 0_+} 0$$

$\Rightarrow$

$$\left(\sin\frac{(s-1)\pi}{2} - i\cos\frac{(s-1)\pi}{2}\right)\Gamma(s)$$

$$\int_0^\infty x^{s-1}e^{ix} = \lim_{\substack{R\to\infty\\ \varepsilon\to 0}} \int_\varepsilon^R x^{s-1}e^{ix}\,dx = \left(i\cos\frac{(s-1)\pi}{2} - \sin\frac{(s-1)\pi}{2}\right)\Gamma(s)$$

$$\Rightarrow \int_0^\infty x^{s-1}\cos x\,dx = -\sin\left(\frac{(s-1)\pi}{2}\right)\Gamma(s)$$

$$\int_0^\infty x^{s-1}\sin x\,dx = \cos\left(\frac{(s-1)\pi}{2}\right)\Gamma(s)$$

14 $\ln z$ rozvoj okolo bodu $z_0 = 1$ $\qquad \ln z = \ln|z| + i\operatorname{Arg}(z) \qquad \operatorname{Arg} z \in (-\pi,\pi)$

máme $(\ln z)' = \frac{1}{z}$

$$\frac{1}{z} = \frac{1}{(z-1)+1} = \sum_{n=0}^\infty (1-z)^n \qquad |z-1| < 1 \ !$$

$$\Rightarrow \qquad \doteq \left(\sum_{n=1}^\infty -\frac{(1-z)^n}{n}\right)'$$

$$\Rightarrow (\ln z)' = \left(\sum_{n=0}^\infty \frac{(-1)^{n+1}(z-1)^n}{n}\right)'$$

z jednoznačnosti primitivní fce $\Rightarrow \ln z = C + \sum_{n=1}^\infty \frac{(-1)^{n+1}(z-1)^n}{n}$

a $\ln 1 = 0 \Rightarrow \qquad \ln z = \sum_{n=1}^\infty \frac{(-1)^{n+1}(z-1)}{n}$

b) $(\ln(1-z))^2$ $\sim$ $z_0=0$.

I $\left(\ln(1-z)\right)^{2\prime} = -\frac{2\ln(1-z)}{1-z}$

$$= 2\sum_{n=1}^{\infty}\frac{z^n}{n}\sum_{m=0}^{\infty} z^m$$

$n+m=k$

$$= 2\sum_{k=1}^{\infty} z^k \sum_{n=1}^{k}\frac{1}{n}$$

$$\Rightarrow \left(\ln(1-z)\right)^2 = \int 2\sum_{k=1}^{\infty} z^k \sum_{n=1}^{k}\frac{1}{n}$$

$$= 2\sum_{k=1}^{\infty}\frac{z^{k+1}}{k+1}\sum_{n=1}^{k}\frac{1}{n}$$

II

$$\ln(1+z) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1} z^n}{n}$$

$$\left(\ln(1+z)\right)^2 = \sum_{n=1, m=1}^{\infty}\frac{(-1)^{n+m} z^{n+m}}{n\,m}$$

$n+m=k$

$$\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}(-z)^k\frac{1}{m(k-m)}$$

$$\frac{1}{m(k-m)} = \frac{1}{k}\left(\frac{1}{m}+\frac{1}{k-m}\right)$$

$$\Rightarrow \sum_{m=1}^{k-1}\frac{1}{m(k-m)} = \frac{2}{k}\sum_{m=1}^{k-1}$$

$$\Rightarrow \left(\ln(1-z)\right)^2 = \sum_{k=2}^{\infty}\frac{2z^k}{k}\left(\sum_{m=1}^{k-1}\frac{1}{m}\right)$$

