**WINTER (babies)**
- $W^{k,p}(\Omega)$
- linear elliptic equations
  - Lax-Milgram
- linear parabolic + hyperbolic eq.
  - Galerkin
- Fredholm

\[ -\Delta u = f \text{ in } \Omega \]
\[ u = 0 \text{ on } \partial \Omega \]
\[ \min_{u \in W_0^{1,2}} \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} f u \]

\[ \Longleftrightarrow \]

**SUMMER (teenagers)**
- prove it
- nonlinearity:
  - 12-14: nonlinearity in lower order term
    - $\Delta u + \sin u = f$
  - 14-18: nonlinearity in leading term
    - $\text{div} (|\nabla u|^2 \nabla u) = f$
    - Minty, monotone operator

\[ \min_{u \in W_0^{1,p}} \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} f u \]

- Regularity theory (introduction to)
0. Lebesgue spaces, fixed point theorems, functional analysis

Luzin theorem: Let \( \Omega \) be measurable and \( f \in L^1_{\text{loc}}(\Omega) \). Then
\[
\forall \varepsilon > 0 \, \exists \, U \subseteq \Omega, \, 1 \|U\| < \varepsilon \text{ and } f \in C(\Omega-U).
\]

Egorov (Jgorov) theorem: Let \( \Omega \) be measurable and \( f_n \in L^1_{\text{loc}}(\Omega) \), \( f_n \to f \) in \( L^1_{\text{loc}}(\Omega) \) \( (\iff \forall \text{ compact } K \subseteq \Omega : \int_K |f_n-f| \to 0) \).
Then \( \forall \varepsilon > 0 \, \exists \, U \subseteq \Omega, \, 1 \|U\| < \varepsilon \text{ and } f_n \to f \) in \( C(\Omega-U) \).

Lebesgue dominated convergence theorem

Vitali convergence theorem: Let \( \Omega \) be measurable \( f_n \) be a sequence of measurable functions, \( f_n(x) \to f(x) \) for a.a. \( x \in \Omega \).
Then \( \lim_{n \to \infty} \int \! f_n = \int \! f \), provided that the sequence \( f_n \) is uniformly equiintegrable \( (\iff \forall \varepsilon > 0 \, \exists \, \delta > 0 \, \forall \, U \subseteq \Omega \, 1 \|U\| < \delta \, \forall \, n \int |f_n| < \varepsilon ) \).

Fatou lemma: \( \inf \lim_{n \to \infty} \int \! f_n \) and \( f \geq 0 \). Then \( \int \! f \leq \liminf_{n \to \infty} \int \! f_n \).

Literature: Lukeš, Maliý: Measure and integral
Kufner, John, Fučík: Function spaces

Regularization

Def: Regularization kernel: \( \eta \in C^\infty(\mathbb{R}^d) \) nonnegative, radially symmetric \( \int_{B(0)} \! \eta(x) dx = 1 \).

Reg. of \( f \): Let \( f \in L^p(\Omega) \) with \( p \in [1,\infty) \). We extend \( f \) by zero outside \( \Omega \) and define \( f_\varepsilon := \varepsilon^{-d} \eta(\varepsilon x) f(x) \), where \( \eta(x) = \frac{1}{d^d} \eta \left( \frac{x}{\varepsilon} \right) \).

\( \forall \, p \in [1,\infty) \): \( \text{if } f \in L^p(\Omega) \text{ then } f_\varepsilon \to f \text{ in } L^p(\Omega) \)
\( p = \infty \) : \( f_\varepsilon \to f \text{ a.e. and } f_\varepsilon \rightharpoonup f \text{ in } L^\infty(\Omega) \) \( (\iff \forall \, g \in L^1(\Omega) : \int \! f_\varepsilon g \to \int \! fg) \).

Reflexivity, separability, weak and weak* convergences

Theorem: \( L^p(\Omega) \) is Banach, separable for \( p \in [1,\infty) \) and reflexive for \( p \in (1,\infty) \).

If \( \{f_n\}_{n=1}^\infty \) is a bounded sequence in \( L^p(\Omega) \) then there exists a subsequence such that
\[
\begin{align*}
\sup_{n} \int \! f_n^p &\to \int \! f^p \text{ in } L^p(\Omega), \, p \in (1,\infty) \quad (\iff \forall g \in L^q : \int \! f_n^p g \to \int \! fg) \\
\sup_{n} \int \! |f_n|^q &\to \int \! |f|^q \text{ in } L^q(\Omega) \\
\sup_{n} \int \! \mu_n &\to \int \! \mu \text{ in } M(\Omega) \text{ (Radon measures)} \, p = 1.
\end{align*}
\]
Fixed point theorems.

1. Let $F$ be continuous from $\mathbb{R}^d \to \mathbb{R}^d$. Assume that $\exists$ convex closed set $\Omega$ such that $F(\Omega) \subseteq \Omega$. Then $\exists x \in \Omega : F(x) = x$.

2. Let $F : X \to X$ (X - Banach space), F is continuous and compact, $\exists$ convex closed $\Omega \subseteq X$, $F(\Omega) \subseteq \Omega$. Then $\exists x \in \Omega : F(x) = x$.

Remark: Note that in infinite dimension (2.) we need compactness.

Nemytskii operator.

Def. & theorem: Let $\Omega \subseteq \mathbb{R}^d$ be open and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$. We say that $f$ is Carathéodory if

1. $\forall y \in \mathbb{R}^n \ f(x, y)$ is measurable wrt $x$

and 2. for a.a. $x \in \Omega \ f(x, \cdot)$ is continuous wrt $y$.

Assume that $\int f(x, y) \leq q(x) + \sum_{i=1}^{N} |y_i|^p_i$ for some $p_i \in [1, \infty)$, $q \in L^q(\Omega)$ with $q \in L^q(\Omega)$.

Then $\forall u \in L^{p_i}(\Omega)$ the function $f(x, u_1(x), u_2(x), \ldots, u_N(x))$ is measurable and the mapping $(u_1, u_2, \ldots, u_N) \to f(., u_1, \ldots, u_N)$ is continuous from $L^{p_1} \times L^{p_2} \times \ldots \times L^{p_N} \to L^p$ ($\Leftrightarrow u_i \to u_i$ in $L^{p_i}$, $i = 1, \ldots, N$, then $f(1 u_1, 2 u_2, \ldots, N u_N) \to f(1 u_1, 2 u_2, \ldots, N u_N)$ and it is called Nemytskii operator.

Proof: a) measurability is obvious

b) $f(., u_1, \ldots, u_N) \in L^p(\Omega)$:

$\int |f(., u_1, \ldots, u_N)|^p \leq \int |q(x)| + \sum_{i=1}^{N} |u_i|^{p_i} \leq c(p) \int |q|^{p} + \sum_{i=1}^{N} |u_i|^{p_i} < \infty$

c) $u_i^n \to u_i$ in $L^{p_i} \Rightarrow f(u_i^n) \to f(u_i)$ in $L^p$

Q: $\limsup_{n \to \infty} \int |f(., u_1^n, \ldots, u_N^n) - f(., u_1, \ldots, u_N)|^p \leq 0$

due to $u_i^n \to u_i$ in $L^{p_i}$, for a subsequence (that we do not relabel) $u_i^n(x) \to u_i(x)$ for a.a. $x \in \Omega$.

$F_{Carath.}$

$\Rightarrow \int |f(x, u^n_1(x), \ldots, u^n_N(x)) - f(x, u_1(x), \ldots, u_N(x))|^p \to 0 \ a.e.$

if we show that the difference is equiintegrable than the use of Vitali's theorem

Uniform equiintegrability:

$u_i^n \to u_i$ in $L^{p_i}(\Omega) \Rightarrow |u_i^n|^{p_i} + |u_i|^{p_i}$ is uniformly equiintegrable

$\int f(x, u^n) - f(x, u)|^p \leq c(p) (\int |f(x, u^n)|^{p} + \int |f(x, u)|^{p})$

$\leq c(p) (2 |g(x)|^{p} + \sum_{i=1}^{N} |u_i|^{p_i} + |u_i|^{p_i}) \Leftarrow$ and this is UEL
1. SOBOLEV SPACES (second reading = with proofs)

You should know \( W^{k,p}(\Omega) \)

**Theorem (local approximation of \( W^{k,p}(\Omega) \) by smooth functions):**

Let \( f \in W^{k,p}(\Omega) \) and extend it by zero outside of \( \Omega \). Define \( f_\varepsilon := \chi_{\varepsilon} \ast f \)

and set \( \Omega_\varepsilon := \{ x \in \Omega : B_\varepsilon(x) \subseteq \Omega \} \). Then \( D^j (f_\varepsilon) = (D^j f)_\varepsilon \) in \( \Omega_\varepsilon \) \( \forall \alpha, \| \alpha \| \leq k \)

and for all \( \Omega' \subseteq \overline{\Omega'} \subseteq \Omega \), \( f_\varepsilon \to f \) in \( W^{k,p}(\Omega') \).

**Proof:**

\[
\frac{\partial}{\partial x_i} (f_\varepsilon (x)) = \frac{\partial}{\partial x_i} \int_{\mathbb{R}^d} \chi_{\varepsilon} (x-y) f(y) \, dy = \int_{\mathbb{R}^d} \frac{\partial}{\partial x_i} \chi_{\varepsilon} (x-y) f(y) \, dy
\]

\[
= - \int_{\mathbb{R}^d} \frac{\partial}{\partial y_i} (\chi_{\varepsilon} (x-y)) f(y) \, dy = - \int_{\mathbb{R}^d} \frac{\partial}{\partial y_i} (\chi_{\varepsilon} (x-y)) f(y) \, dy
\]

\[
= - \int_{\mathbb{R}^d} \frac{\partial}{\partial y_i} (\chi_{\varepsilon} (x-y)) f(y) \, dy \overset{\text{def}}{=} \int_{\Omega} \chi_{\varepsilon} (x-y) \frac{\partial}{\partial y_i} f(y) \, dy = \left( \frac{\partial}{\partial y_i} \chi_{\varepsilon} \right)_x (x)
\]

\( \Omega' \subseteq \overline{\Omega'} \subseteq \Omega \):

Find \( \varepsilon_0 > 0 \) \( \forall \varepsilon < \varepsilon_0, \Omega' \subseteq \Omega_{\varepsilon} \)

then \( \lim_{\varepsilon \to 0^+} \| f_\varepsilon - f \|_{W^{k,p}(\Omega')} \leq \lim_{\varepsilon \to 0^+} \sum_{\alpha} \| D^\alpha (f_\varepsilon) - D^\alpha f \|_{L^p(\Omega')} = \lim_{\varepsilon \to 0^+} \sum_{\alpha} \| (D^\alpha f)_\varepsilon - D^\alpha f \|_{L^p(\Omega')}
\]

\( = 0 \) (Lebesgue spaces and regularization)

**Theorem (composition of Lipschitz and Sobolev functions):** Let \( \Omega \subseteq \mathbb{R}^d \) open and \( f : \mathbb{R} \to \mathbb{R} \) Lipschitz. Assume that \( u \in W_0^{k,p}(\Omega) \). Then \( (f(u) - f(u)) \in W_0^{k,p}(\Omega) \) and

(weak dir.) \( \frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \chi (x_i, u(x)) + S f \)

\( S f \subseteq \{ s \in \mathbb{R} : f'(s) \text{ does not exist} \} \).

Moreover, define \( \Omega_a := \{ x \in \Omega : u(x) = a \} \), then \( a u = a \) a.e. in \( \Omega_a \).

**Example:** \( \frac{\partial |u|}{\partial x_i} = \text{sgn} u \frac{\partial u}{\partial x_i} \chi (x_i, u(x)) + 0 \)

**Proof:** Rademacher said that \( |S f| = 0 \).

1. We prove it for \( f \in C^1 \), \( \mathfrak{f}_{up} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}
\]

\( |f(u) - f(0)| \leq \mathfrak{f}_{up} |u - 0| = \mathfrak{f}_{up} |u| \)

if \( u \in L^p(\Omega) \) \( \Rightarrow (f(u) - f(0)) \in L^p(\Omega) \)

Next we show \( \frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \)

if \( u \in \text{is smooth} \) then \( \frac{\partial f(u)}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \)

\( \psi \in C_0^\infty(\Omega) \)

\( \int_{\Omega} f(u) \frac{\partial \psi}{\partial x_i} = \int_{\Omega} f(u) \frac{\partial \psi}{\partial x_i} + (f(u) - f(u)) \frac{\partial \psi}{\partial x_i}
\]

consider \( \varepsilon \ll 1 \), \( \sup \psi \leq \Omega_{\varepsilon}
\]

\( = - \int_{\Omega_{\varepsilon}} f'(u) \frac{\partial u}{\partial x_i} \psi + \int_{\Omega} (f(u) - f(u)) \frac{\partial \psi}{\partial x_i} \)

\( = 0 \) (x)
\[ (*) \quad \frac{\partial}{\partial x_i} f \leq \frac{\partial}{\partial x_i} f \quad \forall f, \psi \text{ in } \mathcal{D}(\Omega) \]

\[ \| \psi \|_{L^1(\Omega)} \leq C \| \psi \|_{L^1(\Omega)} + \| \psi \|_{L^1(\Omega)} \]

**trivial**

\[ \implies \forall \psi \quad \frac{\partial}{\partial x_i} f \leq \frac{\partial}{\partial x_i} f \]

**extension to **\( C^0(\Omega) \)

\[ f_{up} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{x - y} \]

we know \( f_{up} = f_{up} \in L^p(\Omega) \)

if \( \frac{\partial u}{\partial x_i} = f'(u) \frac{\partial u}{\partial x_i} \chi(\{ u > 0 \}) \), then \( \frac{\partial f}{\partial x_i} \in L^p \)

the formula is true for \( f_{up} := \max (0, f) \)

\[ f_{up}(s) := \begin{cases} \sqrt{s + 3} - 3 & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases} \]

\[ f_{up}'(s) \geq \chi_{\{ s \geq 0 \}} \quad \forall s \in C^0(\Omega) \]

\[ \frac{\partial}{\partial x_i} f \leq \lim_{s \to 0^+} \frac{\int f(x) - f(y)}{x - y} = \lim_{s \to 0^+} \frac{\int f_{up}(x) - f_{up}(y)}{x - y} \]

\[ \lim_{s \to 0^+} \int f_{up}'(u) \frac{\partial u}{\partial x_i} \chi_{\{ u > 0 \}} \quad \text{Lepage} \quad \int f_{up}'(u) \frac{\partial u}{\partial x_i} \chi_{\{ u < 0 \}} \]

\[ \implies \frac{\partial}{\partial x_i} \chi_{\{ u = 0 \}} = 0 \quad \text{a.e. in } \Omega \]

\[ \implies \frac{\partial}{\partial x_i} \chi_{\{ u = 0 \}} = 0 \quad \text{a.e. in } \{ x : u(x) = 0 \} \]

\[ \implies f' \in C(\Omega) : \frac{\partial}{\partial x_i} f' = 0 \quad \text{a.e. in } \{ x : u(x) = 0 \} \]

**full generality:**

\[ f_{up} \in C^0, \quad f_{up} \to f \text{ in } C(\Omega) \]

\[ \| f' \|_{L^1(\Omega)} \leq f_{up} \quad \forall f, f' \in C(\Omega) \text{ except from } S_f \]

\[ \frac{\partial}{\partial x_i} f \leq \lim_{s \to 0^+} \frac{\int f(x) - f(y)}{x - y} \]

\[ \frac{\partial}{\partial x_i} f \leq \lim_{s \to 0^+} \frac{\int f_{up}(x) - f_{up}(y)}{x - y} \]

\[ \implies \frac{\partial}{\partial x_i} f \leq \lim_{s \to 0^+} \frac{\int f_{up}'(u) \frac{\partial u}{\partial x_i} \chi_{\{ u \neq 0 \}}}{x - y} \]

\[ - \text{easy if } u \notin S_f \]

\[ - \text{if } u \in S_f \text{ then } \frac{\partial}{\partial x_i} = 0 \]
Theorem (equivalent characterization of Sobolev functions): Let $\Omega \subset \mathbb{R}^d$ be an open set. Denote $\Omega_\delta := \{x \in \Omega, B_\delta(x) \subseteq \Omega\}$ and set $u^h(x) := \frac{u(x + he_1) - u(x)}{h}$. Then

1. If $u \in W^{1,p}(\Omega)$ then $\forall \delta > 0$ $\forall h < \frac{\delta}{2} \implies \|u^h\|_{L^p(\Omega_\delta)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)}$.

2. If $p \in (1, \infty)$ and $\sup_{\delta > 0} \sup_{h > 0} \|u^h\|_{L^p(\Omega_\delta)} \leq K$ then $\frac{\partial u}{\partial x_i}$ exists and $\left\| \frac{\partial u}{\partial x_i} \right\|_{L^p(\Omega)} \leq K$.

3. For $p \in [1, \infty)$ if $u \in W^{1,p}(\Omega)$ then $u^h \rightharpoonup u$ $\quad$ in $L^p(\Omega)$.

Proof: 2. Fix $\Omega_1 \subseteq \Omega$. Find $\delta_0 : \Omega_\delta \subseteq \Omega_1$. Then $\|u^h\|_{L^p(\Omega_\delta)} \leq K \forall h \leq \frac{\delta_0}{2}$.

For $p \in (1, \infty)$ we have $L^p$ is reflexive $\implies$ find a subsequence $u^h \rightharpoonup u$ in $L^p(\Omega_1)$.

From weak lower semi-continuity $\|u^h\|_{L^p(\Omega)} \leq \liminf_{h \to 0} \|u^h\|_{L^p(\Omega_\delta)} \leq K$.

Last: we need $\tilde{u} = \frac{\partial u}{\partial x_i}$ a.e. in $\Omega_1$. For $\varphi \in C_c^\infty(\Omega_1)$,

\[ \int_{\Omega} u \varphi = \lim_{h \to 0} \int_\Omega u^h \varphi = \lim_{h \to 0} \int_\Omega \left( \frac{u(x + he_1) - u(x)}{h} \right) \varphi(x) \, dx \]

\[ = \lim_{h \to 0} \int_\Omega u(x) \left( \frac{\varphi(x - he_1) - \varphi(x)}{h} \right) \, dx = \int_\Omega u(x) \frac{\partial \varphi}{\partial x_i}(x) \, dx \quad \text{uniformly $\to \frac{\partial \varphi}{\partial x_i}$} \]

The case $p = \infty$ is the same, just replace weak by weak*.

$\Rightarrow$ we know $\frac{\partial u}{\partial x_i}$ exists $\forall \Omega \subseteq \Omega_1$ with $\|\frac{\partial u}{\partial x_i}\|_{L^p(\Omega_\delta)} \leq K$ $\Rightarrow$ let $\Omega_\delta \subseteq \Omega$.

1. $u \in W^{1,p}(\Omega)$ ($p = \infty$), extend $u$ by 0.

$u_\infty := u \cdot \mathbb{1}

\quad \delta \quad \text{in} \quad \Omega_\infty

\quad \delta \quad \text{in} \quad L^p(\Omega_\delta)

\quad \delta \quad \text{in} \quad L^p(\Omega_\delta)

\frac{u_\infty(x + he_1) - u_\infty(x)}{h} = \mathfrak{a} \int_0^1 \frac{dt}{h} u_\infty(x + he_1) \, dt = \mathfrak{a} \int_0^1 \frac{dt}{h} \frac{\partial u_\infty}{\partial x_i}(x + he_1) \, dt$

$\left\| \frac{u_\infty(x + he_1) - u_\infty(x)}{h} \right\|_p \leq \int_\Omega \left\| \frac{\partial u_\infty}{\partial x_i}(x + he_1) \right\|_p \, dx$

$p = \infty$:

Define $\Omega_{\infty} := \Omega \cap B_r$, where $B_r$ is a ball.

$u \in W^{1,\infty}(\Omega)$ $\Rightarrow$ $u \in W^{1,\infty}(\Omega_k)$ $\quad$ bounded $\quad$ $\Rightarrow$ $u \in W^{1,p}(\Omega_k)$ $\quad \forall \quad p \in [1, \infty)$.

$\|u^h\|_{L^\infty(\Omega_k)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty(\Omega)} \quad \text{lim}_{h \to 0} \|f\|_{L^1} = \|f\|_1$

\[ \|u^h\|_{L^\infty(\Omega_k)} \leq \left\| \frac{\partial u}{\partial x_i} \right\|_{L^\infty(\Omega)} \quad \text{lim}_{h \to 0} \|f\|_{L^1} = \|f\|_1 \]

3. Easy homework. Hint: show that $u^h$ is Cauchy w.r.t. $h$ in $L^p$.
Properties up to the boundary and extensions

Theorem (approximation by smooth functions). Let $\Omega \subset \mathbb{R}^d$ be open and $p \in (1, \infty)$. Then for all $m \in \mathbb{N}$ there exists $u_m \in C^\infty(\Omega)$ such that $\|u_m - u\|_{W^{m,p}(\Omega)} \to 0$.