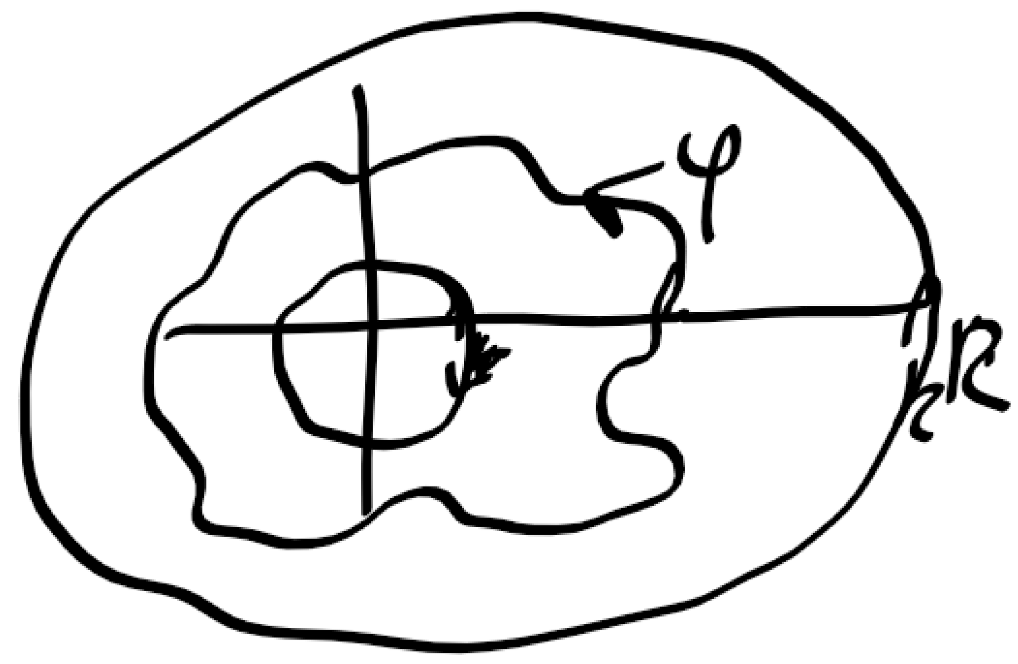
LAURENTOVY ŘADY

f - holomorfní na $r < |z - z_0| < R$ (mezikruží)

konvergence a platí

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \int_{\varphi} \frac{f(z)}{(z - z_0)^{n+1}} dz$$



$$R = \lim_{n \to \infty} \frac{1}{(a_n)^{1/n}}$$

$$r = \lim_{n \to \infty} (a_{-n})^{1/n}$$

okolí "∞" znamená $R = \infty$, $r > 0$

① $f(z) = \frac{1}{z-2}$ $z_0 = 0$, $z_0 = \infty$

$$\frac{1}{z-2} = -\frac{1}{2} \left(\frac{1}{1 - \frac{z}{2}} \right) = -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{n} -\frac{1}{2^{n+1}} z^n$$

$r = 0$, $R = 2$

o) $z_0 = \infty$
$$\frac{1}{z-2} = \frac{1}{z} \left(\frac{1}{1 - \frac{2}{z}} \right) = \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$= \sum_{n=-1}^{\infty} \frac{2^{n+1}}{z^n}$$

$r = 2$, $R = \infty$

② $f(z) = \dfrac{z^2-2z+5}{(z-2)(z^2+1)}$ $\quad z_0 = 2 \, ; \quad 1 < |z| < 2$

a) $\dfrac{z^2-2z+5}{(z^2+1)(z-2)} = \dfrac{z^2+1 \;+2(2-z)}{(z^2+1)(z-2)} = \dfrac{1}{z-2} - \dfrac{2}{z^2+1}$

$$= \frac{1}{z-2} - \frac{i}{z+i} + \frac{i}{z-i} =$$

$$= \frac{1}{z-2} - \frac{i}{(z-2)+2+i} + \frac{i}{(z-2)+2-i}$$

$$= \frac{1}{z-2} - \frac{i}{2+i} \frac{1}{\frac{(z-2)}{2+i}+1} + \frac{i}{(2-i)} \frac{1}{\frac{z-2}{2-i}+1}$$

$$= \frac{1}{z-2} - \frac{i}{2+i} \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{(2+i)^n} + \frac{i}{(2-i)} \sum_{n=0}^{\infty} \frac{(-1)^n (z-2)^n}{(2-i)^n}$$

$r = 0 \qquad R = |2+i| = \sqrt{5}$

b) $1 < |z| < 2$

$$\frac{z^2-2z+5}{(z^2+1)(z-2)} = \frac{1}{z-2} - \frac{i}{z+i} + \frac{i}{z-i} =$$

$$= -\frac{1}{2} \left(\frac{1}{1-\frac{z}{2}} \right) - \frac{i}{z} \left(\frac{1}{1+\frac{i}{z}} - \frac{1}{1-\frac{i}{z}} \right)$$

$$= -\frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n - \frac{i}{z} \left(\sum_{n=0}^{\infty} \left(-\frac{i}{z}\right)^n - \left(\frac{i}{z}\right)^n \right)$$

$$= -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n + \sum_{n=0}^{\infty} \left(-\frac{i}{z}\right)^{n+1} + \left(\frac{i}{z}\right)^{n+1} \qquad r=1 \quad R=2$$

(3) $f(z) = \frac{1}{(z^2+1)^2}$ $\qquad z_0 = i, \quad z_0 = \infty$

$$= \frac{1}{(z-i)^2(z+i)^2} = \frac{1}{(z-i)^2} \cdot \frac{1}{(z-i+2i)^2}$$

$$= \frac{1}{(z-i)^2 \cdot (2i)^2 \left(\frac{z-i}{2i}+1\right)^2}$$

$$= -\frac{1}{(z-i)^2 4} \left(\sum_{n=0}^{\infty} \left(\frac{z-i}{2i}\right)^n \right)^2$$

$$= -\left(\sum_{n=0}^{\infty} i^{-n} \frac{(z-i)^{n-2}}{2^{n+1}} \right)^2$$

$$= -\sum_{n,m=0}^{\infty} \frac{i^{-n-m} (z-i)^{n+m-4}}{2^{n+m+2}} \quad \overset{n+m=k}{=} \quad -\sum_{k=0}^{\infty} \sum_{m=0}^{k} \frac{i^{-k}}{2^{k+2}} (z-i)^{k-4}$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)\, i^{k} (k+1)}{2^{k+1}} (z-i)^{k-4}$$