Proof: 1st by picture

- Open covering of $\Omega$
- Introduce partition of unity
  $\mu = \sum \mu_i \text{ (cut-off)}$
- Mollification

Proof: 2nd rigorous.

1. Description of $\Sigma$: There exists $M$ orthogonal transformations $T_r$, $r = 1, \ldots, M$, continuous functions $a_r, a_{i, j} : [0, 1] \to \mathbb{R}$ such that

$$V_r = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \quad \sum_{i=1}^d \left( a_{i,j}(x_i) - x_i a_j(x_i) \right) < \epsilon \}$$

$$\Lambda_r = \{ 1 - \frac{x_i}{a_{i,j}(x_i)} < x_i < a_{i,j}(x_i) \}$$

$$T_r(\Omega) = \Omega_r \mapsto \Omega$$

Define $\Sigma_r : T_r(\Omega) \mapsto [0, 1]^d$. We can find open $\Omega_{r+1} \supset \Omega_r$, $\Sigma_r \Omega_r \subset \Omega_{r+1}$.

Then we have finite open covering of a compact set.

Lemma: $\forall r \in C^\infty_0(\Omega_r), \ r = 1, \ldots, M+1$ such that $\forall x \in \Omega$, $\sum_{r=1}^{M+1} a_r(x) = 1$. Partition of unity

For given $u \in W^{m,p}(\Omega)$ and arbitrary $\epsilon > 0$ we want to find $u_\epsilon \in C^\infty(\overline{\Omega})$, $\|u - u_\epsilon\|_{W^{m,p}(\Omega)} \leq \epsilon$ (Dream 1)

Define $u_\epsilon(x) = u(x)^{\sum_{r=1}^{M+1} a_r(x)}$, show that $\forall \epsilon > 0$ $\exists u_\epsilon \in C^\infty(\overline{\Omega})$ such that $\|u_\epsilon - u\|_{W^{m,p}(\Omega)} \leq \epsilon$ (Dream 2)

Dream 2 $\Rightarrow$ Dream 1

Define $u_\epsilon = \sum_{r=1}^{M+1} u_r^\epsilon$, then $\|u - u_\epsilon\|_{W^{m,p}(\Omega)} = \|\sum_{r=1}^{M+1} (u_r - u_r^\epsilon)\|_{W^{m,p}(\Omega)} \leq \sum_{r=1}^{M+1} \frac{\epsilon}{M+1} = \epsilon$

Mollification - on the $\Omega_r$

$\Omega_{r+1} = \Omega \cup \Omega_{r+1} \supset \Omega_r$, $\Omega_{r+1} = \Omega_r + \Omega_0$

Mollification - on the $\partial \Omega_{r+1}$

$\Omega_r$, without loss of generality $T_r = I$. Then $\Omega_r = V_r$ and $(x_1, x_d) = (x_1, x_d)$. 

\[\begin{array}{c}
\end{array}\]
\[ u_1 = \sum_{r} u_r \quad \psi_r \in \mathcal{C}_0^\infty(V_1) \quad u_r^n(x) := \psi_r(x_1, \ldots, x_{d-1}, x_d + n) \]

\[ \forall \varepsilon > 0 \exists \delta, \eta > 0 \text{ s.t. } \| u_1 - u_1^n \|_{L^1} \leq \frac{\varepsilon}{2} \]

\[ u_1^n := u_1 \ast \eta_\delta \]

\[ V_1^+ := \{ (x_1, x_d) \mid a_1(x') < x_d < a_1(x') + \delta \} \]

\[ V_1^- := \{ (x_1, x_d) \mid a_1(x') - \delta < x_d < a_1(x') \} \]

\[ \psi_r \in \mathcal{C}_0^\infty(V_1) \quad \exists \delta > 0 \quad \psi_r = 0 \text{ on the set } x_d > a_1(x') + (\beta - \delta) \quad \text{and} \quad x_d < a_1(x') - (\beta + \delta) \]

\[ \Rightarrow \delta \text{ must be less than } \delta' \]

Take point \((x_1, \ldots, x_{d-1}, x_d - h)\) where \((x_1, \ldots, x_d) \in \delta \Omega\)

we need to check that \( \text{dist} \left( (x_1, \ldots, x_{d-1}, x_d - h), \partial \Omega \right) < \delta \)

\[ \delta > 0 \quad a_1 \text{ is continuous} \Rightarrow \exists h_{\text{max}} \quad \forall h < h_{\text{max}} \quad \text{is true} \]

take \( h < \min h_{\text{max}} \) \( h_{\text{max}}, \eta > 0 \) \( \| u_1^n - u_1 \|_{L^1} \leq \frac{\varepsilon}{\eta} \)

Theorem (Extension): Let \( \Omega \subset \mathbb{C}^{d+1} \) and \( \mu \in [1, \infty] \). Then there exists continuous linear operator \( E : W^{1,p}_0(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d) \) such that \( E \mu \in W^{1,p}_0(\Omega) \)

1. \( E \mu = \mu \) in \( \Omega \)

2. \( \exists B \subset \mathbb{R}^d \) such that \( E \mu = 0 \) on \( \mathbb{R}^d \setminus B \)

3. \( \| E \mu \|_{W^{1,p}(\mathbb{R}^d)} \leq c(d, p, \Omega) \| \mu \|_{W^{1,p}_0(\Omega)} \)

Proof: picture

\[ \text{partition of unity } \mu = \sum_{r} \mu_r \]

extend \( \mu_r \)

\[ \text{flatten the boundary} \quad \text{extension by symmetry} \quad \text{uniflation} \]

\[ u = \sum_{r} \mu_r \] \( \in W^{1,p}_0(\Omega) \) to extend \( u_1 \) to \( W^{1,p}(\mathbb{R}^d) \)

\[ V_1 \cap T_1 = \Gamma \quad F : V_1 \rightarrow \overline{V_1} \left\{ \begin{array}{l} y_1 = x' \\
 y_d = x_d - a(x') \end{array} \right. \]

\[ V_1 : x_1 \rightarrow x_1 \quad \overline{V_1} : y' \rightarrow y_1' \]

\[ F \text{ is Lipschitz} \quad \det \nabla F = 1 \]

\[ F^{-1} : \overline{V_1} \rightarrow V_1 \left\{ \begin{array}{l} x' = y' \\
 x_d = y_d + a(y') \end{array} \right. \]

\[ F^{-1} \text{ is Lipschitz} \quad \det \nabla F^{-1} = 1 \]
\[ m_1 \in W^{1,p}(\Omega_1 \cap \Omega_2) \]
\[ n_{1}(y) := m_1 (F^{-1}(y)) \quad \text{defined where } y_d > 0 \]

Easy homework: check that \( m_1 \in W^{1,p}(\Omega_2) \), \( \| m_1 \|_{W^{1,p}(\Omega_2)} \leq \| m_1 \|_{W^{1,p}(\Omega_1)} c(\Omega_2, \Omega) \)

\[ \frac{\partial n_{1}}{\partial y_d} \sim \| \partial y (F^{-1}(y)) \| \| \nabla F \| \]

Composition Sobolev (Lipschitz) = Sobolev

Define \( E_{n_{1}}(y) := \begin{cases} n_{1}(y) & \text{if } y_d > 0 \\ n_{1}(y_{1}, \ldots, y_{d-1}) & \text{if } y_d \leq 0 \end{cases} \)

\( E_{n_{1}} \) is compactly supported in \( \Omega_2 \), Sobolev in \( \Omega_2 \) and \( \Omega_2^* \). What happens when \( y_d = 0 \)?

Check \( \frac{\partial E_{n_{1}}}{\partial y_d} : \quad \psi \in C_{c}^{\infty}(\Omega_2) \)

\[ \int_{\Omega_2} \frac{\partial E_{n_{1}}}{\partial y_d} \frac{\partial}{\partial y_d} \psi \, dy = - \int_{\Omega_2} \frac{\partial n_{1}}{\partial y_d} \frac{\partial}{\partial y_d} \psi \, dy + \int_{\Omega_2} \frac{\partial n_{1}}{\partial y_{d-1}} \frac{\partial}{\partial y_{d-1}} \psi \, dy \]

\[ = \int_{\Omega_2} \left( \frac{\partial n_{1}}{\partial y_d} - \frac{\partial n_{1}}{\partial y_{d-1}} \right) \frac{\partial}{\partial y_d} \psi \, dy \]

\( E_{n_{1}}(x) := E_{n_{1}}(F(x)) \)

again \( \| E_{n_{1}} \|_{L^p} \leq c \| E_{n_{1}} \|_{L^p} \leq c \| m_1 \|_{L^p} \leq c \| m_1 \|_{L^p} \)

**Embeddings**

We know, \( \Omega \in C^{\alpha} : \quad W^{1,p}(\Omega) \subset L^{q}(\Omega) \) if \( p < d \quad q < \frac{d}{d-1} \)

\( W^{1,p}(\Omega) \subset C^{\alpha}(\Omega) \) if \( p > d \quad \alpha = 1 - \frac{d}{p} \)

General scheme: take \( m \in W^{1,p}(\Omega) \), extend to \( E_{m} \in W^{1,p}(\mathbb{R}^d) \), compactly supported, prove embeddings for \( E_{m} \) in \( W^{1,p}(\mathbb{R}^d) \) and then go back to \( m \in W^{1,p}(\Omega) \)

1. Embeddings of type \( W^{1,p} \subset C^{\alpha} \) (Horrey)

**Lemma:** Let \( m \in W^{1,p}(\mathbb{R}^d) \) and \( 0 \) be a Lebesgue point of \( m \). Then

\[ |B_{R}f_{m} - m(0)| \leq R^{A} C(A, \alpha, d) \sup_{\| \phi \|_{E_{2}} = 1} \frac{|\int_{\mathbb{R}^d} \phi(x) dx|}{\int_{|x| < R} \phi^{d+\alpha} dx} \]

\( (A > 0) \)

**Proof:**

\[ |B_{R}f_{m} - m(0)| = \lim_{R \rightarrow 0^{+}} |B_{R}f_{m} - B_{R}m(0)| = \lim_{R \rightarrow 0^{+}} | \int_{\mathbb{R}^d} \frac{d}{dp} f_{m} \, dp | = \lim_{R \rightarrow 0^{+}} | \int_{|x| < R} \phi^{d+\alpha} dx | \]

\[ \leq \lim_{R \rightarrow 0^{+}} \int_{|x| < R} \frac{d}{dp} f_{m} \, dx \, dp \leq \lim_{R \rightarrow 0^{+}} \int_{|x| < R} |m(x)| dx \, dp \]

\[ = \int_{|x| < R} |m(x)| dx \, dp \leq \int_{|x| < R} \frac{d}{dp} f_{m} \, dx \, dp \]

\[ \leq \left( \sup_{R \leq |x| < R} \int_{|x| < R} |m(x)| \right) \lim_{R \rightarrow 0^{+}} \int_{|x| < R} \phi^{d+\alpha} dx \, dp = \frac{A^{\alpha}}{A} C(d) R^{A-\alpha} \]

\( \sup_{R \leq |x| < R} \int_{|x| < R} |m(x)| \)

\( \frac{A^{\alpha}}{A} C(d) R^{A-\alpha} \)
We want $W^{0,p}(\Omega) \subset C^{0,\alpha}(\Omega)$ if $p > d$, $\alpha = 1 - \frac{d}{p}$.

We know: If $x$ is a Lebesgue point of $u$, then $|u(x) - \int_B u(y)dy| \leq R^d \|\nabla u\|_{L^p(B)}$ $\sup_{B(0,x)} \int_{B(x, \epsilon)} |u(y) - u(x)|dy \leq 0$.

Proof of $W^{0,p} \subset C^{0,\alpha}$:

1. We extend $u$ by $Eu : u \in W^{0,p}(\Omega) \Rightarrow Eu \in W^{0,p}(\mathbb{R}^d)$, $Eu$ is compactly supported in $\mathbb{R}^d$.

$Eu = u$ in $\Omega$, $\|Eu\|_{W^{0,p}(\mathbb{R}^d)} \leq C(\Omega, p) \|u\|_{W^{0,p}(\Omega)}$.

2. To show that if $xy$ are Lebesgue points of $Eu$, then

$$|u(x) - u(y)| \leq |x - y|^{\alpha} \max\{1, \frac{|x - y|}{|B(x, \epsilon)|}\} \leq c(\alpha) R^d \max\{|x - y|, \frac{|x - y|}{|B(x, \epsilon)|}\}.$$

Proof of 2: set $R := |x - y|$, use (1):

$$|u(x) - u(y)| \leq |u(x) - f_{B(x)} u(y)| + |f_{B(x)} u(y) - f_{B(y)} u(y)| + |f_{B(y)} u(y) - f_{B(y)} u(y)| \leq c(\alpha) R^d \max\{|x, |y|\} + |f_{B(x)} u(y) - f_{B(y)} u(y)| \leq c(\alpha) R^d \max\{|x, |y|\} + \frac{|u(x) - u(y)|}{|B(x, \epsilon)|} \leq c(\alpha) R^d \max\{|x, |y|\}.$$

3. Morrey embedding: $\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq c(\alpha, p) \sup_{x \neq y} \frac{R^d}{|x - y|^{\alpha}}$.

Proof: Step 2.

Note: true for Lebesgue points, but from above we can redefine $Eu$ to be continuous.

4. End of the proof

$\sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \leq c \|u\|_{L^p(\Omega)} \sup_{x \neq y} \frac{R^d}{|x - y|^{\alpha}} \leq c \|u\|_{W^{0,p}(\Omega)} \sup_{x \neq y} \frac{R^d}{|x - y|^{\alpha}} \leq c \|u\|_{W^{0,p}(\Omega)} \sup_{x \neq y} \frac{R^d}{|x - y|^{\alpha}} \leq c \|u\|_{W^{0,p}(\Omega)} \sup_{x \neq y} \frac{R^d}{|x - y|^{\alpha}}$.

what remains: $\|u\|_{L^p(\Omega)} \leq c \|u\|_{W^{0,p}(\Omega)}$.

5. You should know $C^{0,\alpha} \subset C^{0,\beta}$ if $\alpha \leq \beta$.

Note: What if $p = d^2$, $W^{0,p}(\Omega) \subset BMO(\Omega)$. $BMO(\Omega) = \{u \in L^{d}(\Omega) : \|u - \int_{\Omega} u\|_{L^{d}} \leq c\}$. PDE people like it, function spaces people don't.
Embedding \( W^{1,p}(\Omega) \hookrightarrow L^{q}(\Omega) \), \( q = \frac{dp}{d-p} \)
\( W^{1,p}(\Omega) \hookrightarrow L^{q,\infty}(\Omega) \)  


Lemma (Gagliardo–Nirenberg inequality): \( \exists c(d) \forall \mu \in C_{0}^{\infty} \mathbb{R}^{d} \)

1. \( \| u \|_{L^{d}(\mathbb{R}^{d})} \leq c(d) \| \nabla u \|_{L^{d}(\mathbb{R}^{d})} \)
2. \( \| u \|_{L^{q}(\mathbb{R}^{d})} \leq c(d(p), p) \| \nabla u \|_{L^{p}(\mathbb{R}^{d})} \quad p < d \)

Proof: "1. \( \Rightarrow \) 2."

Define \( \nu = \| u \|_{L^{d}(\mathbb{R}^{d})} \), apply \( \| u \|_{L^{d}(\mathbb{R}^{d})} \leq c(d) \| \nabla u \|_{L^{d}(\mathbb{R}^{d})} \quad q > 1 \)

\[
\left( \int_{\mathbb{R}^{d}} |u| \frac{dq}{d-1} \right)^{\frac{d-1}{q}} = \| u \|_{L^{d}(\mathbb{R}^{d})} \leq c(d) \| \nabla u \|_{L^{d}(\mathbb{R}^{d})} \leq c(d) \int_{\mathbb{R}^{d}} |\nabla u|^{q-1} \leq c(d) q \int_{\mathbb{R}^{d}} |\nabla u|^{q-1} \quad \forall u \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d})
\]

Choose \( d, q, \frac{d(p)}{d-1} = p(q-1) \), \( q = \frac{d(p-1)}{d-p}, q-1 = \frac{d-p}{d-1} = \frac{d(p-1)}{d-1} \)

\[
\Rightarrow \left( \int_{\mathbb{R}^{d}} |u| \frac{dq}{d-1} \right)^{\frac{d-1}{q}} \leq c(d) \left( \frac{d(p)}{d-1} \right)^{\frac{d-1}{q}} \| \nabla u \|_{L^{p}(\mathbb{R}^{d})} \left( \int_{\mathbb{R}^{d}} |u| \frac{dq}{d-1} \right)^{\frac{d-1}{q}}
\]

Then \( \mathcal{F}(\mathbb{R}^{d}) = \left( \int_{\mathbb{R}^{d}} |u| \frac{dq}{d-1} \right)^{\frac{d-1}{q}} \quad \forall u \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d}) \)

Proof of 1.

Lemma (Gagliardo): Let \( u_{i} \in C_{0}^{\infty}(\mathbb{R}^{d}) \) for \( i = 1, \ldots, d \). Define \( v_{i}(x_{1}, \ldots, x_{d}) := u_{i}(x_{1}, x_{2}, \ldots, x_{d}) \).

Then \( \mathcal{F}(\mathbb{R}^{d}) = \int_{\mathbb{R}^{d}} \frac{d^{d}}{d^{i}} |v_{i}(x_{d})| \quad \forall u_{i} \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d}) \)

Proof: by induction w.r.t. \( d \)

1. If \( d = 2 \), \( \mathcal{F}(\mathbb{R}^{d}) = \int_{\mathbb{R}^{d}} \frac{d^{d}}{d^{i}} |v_{i}(x_{d})| \quad \forall u_{i}(x_{1}, x_{2}) \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d}) \)
2. \( d \Rightarrow d+1 \), \( \mathcal{F}(\mathbb{R}^{d}) = \int_{\mathbb{R}^{d}} \frac{d^{d}}{d^{i}} |v_{i}(x_{d})| \quad \forall u_{i}(x_{1}, \ldots, x_{d}) \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d}) \)

\[
\quad \leq \| u_{i} \|_{L^{1}(\mathbb{R}^{d})} \int_{\mathbb{R}^{d}} \left( \frac{\sum_{i=1}^{d} |v_{i}(x_{d})|}{\sum_{i=1}^{d} |v_{i}(x_{d})|} \right)^{d(d+1)} \quad \forall u_{i} \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d})
\]

\[
\quad \leq \| u_{i} \|_{L^{1}(\mathbb{R}^{d})} \int_{\mathbb{R}^{d}} \left( \sum_{i=1}^{d} \left| v_{i}(x_{d}) \right| \frac{d^{d}}{d^{i}} \right)^{\frac{d}{d-1}} \quad \forall u_{i} \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d})
\]

Induction for \( x_{d} \), \( \mathcal{F}(\mathbb{R}^{d}) = \int_{\mathbb{R}^{d}} \frac{d^{d}}{d^{i}} |v_{i}(x_{d})| \quad \forall u_{i} \in \mathcal{C}_{0}^{\infty}(\mathbb{R}^{d}) \)
Use of $G-L$: $u \in C^0_c(\mathbb{R}^d)$

\[ u(x) = \int_{-\infty}^{x_i} \frac{\partial u}{\partial x_i}(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_d) \, ds \]

\[ |u(x)| \leq \int_{-\infty}^{x_i} |u(x_1, \ldots, x_{i-1}, s, x_{i+1}, \ldots, x_d)| \, ds \]

\[ |u(x)|^q \leq \sum \frac{1}{i} \left( \int_{-\infty}^{x_i} |u(x_1, \ldots, x_{i-1}, S, x_{i+1}, \ldots, x_d)| \, ds \right)^{\frac{q}{i}} \]

\[ \|u\|_{L^q_c(\mathbb{R}^d)} = \sum \frac{1}{i} \left( \int_{-\infty}^{x_i} |u(x_1, \ldots, x_{i-1}, S, x_{i+1}, \ldots, x_d)| \, ds \right)^{\frac{q}{i}} \]

\[ \|u\|_{L^q_c(\mathbb{R}^d)} \leq \sum \left( \int_{-\infty}^{x_i} |u(x_1, \ldots, x_{i-1}, S, x_{i+1}, \ldots, x_d)| \, ds \right)^{\frac{q}{i}} \]

Proof $a)$ $W^{0}(\Omega) \subset L^{p}_{\text{loc}}(\Omega)$ $\quad \forall p \in (1, \infty)$ and $\Omega \subset C^{1,\alpha}$

\[ \|u\|_{L^{p}_{\text{loc}}(\Omega)} \leq \|u\|_{L^{p}_{\text{loc}}(\mathbb{R}^d)} \leq \|u\|_{W^{0}(\Omega)} \leq \|u\|_{W^{0}(\mathbb{R}^d)} \]

$b)$ $W^{0}(\Omega) \subset L^{q}(\Omega) \quad q < \frac{d}{p}$

\[ \|u\|_{L^{q}(\Omega)} \leq \|u\|_{L^{q}(\mathbb{R}^d)} \leq \|u\|_{W^{0}(\mathbb{R}^d)} \]

Compact embedding $W^{0}(\Omega) \subset L^{q}(\Omega)$ if $q < \frac{d}{p}$

1. Step: Show $W^{1}(\Omega) \subset L^{q}(\Omega)$

2. Step: Show $W^{2}(\Omega) \subset L^{q}(\Omega)$

"$1 \rightarrow 2"$ $\quad \forall u \in W^{1}$ $\quad \|u\|_{L^{q}(\Omega)} \leq c \|u\|_{W^{1}}$

Lebesgue interpolation $\|u\|_{L^{q}} \leq \|u\|_{W^{1}} \|u\|_{L^{q}}$ $\quad \forall p < q \leq 2 \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{2}$

\[ \|u\|_{L^{q}} \leq \|u\|_{W^{1}} \|u\|_{L^{q}} \]

Assume 1. holds, let $B$ be bounded subset of $W^{0}(\Omega)$.