$r=0 \qquad R=2$

(4) $f(z) = z^2 \sin \frac{1}{z-1}$ $\qquad z_0 = 1$

$$= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z-1} \right)^{2n+1} = \left[(z-1)^2 + 2(z-1)+1\right] \sum_{n=0} \frac{(-1)^n}{(2n+1)!} \frac{1}{(z-1)^{2n+1}}$$

$$= \sum_{n=0} \frac{(-1)^n}{(2n+1)!} (z-1)^{-2n+1} + \frac{(-1)^n 2}{(2n+1)!} (z-1)^{-2n} + \frac{(-1)^n}{(2n+1)!} (z-1)^{-2n-1}$$

$r=0 \qquad R=\infty$

(5) $f(z) = e^{z+\frac{1}{z}} = e^z \cdot e^{\frac{1}{z}}$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{m=0}^{\infty} \frac{1}{m!\, z^m} = \sum_{m,n=0}^{\infty} \frac{z^{n-m}}{n!\, m!} =$$

$n-m=k$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $n=k+m$

$$= \sum_{k=-\infty}^{\infty} \sum_{n=\max(0,k)}^{\infty} \frac{z^k}{n!\,(n-k)!}$$

$$= \sum_{k=-\infty}^{\infty} z^k \left(\sum_{n=\max(0,k)}^{\infty} \frac{1}{n!\,(n-k)!} \right)$$

(6) podobně jako (5)

(7) $$\ln \frac{z-a}{z-b} = \ln\left(1 + \left(\frac{z-a}{z-b} - 1\right)\right)$$

$$= \ln\left(1 + \frac{b-a}{z-b}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(b-a)^n}{(z-b)^n}$$

$$|z-b| > |b-a|$$

(8) $$\cos\frac{1}{z} = \frac{e^{\frac{i}{z}} + e^{-\frac{i}{z}}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \left(\left(\frac{i}{z}\right)^n + \left(\frac{-i}{z}\right)^n \right) \frac{1}{n!}$$

(9) $\frac{1}{\cos z}$ $\qquad$ $z_0 = 0$

$$= \frac{1}{\sum_{n=0} (-1)^n \frac{z^{2n}}{(2n)!}} = \frac{1}{1 - \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n}}{(2n)!}}$$

$$= \sum_{m=0}^{\infty} \left(\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n}}{(2n)!} \right)^m = \ldots\ldots$$

(10) podobně

(11) $\sqrt{z}$ $\quad z_0=0$

$$\sqrt{z} = e^{\frac{1}{2}\ln|z| + \frac{i}{2}(\mathrm{Arg}\, z + 2k\pi)} \qquad z = re^{i\theta}$$

dvě větve
$$\begin{cases} e^{\frac{1}{2}\ln|z| + \frac{i}{2}\mathrm{Arg}\, z} \\ e^{\frac{1}{2}\ln|z| + i\left(\frac{\mathrm{Arg}\, z}{2} + \pi\right)} \end{cases}$$

ať zvolím interval $[a, a+2\pi) \ni \mathrm{Arg}\, z$, tak funkce bude mít skok na přechodu $\mathrm{Arg}\, z \to a_+$!! $\Rightarrow \sqrt{z}$ není holomorfní na žádném prstenci $r<|z|<R \Rightarrow$ není Laur. řada v 0

(12) $f(z) = \sqrt{1+\sqrt{z}}$ $\quad z_0 = 1$ $\quad$ holomorfní na okolí $z_0=1$

$\mathrm{Arg}\, z \in (-\pi, \pi)$

možné možnosti $\sqrt{z} := \begin{cases} \sqrt{|z|}\, e^{\frac{i}{2}\mathrm{Arg}\, z} \\ \sqrt{z}\, e^{\frac{i}{2}\mathrm{Arg}\, z} \cdot e^{i\pi} \end{cases}$

ln je hlavní větev

$$= \begin{cases} e^{\frac{\ln z}{2}} \\ e^{\frac{\ln z}{2} + \pi i} \end{cases}$$

$k, l = 0, 1$

$$\sqrt{1+\sqrt{z}} = e^{\frac{1}{2}\ln\left(1 + e^{\left(\frac{\ln z}{2} + \pi k i\right)}\right) + l i\pi}$$

čtyři možné větve, pro $k=1$ ale singularita v "0" – nelze rozvinout.
$\Rightarrow k=0$

$l = 0, 1$