1. $\forall \varepsilon > 0 \exists \{\mu_i\} \subset W^{0}(\Omega)$ $\forall \mu \in B \quad \min \|u - \mu\|_{L^{\infty}} \leq \varepsilon$

\[ \|u - \mu\|_{L^{q}} \leq c \|u - \mu\|_{L^{\infty}} \|u - \mu\|_{W^{1}} \leq c \|u - \mu\|_{W^{1}} \]

\[ \min \|u - \mu\|_{L^{q}} \leq c \varepsilon \]

Proof of 1. $B$ a ball subset of $W^{0}(\Omega)$, $EB$ a ball subset of $W^{0}(\mathbb{R}^d)$ (created by extension)

\[ u \in EB \quad u_E = u \ast \eta_b \quad (u_E(x) = \int_{B} u(x+y) \eta_b(y) \, dy) \]

try to estimate $u - u_E$ in $L^2$

\[ \|u - u_E\|_{L^2} \leq \int_{E} \int_{\mathbb{R}^d} |u(x+y) - u_E(x+y)| \, dy \, dx \]

\[ \leq \|\nabla u\|_{L^2} \int_{\mathbb{R}^d} \eta_b(y) \, dy \leq c \|\nabla u\|_{L^2} \int_{\mathbb{R}^d} \eta_b(y) \, dy \leq \frac{c}{\varepsilon} \|u\|_{L^{\infty}}(\mathbb{R}^d) \]

\[ \|\nabla u\|_{L^2} \leq \varepsilon \|u\|_{L^{\infty}}(\mathbb{R}^d) \]

\[ \|u\|_{L^{\infty}}(\mathbb{R}^d) \leq \varepsilon \|u\|_{L^{\infty}}(\mathbb{R}^d) \]
Give me $\varepsilon > 0$, set $\delta := \frac{\varepsilon}{2}$, so $\|w_0\|_{W^p(\Omega)} \leq C(\delta, \|w_0\|)$ find finite covering $\{\omega_i\}_{i=1}^N$ of $W^p(G)$ so $\min \|w - w_i\|_{L^p(\Omega)} \leq \frac{\varepsilon}{2}$

\[
\Rightarrow \|w - w_i\|_{L^p} + \|w - w_i\|_{L^p(\Omega)} \leq \varepsilon
\]

**Trace theorems**

1. on cube for smooth functions

\[\Omega = (-1,1)^d \times (0,1) \quad \text{if } \Omega \subseteq \mathbb{R}^d \quad \text{and } u(x,1) = 0\]

Question: What is the best $q$ such that $(-1,1)^d \times (0,1) \ni x \to \int_{x_d = 0} \int_{x_d = 1} |u(x, y)|^q \, dy \, dx$ is finite?

\[
q \leq \frac{d}{d-1} \quad \text{where } q = \frac{d}{d-1}
\]

\[
\|u\|_{L^q((-1,1)^d \times (0,1))} \leq c \|u\|_{L^q(G)}
\]

Last time: On cube $(-1,1)^d \times (0,1)$ if $u$ is smooth

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\[
\|u\|_{L^q((-1,1)^d \times (0,1))} \leq c \|u\|_{L^q(G)}
\]

We define \( \{ \psi_i \} = \text{partition of unity} \)

\[
\int \sum \psi_i \, f(x) \, dx = \sum \int \psi_i \, f(x) \, dx
\]

For $T \in L^q((-1,1)^d)$

\[
\frac{d}{dx} T(x) = \int T'(x) \, \mathbf{n} \cdot ds
\]

2. Definition: Let $\Omega \in \mathbb{C}^n$ and $f : \partial \Omega \to \mathbb{R}$, we say that $f \in L^p(\partial \Omega)$ if $\forall i = 1, \ldots, N$

for $T \in L^q((-1,1)^d)$

\[
\int T(x) \, ds
\]

We define \( \{ \psi_i \} \)

\[
\int f \, ds = \sum_{i=1}^N \int \psi_i \, f \, ds
\]

\[
\int f \, ds \quad \text{is independent of the covering!}
\]

Lemma (integration by parts): Let $\Omega \in \mathbb{C}^n$ and $f \in C^1(\overline{\Omega})$

\[
\int \frac{\partial f}{\partial x_i} \, dx = \int f \, \mathbf{n} \cdot ds
\]

1. Trace theorem: Let $\Omega \in \mathbb{C}^n$. Then there exists a linear operator $\text{Tr} : W^p(\Omega) \to L^q(\partial \Omega)$

for all $p \leq d$ such that for all $u \in C^1(\overline{\Omega})$

\[
\text{Tr}_{u} = u|_{\partial \Omega}
\]

Proof: \( a < p \leq d \) $W^p(\Omega) \cap C(\overline{\Omega})$

b) $p \leq d$, we have $C^1(\overline{\Omega})$ is dense in $W^p(\Omega)$, $u \in W^p(\Omega)$, $f \in C^1(\overline{\Omega})$

\[
\int u^q \, ds \leq c \int u^q \, dx
\]

estimate on cube $\Rightarrow u^q$ is Cauchy in $L^q(\partial \Omega)$, Banach space, $u^q_{\text{Tr}} := \lim_{u \to \infty} u^q|_{\partial \Omega}$
Theorem (Integration by parts for Sobolev functions): Let $\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)$, $v \in W^{1,q}(\Omega)$.

Let $W^{1,p} \subset L^p$ and $W^{1,q} \subset L^q$. Then

$$
\begin{aligned}
\int_\Omega \frac{\partial u}{\partial x_i} v &= \int_\Omega \frac{\partial v}{\partial x_i} u + \int_{\partial \Omega} Tr u Tr v\ n_i
\end{aligned}
$$

Second part of the diff. hw - prove it.

Inverse trace operator (* functions with non-integer derivative*)

What is the target of $\text{Tr}^\ast$? Warning! Not $L^{p-\frac{1}{q}}(\partial \Omega)$!

Theorem: Let $\Omega \subset \mathbb{R}^n$, $p \in (1, \infty)$, $q \in (\frac{1}{p}, 1]$. Then $\text{Tr}$ is bounded from $W^{p}(\Omega) \to W^{1,q}(\partial \Omega)$.

Moreover, $\exists$ $\text{Tr}^\ast$: $W^{1,q}(\partial \Omega) \to W^{p}(\Omega)$, linear, bounded and $\text{Tr}^\ast(\text{Tr} u) = u$.

(For $p = 1$, $\exists$ $\text{Tr}^\ast$: $W^{1,q}(\partial \Omega)$ ($= L^q(\partial \Omega)$) $\to W^{1,q}(\partial \Omega)$ ($= L^q(\partial \Omega)$), which is nonlinear.)

Definition (Sobolev-Slobodetskii): We say that $u \in W^{p}(\Omega)$, $s \in (0, 1)$ if

$$
\int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{n+sp}}\ dx\ dy < \infty.
$$

Remark: Similarly on $\partial \Omega$.

Definition (Nikolskii): Let $u \in L^p(\Omega)$, we say that $u \in W^{s,p}(\Omega)$ if $\int_\Omega \frac{|u(x) - u(y)|}{|x - y|^{n+sp}}< \infty$.

$r > 0$, $u$ compactly supported in $\Omega \setminus \{|x| \geq r\}$.

Theorem: $W^{p}(\Omega) \subset W^{s,p}(\Omega) \subset W^{s,p}(\Omega)$ $\forall \varepsilon > 0$.

2. **Nonlinear Elliptic Equations as Compact Perturbations**

Example: $-\Delta u + g(u) = f$ in $\Omega$

$u = 0$ on $\partial \Omega$

$g: \mathbb{R} \to \mathbb{R}$, continuous, $|g(s)| \leq c_s(1+|s|)^k$ $\forall s \in \mathbb{R}, 0, 1$

Nemytskii: $g: L^q(\Omega) \to L^p(\Omega)$

"A priori estimates" for a smooth solution,

$\begin{aligned}
\int \Delta u\ |u|_2^2 &\leq \int \frac{\partial u}{\partial x_i}\ u\ \frac{\partial u}{\partial x_i}\ \frac{\partial u}{\partial x_i} + \int \frac{\partial u}{\partial x_i}\ u\ \frac{\partial u}{\partial x_i}\ \frac{\partial u}{\partial x_i} + \int c_s\ (1+|u|)^k|u|_2
\end{aligned}$

$\begin{aligned}
\int \frac{\partial u}{\partial x_i}\ u\ \frac{\partial u}{\partial x_i} &\leq c\ \int (1+|u|)^k |u|_2\ |u|_2 + \frac{k\ c_s}{2}\ \int (\frac{1}{2}\ |u|_2^2) + \frac{c_s}{2}\ \int (\frac{1}{2}\ |u|_2^2)
\end{aligned}$

Choose $0 < \varepsilon < 1$

$\begin{aligned}
\|u\|_2^2 &\leq c\ (\|u\|_2^2 + (1+\|u\|_2^2))
\end{aligned}$
Lemma: If \( f \in L^2(\Omega) \) then \( \exists u \in W_0^{1,2}(\Omega) \) s.t. \( \forall \varphi \in W_0^{1,2}(\Omega) \), \( \int_\Omega \varphi \cdot \nabla u + g(\varphi) \cdot u = \int_\Omega f \varphi \) (weak form)

Proof: by fixed point \( w = L^2(\Omega) \) - look for \( u \in W_0^{1,2}(\Omega) \)

\[
\text{winter } \Rightarrow \forall \varphi \in L^2(\Omega) \exists ! u \in W_0^{1,2}(\Omega) \quad - \Delta u = f - g(\varphi) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega
\]

\( M: L^2(\Omega) \to C(\Omega) \)

\( \varphi \mapsto u \). To show that \( M \) has a fixed point Schauder.

1. \( M \) is continuous \( \quad (\text{Yes - winter semester + Nemytskii}) \)

2. \( M \) is compact \( \quad (\text{Yes - } W^{1,2} \text{ in } L^2) \)

3. \( \int_\Omega |u|^2 \leq c_0 \int_\Omega |\nabla u| + \int_\Omega |g(\varphi)| |u| \\leq \varepsilon \|u\|_{L^2}^2 + C(\|\nabla f\|_{L^2}^2 + \|g(\varphi)\|_{L^2}^2) \)

\[
\Rightarrow \quad \|u\|_{H^1}^2 \leq C \left( \|\nabla f\|_{L^2}^2 + \|g(\varphi)\|_{L^2}^2 \right)
\]

\[
\|g(\varphi)\|_{L^2}^2 \leq c_0 \left( 1 + \|\varphi\|_{L^2}^2 \right) \leq \delta \|u\|_{H^1}^2 + C(\delta)
\]

\[
\|u\|_{L^2}^2 \leq C(\delta) \left( \|\nabla f\|_{L^2}^2 + 1 \right) + 5 \|u\|_{H^1}^2 \leq R \]

\[
\|u\|_{L^2}^2 \leq C(\delta) \left( \|\nabla f\|_{L^2}^2 + 1 \right) + 5 \delta \leq R^2 \]

Set \( \delta = \frac{1}{5} \) and assume \( R^2 \leq 2C(\|\nabla f\|_{L^2}^2 + 1) \) \( \Rightarrow \) \( \|u\|_{L^2} \leq R \). Hence \( M: B_R \to B_R \) where \( B \) is a ball in \( L^2(\Omega) \)

Schauder \( \Rightarrow \exists u \in W_0^{1,2} \) a fixed point

Uniqueness: \( u_1, u_2 \) solutions:

\[
\forall \varphi \quad \int_\Omega (u_1 - u_2) \nabla \varphi + \int_\Omega (g(u_1) - g(u_2)) \varphi = 0 \quad \forall \varphi \in W_0^{1,2}(\Omega)
\]

\( \Rightarrow \quad u_1 = u_2 \)

a) if \( g \) is nondecreasing \( \Rightarrow \quad (g(s_1) - g(s_2)) (s_1 - s_2) \geq 0 \) \( \Rightarrow \) uniqueness

b) if \( g \) is \( C(\Omega) \) and \( \sup_{s \in \Omega} \quad -g(s) < C(\Omega) \) then \( \Rightarrow \quad u_1 = u_2 \)

Another example: \( -\Delta u + e^u = f \quad \text{in } \Omega \)

\( u = 0 \quad \text{on } \partial \Omega \)

Estimates: \( \quad \int_\Omega \|u\|^2 + \int_\Omega e^u \|u\|^2 = \int_\Omega f \|u\|^2 + C(\|\nabla f\|_{L^2}^2 + \|g(\varphi)\|_{L^2}^2)\)

\[
\|u\|_{L^2}^2 + \int_\Omega e^u \|u\|^2 \leq C \|\nabla f\|_{L^2}^2 + C(\|g(\varphi)\|_{L^2}^2) \]

\[
\|u\|_{L^2}^2 + \int_\Omega e^u |u| \leq C \left( 4 \|\nabla f\|_{L^2}^2 \right) \quad (E \ll 1 \quad \text{and Poincare})
\]
Lemma: Let $\Omega \subseteq \mathbb{R}^d$, $f \in L^2$. Then $\exists! u \in W_0^{1,2}(\Omega)$ and $v \in H^1(\Omega)$ such that $\forall \sigma \in W_0^{1,2}(\Omega)$

\[ \int_{\Omega} \nabla u \cdot \nabla \sigma \, dx + \int_{\Omega} e^{u} \sigma \, dx = \int_{\Omega} f \sigma \, dx \quad \forall \sigma \in W_0^{1,2}(\Omega) \]

Proof: 1. uniqueness

\[ \int_{\Omega} |\nabla (u_1 - u_2)|^2 \, dx + \int_{\Omega} e^{u_1} - e^{u_2} \left( \frac{u_1}{u_1 - u_2} - \frac{u_2}{u_2 - u_1} \right) \min(k, |u_1 - u_2|) \, dx = 0 \]

where $k = \min(k_1, |u_1 - u_2|)$ (such that $u \in W_0^{1,2}(\Omega)$)

\[ \int_{\Omega} |\nabla (u_1 - u_2)|^2 \, dx = 0 \]

$u_1 = u_2$ for all $x \in \Omega$.

2. existence by approximation

$-\Delta u + e^{\min(n, u)} = f$ in $\Omega$, $u = 0$ on $\partial \Omega$ by first example

\[ \int_{\Omega} \nabla u_n \cdot \nabla u_n \leq \int_{\Omega} |f| \, dx \]

$u_n \rightharpoonup u$ in $H^1(\Omega)$

$u_n \rightarrow u$ in $L^2(\Omega)$

$u_n \rightarrow u$ a.e. in $\Omega$

1. $\int_{\Omega} e^{u_n} \leq \liminf_{n \to \infty} \int_{\Omega} e^{\min(n, u_n)} \leq \limsup_{n \to \infty} \int_{\Omega} \left( 2 + e^{\min(n, u_n)} \right) u_n^2 < \infty$

2. weak formulation

$u \in W_0^{1,2}(\Omega)$ and $L^\infty(\Omega)$

\[ \int_{\Omega} \nabla u \cdot \nabla \sigma \, dx + \int_{\Omega} e^{u} \sigma \, dx = \int_{\Omega} f \sigma \, dx \quad \forall \sigma \in W_0^{1,2}(\Omega) \]

justification of second limit:

Vitali: if $q^n \rightarrow q$ a.e. and $\forall \varepsilon > 0 \exists \delta > 0 \forall s \in \Omega \ |s| \leq \delta \Rightarrow |S g| \leq \varepsilon$ then $S g = \lim S g_n$

\[ \int_{\Omega} e^{\min(n, u)} \leq \liminf_{n \to \infty} \int_{\Omega} e^{\min(n, u)} \leq C (|S| + \int_{\Omega} e^{\min(n, u)}) \]

\[ \leq C (|S| + e^{|S| + \sum_{k \geq 1} \int \min(n, u) \, dx} + \sum_{k \geq 1} \int \min(n, u) \, dx \leq |S| + \sum_{k \geq 1} \int \min(n, u) \, dx \leq |S| + \sum_{k \geq 1} \int \min(n, u) \, dx \]

$\varepsilon := \frac{\varepsilon}{k}$ for $|S| + e^{\frac{\varepsilon}{k}} < \varepsilon$, choose $\delta : C \delta (1 + e^{\frac{\varepsilon}{k}}) < \varepsilon$, then

for $|S| \leq \delta \Rightarrow \int_{\Omega} e^{\min(n, u)} \leq \varepsilon$ and we can use the Vitali theorem for the second limit
**Last example:** \[-\Delta u + b(\nabla u) = f \quad \text{in } \Omega\]
\[u = 0 \quad \text{on } \partial \Omega\]

**b is continuous and bounded.**

**Lemma:** \( \exists u \in W_0^{1,2}(\Omega) \) s.t. \( \forall \varphi \in W_0^{1,2}(\Omega), \bullet \int_{\Omega} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi - \int_{\Omega} b(\nabla u) \varphi \)

Define mapping \( M : W_0^{1,2}(\Omega) \to W_0^{1,2}(\Omega); \varphi \mapsto u \), where \( -\Delta u = f - b(\nabla u) \) in \( \Omega \)

**Fixed point:** \( u = 0 \) on \( \partial \Omega \).

1. winter semester + Nemitskii \( \Rightarrow M \) is continuous
\[\|M(u)\|_{W_0^{1,2}(\Omega)} \leq \int_{\Omega} |f| \leq \int_{\Omega} |f| + \int_{\Omega} |b(\nabla u)| u \leq \int_{\Omega} \frac{1}{2} \|u\|_{W_0^{1,2}(\Omega)}^2 + C \|f\|_{L^2(\Omega)} \]

2. \( \|M(u)\|_{W_0^{1,2}(\Omega)} \leq C \|u\|_{W_0^{1,2}(\Omega)} \)

3. compactness: if \( \{u_n\} \) bounded in \( W_0^{1,2}(\Omega) \) is \( \text{LLC}^0 \) precompact in \( W_0^{1,2}(\Omega) \).
\[\int_{\Omega} \text{v}(u_n - u_m) = \int_{\Omega} \left( b(\nabla u_n) - b(\nabla u_m) \right) v \]
\[\|u_n - u_m\|_{W_0^{1,2}(\Omega)} \leq C \|u_n - u_m\|_{L^2(\Omega)} \]
\( \{u_n\} \) is bounded in \( W_0^{1,2}(\Omega) \) \( \Rightarrow \{u_n\} \) is Cauchy in \( L^2(\Omega) \) \( \Rightarrow \{u_n\} \) Cauchy in \( W_0^{1,2}(\Omega) \)
\( \Rightarrow u_n \to u \) in \( W_0^{1,2}(\Omega) \) \( \Rightarrow M \) is compact

**Schauder** \( \Rightarrow \exists \) a fixed point \( u \)

**Homework:** \( \Omega \subset C^{0,1} \)
\[-\Delta u - \frac{A}{4} u = f \quad \text{in } \Omega \]
\[u = 0 \quad \text{on } \partial \Omega \]

Show that \( \forall f \in L^2(\Omega), f \geq 0 \exists ! u \in W_0^{1,2}(\Omega) \), \( u \geq 0 \)

### 3. **NONLINEAR ELLIPTIC EQUATIONS - MONOTONE OPERATOR THEORY**

#### 3.1 **Motivation**

Find \( \min_{W_0^{1,2}(\Omega), u \geq 0 \text{ on } \partial \Omega} \int_{\Omega} (\nabla u)^p \) for \( u \in W_0^{1,2}(\Omega) \) given

**a)** minimum exists (due to Convexity - will be proven later)

**b)** Euler - Lagrange: \( \exists ! \phi \in W_0^{1,2}(\Omega), \|u\|_{W_0^{1,2}(\Omega)}^p \leq \phi \|u\|_{W_0^{1,2}(\Omega)}^p \)
\( \Rightarrow 0 \leq \phi \int_{\Omega} \nabla u \cdot \nabla \phi \leq \int_{\Omega} (u + \varepsilon \nabla \phi)^p \)

**algebra:** \( z \in \mathbb{R}^d, \quad 1 + z \cdot \varepsilon \nabla \phi - |z|^p \)
\( \leq \varepsilon \int_{\Omega} \frac{z}{dt} |t(z + \varepsilon \nabla \phi) + |(1-t)z|^p \]
\( \leq \varepsilon \int_{\Omega} \frac{z}{dt} \left( (t(z + \varepsilon \nabla \phi) + (1-t)z) \right)^p \]
\( \leq \varepsilon \int_{\Omega} t(z + \varepsilon \nabla \phi) + (1-t)z |\nabla \phi|^p = \varepsilon \int_{\Omega} t(z + \varepsilon \nabla \phi) + (1-t)z \nabla \phi |\nabla \phi|^p \)
\( \leq (t(z + \varepsilon \nabla \phi) + (1-t)z) \nabla \phi |\nabla \phi|^p \)
\[ \begin{align*}
&\text{Definition: Let } E : \mathbb{R}^n \to \mathbb{R}^n \text{ be a mapping. We say that} \\
&\text{1. } E \text{ is monotone } \iff \forall x, y \in \mathbb{R}^n \quad (E(x) - E(y)) \cdot (x - y) \geq 0 \\
&\text{2. } E \text{ is strictly monotone } \iff \forall x, y \in \mathbb{R}^n, x \neq y \quad (E(x) - E(y)) \cdot (x - y) > 0 \\
&\text{Example: } E(x) := (\delta + |x|^2)^{p/2} \cdot x, \quad \delta \geq 0 \\
&\text{E is strictly monotone.}
\end{align*} \]

Proof. \[ \begin{align*}
&\text{Let } f : \mathbb{R}^n \to \mathbb{R}^n, \quad A = \{ x \in \mathbb{R}^n : |x|^2 > 0 \}, \quad B = \{ x \in \mathbb{R}^n : |x|^2 = 0 \}. \\
&\text{Formulation of the problem.}
\end{align*} \]
Weak formulation.

Let $A$ and $B$ be Carathéodory, $\exists c \in \mathbb{R}, c_t \in L^p(\Omega)$ such that
\[
\begin{align*}
|A(x, u, \xi)| & \leq c (1 + |u|^{p-1} + |\xi|^{p-1}) + c_t(x) \\
|B(x, u, \xi)| & \leq c (1 + |u|^{p-1} + |\xi|^{p-1}) + c_t(x)
\end{align*}
\]
$p \in (1, \infty)$, $u \in W^{1,p}(\Omega)$, $\varphi \in L^p(\Gamma_0)$, $\psi \in L^p(\Omega)$ (enough $\psi \in (W^{1,p}(\Omega))^*$).

We say that $\mu \in W^{1,p}(\Omega)$ is a weak solution if $\forall \varphi \in W^{1,p}(\Omega)$, $\varphi \in C_0^\infty(\Omega)$,
\[
\int_\Omega A(x, u(x), \nabla u(x)) \cdot \nabla \varphi(x) + B(x, u(x), \nabla u(x)) \varphi(x) = \int_{\Gamma_0} f(x) \varphi(x) + \int_\Omega g(x) \varphi(x).
\]
Definition is meaningful:

$\mu \mapsto A(\cdot, u, \cdot) \Rightarrow$ growth assumptions on $A +$ Carathéodory by Nemytskii it is continuous mapping from $W^{1,p}(\Omega) \Rightarrow \overrightarrow{L^p(\Omega) \times \ldots \times L^p(\Omega)}$.

$\mu \mapsto B(\cdot, u, \cdot) \in W^{1,p}(\Omega) \Rightarrow L^p(\Omega)$

\[
\int_{\Omega} A(x, u(x), \nabla u(x)) \cdot \nabla \varphi \, d\Omega \leq \infty \quad \text{by Hölder}
\]

Exercise 1.2: Show that if $u, f, A, B, g, \mu_0$ are smooth and $\mu$ is a weak solution,
then it is a classical solution.

Existence (and uniqueness) of weak solution (for $\Gamma_0 = \emptyset$

Assumption (coercivity):
\[
\exists a > 0 \text{ s.t. } A(x, u, \xi) \cdot \xi + B(x, u, \xi) \cdot \xi \geq a|\xi|^p - b(x)
\]

Assumption (monotonicity of the leading term):
For a.a. $x \in \Omega$, $\forall \mu \in \mathbb{R}$, $\forall \xi_1, \xi_2 \in \mathbb{R}^d$: $(A(x, u, \xi_1) - A(x, u, \xi_2)) \cdot (\xi_1 - \xi_2) \geq 0$

$(A(u, \xi))$ is monotone w.r.t. $\xi$

Assumption (strict monotonicity of the leading term):
$A(x, u, \xi)$ is strictly monotone w.r.t. $\xi$

Assumption (the whole operator is monotone):
For a.a. $x \in \Omega$, $\forall u, u_1, u_2 \in \mathbb{R}$, $\forall \xi_1, \xi_2 \in \mathbb{R}^d$: $(A(x, u_1, \xi_1) - A(x, u_2, \xi_2)) \cdot (\xi_1 - \xi_2) + (B(x, u_1, \xi_1) - B(x, u_2, \xi_2)) (u_1 - u_2) \geq 0$

$E : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^{d+1}, \xi = (\xi_1, \ldots, \xi_d)$

\[
(\mu, \xi) \mapsto (B(x, u, \xi_1), A_1(x, u, \xi_1), \ldots, A_d(x, u, \xi_d)) , \quad E \text{ is monotone mapping}
\]

\[
\left(\text{div}(\xi + |\xi|^2)^{\frac{p-2}{2}} \xi = 0\right) \text{ for } p = 1, S = 1, \xi = \nabla u \text{ corresponds to the minimal surface eqn.}
\]
Theorem: Let $\Omega \subseteq \mathbb{R}^d$, $\Omega \in C^1$, $u_0 \in W^{1,p}(\Omega)$, $p \in (1,\infty)$, $A$ and $B$ are Carathéodory and satisfy the growth assumptions $f \in W^{1,p}_0(\Omega)^*$. Then there exists a weak solution $u \in W_0^{1,p}(\Omega)$, $u - u_0 \in W_0^{1,p}(\Omega)$, provided that at least one of the following holds:

a) the whole operator is monotone
b) $A$ is monotone w.r.t. $\xi$ and $B$ depends on $\xi$ linearly
c) $A$ is strictly monotone w.r.t. $\xi$
and, $A$ and $B$ are coercive. Moreover, if the whole operator is strictly monotone, then $\exists! u \in W_0^{1,p}$.

Proof: Uniqueness. Let $u_1, u_2$ be two solutions. Then $\nabla (A(\cdot, u_1, \nabla u) - A(\cdot, u_2, \nabla u_2)) \cdot \nabla \varphi + (B(\cdot, u_1, \nabla u) - B(\cdot, u_2, \nabla u_2)) \varphi = 0$

Set $\psi := u_1 - u_2 \in W_0^{1,p}(\Omega)$

$\nabla (A(\cdot, u_1, \nabla u_1) - A(\cdot, u_2, \nabla u_2)) \cdot (u_1 - u_2) + (B(\cdot, u_1, \nabla u_1) - B(\cdot, u_2, \nabla u_2)) (u_1 - u_2) = 0$

& $M \geq 0 \Rightarrow a.e.$ in $\Omega$ 

Existence.

Step 1. Galerkin approximation (we use fixed point, $A, B$ are Carathéodory & coercive)

Step 2. Uniform estimates (independent of approximation, coercivity is used)

Step 3. Limit passage (monotonicity is used, reflexivity)

Step 4. $W_0^{1,p}$ is separable, $\exists$ linearly independent $\{w_i\}_{i=1}^{\infty}$ dense subset

look for $G \mu \Rightarrow \mu = \mu_0 + \sum_{i=1}^{\infty} \alpha_i \cdot w_i$

Define $F : \mathbb{R}^n \to \mathbb{R}^n$

$[F(x)]_i := \int_{\Omega} A(\cdot, \mu, \nabla w_i) \cdot \nabla w_i + B(\cdot, \mu, \nabla w_i) w_i - \langle f, w_i \rangle$ where $\mu = \mu_0 + \sum_{i=1}^{\infty} \alpha_i \cdot w_i$

look for $\alpha \in \mathbb{R}^n$ s.t. $F(\alpha) = 0$

Lemma: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and $\exists R > 0$ s.t. $\forall \alpha \in \mathbb{R}^n, \|\alpha\| > R : F(\alpha) \notin \mathbb{R}$.

Then $\exists \alpha, \|\alpha\| < R$ s.t. $F(\alpha) = 0$.

(without proof, consequence of Browder fixed point theorem)
Use lemma:

1. F is continuous (because continuous dependence of integrand on $u^i$ and $m^0$ on $\alpha$)
2. $F(\alpha), \alpha = \sum_{i=1}^{2} \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0 = \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0 - \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0$

Hölder

$\geq \sum_{i=1}^{2} \int_{\tau} \left[ \frac{1}{2} \nabla m^0 \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0 \right] \nabla m_i + B(\alpha, m^0) \cdot \nabla m^0 - \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0$

Coercivity

$\geq C_1 \int_{\tau} (\nabla m^0)^p - C - \left( \| A \|_{p \rightarrow \infty} + \| B \|_{p \rightarrow \infty} \right) \| \nabla m^0 \|_{L^p}$

Growth

$\geq C_2 \int_{\tau} (\nabla m^0)^p - C(1 + \| \nabla m_i \|_{L^p}) - \| \nabla m^0 \|_{L^p} - \| \nabla m^0 \|_{L^p} - \| \nabla m^0 \|_{L^p}$

Poincaré

$\geq C_{\text{Poincaré}} \| \nabla m^0 \|_{L^p} - C(1 + \| \nabla m_i \|_{L^p}) - \| \nabla m^0 \|_{L^p} - \| \nabla m^0 \|_{L^p} - \| \nabla m^0 \|_{L^p}$

If $\| \nabla m^0 \|_{L^p} \geq R_0 \Rightarrow \| \nabla m^0 \|_{L^p} \geq R_0 \Rightarrow F(\alpha), \alpha \geq 0.

$\Rightarrow \exists \alpha, F(\alpha) = 0 \Rightarrow \exists \{ \alpha \}$ the Galerkin approximation.

Step 2. Uniform estimates

$\int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0 = \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0$

multiply by $\alpha_i$ and sum $m^0_i$, $i$

$(ET) \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0 = \int_{\tau} A(\alpha, m^0) \cdot \nabla m^0 + B(\alpha, m^0) \cdot \nabla m^0$

repeat Step 1: $\| m^0 \|_{L^p} \leq C(1 + \| m^0 \|_{L^p} + \| \nabla m^0 \|_{L^p}) \leq K_1$

Step 3. Use Nemytskii:

$\| A(\alpha, m^0) \|_{L^p} \leq K_2$

$\| B(\alpha, m^0) \|_{L^p} \leq K_3$

$W^0$ is reflexive, $L^p$ is reflexive ($p \in (1, \infty)$)

find subsequences $m^0 \rightarrow m$ in $W^0(C)$ ($m^0 \in W^0(C)$)

$A(\alpha, m^0) \rightarrow \bar{A}$ in $L^p$ ($\bar{A}$)

$B(\alpha, m^0) \rightarrow \bar{B}$ in $L^p$ ($\bar{B}$)
\[ \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m) + B_i(u^n,\eta^m)v^n_i = <f_i,w_i> \quad \text{for some } i \]

\[ \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m) + B_i(u^n,\eta^m)v^n_i = <f_i,w_i> \quad \forall i \in \mathbb{N} \quad \text{and } \{f_i,w_i\} \text{ is dense in } W^{1,p}_0 \]

\[ \sum_{i \in \mathbb{N}} \bar{A}_i u^n + \bar{B}_i w^n_i = <f_i,w_i> \quad \forall w \in W^{1,p}_0(S^2) \]

\[ u^n \rightharpoonup u \quad \text{in } W^{1,p}_0(S^2), \quad \lambda_i(u^n,\eta^m) \rightarrow \lambda \text{ in } L^1(\Omega_1,\mathbb{R}^d) \]

\[ B(u^n,\eta^m) \rightarrow B \text{ in } L^p(S^2) \]

Step 3a: We show that \[ \lim_{n \to \infty} \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m)\eta^n_i + B_i(u^n,\eta^m)\eta^m_i = \sum_{i \in \mathbb{N}} \bar{A}_i u^n + \bar{B}_i w^n \]

we are able to interchange the limit and the product of 2 weakly converging seq.

Proof: \[ \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m)\eta^n_i + B_i(u^n,\eta^m)\eta^m_i = \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m) + B_i(u^n,\eta^m) \]

\[ = <f_1 u^n - \eta^m_0> + \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m)\eta^n_i + B_i(u^n,\eta^m)\eta^m_i \]

\[ \rightarrow <f_1 u^n - \eta^m_0> + \sum_{i \in \mathbb{N}} \bar{A}_i u^n + \bar{B}_i w^n \]

\[ \sum_{i \in \mathbb{N}} \bar{A}_i u^n + \bar{B}_i w^n = \sum_{i \in \mathbb{N}} \left( \bar{A}_i (u^n - \eta^m_0) + \bar{B}_i (w^n - \eta^m_0) + \sum_{i \in \mathbb{N}} \bar{A}_i \eta^n_i + \bar{B}_i \eta^m_i \right) \]

set \( \bar{w} = u^n - \eta^m_0 \)

Step 3b: We will show that \[ \lim_{n \to \infty} \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m)\eta^n_i = \sum_{i \in \mathbb{N}} \bar{A}_i u^n \]

Proof: \[ \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m)\eta^n_i = \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m)\eta^n_i + \sum_{i \in \mathbb{N}} B_i(u^n,\eta^m)\eta^m_i = \sum_{i \in \mathbb{N}} \bar{A}_i u^n + \bar{B}_i w^n \]

Step 3c. I want to identify \( \bar{A} \) & \( \bar{B} \), monotonicity comes into game.

Case 1. The whole operator is monotone

Take arbitrary \( V \in L^p(\Omega_1,\mathbb{R}^d) \), \( \eta^m \in L^p(S^2) \)

\[ 0 \leq \sum_{i \in \mathbb{N}} \left( A_i(u^n,\eta^m) - A_i(\eta^m,\eta^m) \right) + \left( B_i(u^n,\eta^m) - B_i(\eta^m,\eta^m) \right) \]

\[ \rightarrow \lim_{n \to \infty} \sum_{i \in \mathbb{N}} A_i(u^n,\eta^m) - A_i(\eta^m,\eta^m) + \lim_{n \to \infty} \sum_{i \in \mathbb{N}} B_i(u^n,\eta^m) - B_i(\eta^m,\eta^m) \]

3a+ weak conv: \[ = \sum_{i \in \mathbb{N}} A_i(u^n) - A_i(\eta^m) + \sum_{i \in \mathbb{N}} B_i(u^n) - B_i(\eta^m) \]

\[ = \sum_{i \in \mathbb{N}} \left( \bar{A}_i u^n + \bar{B}_i w^n \right) \]

\[ = \sum_{i \in \mathbb{N}} \left( \bar{A}_i u^n + \bar{B}_i w^n \right) \left( \eta^m - \eta^m_0 \right) \]

\[ = \sum_{i \in \mathbb{N}} \left( \bar{A}_i u^n + \bar{B}_i w^n \right) \left( \eta^m - \eta^m_0 \right) \]

\[ = \sum_{i \in \mathbb{N}} \left( \bar{A}_i u^n + \bar{B}_i w^n \right) \left( \eta^m - \eta^m_0 \right) \]

\[ = \sum_{i \in \mathbb{N}} \left( \bar{A}_i u^n + \bar{B}_i w^n \right) \left( \eta^m - \eta^m_0 \right) \]

\[ = \sum_{i \in \mathbb{N}} \left( \bar{A}_i u^n + \bar{B}_i w^n \right) \left( \eta^m - \eta^m_0 \right) \]
Minty knocks: \( V = \Omega - \varepsilon W \) \( \varepsilon > 0 \) \( W \in L^p(\Omega, \mathbb{R}^d) \)
\( N = \mu - \varepsilon W \) \( \mu \in L^p(\Omega) \)

\[ (\varepsilon^0) S \left( (A - A(u, u), \eta_{\varepsilon} - EW) \right) W + (B - B(u, \eta_{\varepsilon} + \varepsilon W)) N \]

\[ \Rightarrow \quad \sum_{\eta_{\varepsilon}} S \left( (A - A(u, u), \eta_{\varepsilon} - EW) \right) W + (B - B(u, \eta_{\varepsilon})) N \]

\[ \Rightarrow \quad W := - \frac{A - A(u, u)}{1 + \|A - A(u, u)\|} \quad N := - \frac{B - B(u, \eta_{\varepsilon})}{1 + \|B - B(u, \eta_{\varepsilon})\|} \]

\[ \Rightarrow \quad \sum_{\eta_{\varepsilon}} \frac{|A - A(u, u)|^2}{1 + |A - A(u, u)|} + \frac{|B - B(u, \eta_{\varepsilon})|^2}{1 + |B - B(u, \eta_{\varepsilon})|} \leq 0 \quad \& \text{ integrand } \geq 0 \]

Case 2. \( A \)-monotone \( \& \) \( B \)-linear w.r.t. \( \eta_{\varepsilon} \)

Identification of \( \tilde{A} \) (use step 3b) \( \forall \Psi \in L^p(\Omega, \mathbb{R}^d) \) arbitrary

\[ \sum_{\eta_{\varepsilon}} S \left( (A(u, u) - A(u, V)) \right) \Psi = 0 \]

\[ \Rightarrow \quad \sum_{\eta_{\varepsilon}} S \left( (A(u, u) - A(u, V))\Psi \right) = \sum_{\eta_{\varepsilon}} \left( A(u, u) - A(u, V) \right) \Psi \rightarrow 0 \quad \text{a.e.} \]

Vitali \( \forall \varepsilon \rightarrow 0 \) \( \exists \beta \) \( \forall u \in L^1 \) \( \exists \delta \in L^1 \) \( \sum_{\eta_{\varepsilon}} |A(u, V) - A(u, V)| \leq \varepsilon \)

Use growth assumptions on \( A \)

\[ \sum_{\eta_{\varepsilon}} |A(u, V) - A(u, V)| \leq \sum_{\eta_{\varepsilon}} C \left( |u|^p + |V|^p \right) \quad G \in L^{1}(\Omega) \Rightarrow A(u, V) \sim |u|^p; |V|^p \]

\[ \Rightarrow \quad \sum_{\eta_{\varepsilon}} \left( \sum_{\eta_{\varepsilon}} |u|^p + |V|^p \right) \leq \sum_{\eta_{\varepsilon}} S \left( C |u|^p + |V|^p + G \right) \]

\[ \Rightarrow \quad \sum_{\eta_{\varepsilon}} |u|^p \leq C \left( \sum_{\eta_{\varepsilon}} |V|^p + G \right) \]

\[ \Rightarrow \quad \sum_{\eta_{\varepsilon}} |V|^p \leq \frac{C}{\varepsilon} \sum_{\eta_{\varepsilon}} |u|^p \]

\[ \Rightarrow \quad \delta \varepsilon \leq \frac{C}{\varepsilon} \Rightarrow \quad \sum_{\eta_{\varepsilon}} |A(u, V)| \leq \delta \varepsilon \]

Vitali ok

\[ \Rightarrow \quad 0 \leq \sum_{\eta_{\varepsilon}} \left( (A - A(u, u), \eta_{\varepsilon} - EW) \right) \Psi \quad \forall \Psi \in L^p(\Omega, \mathbb{R}^d) \]

\[ \Rightarrow \quad \tilde{A} = A(u, u) \]

homework: if \( B \) is linear w.r.t. \( \eta_{\varepsilon} \), \( B(u, \eta_{\varepsilon}) = \sum_{i=1}^{d} b_i(u) \eta_{\varepsilon} \]

\[ \Rightarrow \quad \tilde{b} \left( \sum_{i=1}^{d} b_i(u) \right) \in L^1 \quad \tilde{b} \left( \sum_{i=1}^{d} b_i(u) \right) \in L^p \]
Case 3. A is strictly monotone but B is general

We show that $u^n \rightarrow u$ a.e.

homework: $u^n \rightarrow u$ a.e.

\[ \mu^n \rightarrow \mu \text{ in } W^{1,\infty}(\Omega) \]

\[ \Rightarrow \quad B(u^n, \nu^n) \rightarrow B(u, \nu) \text{ in } L^q \quad \forall \, q < p' \]

We know $\bar{A} = A(u, \nu)$

\[
\begin{align*}
0 \leq & \int (A(u^n, \nu^n) - A(u^n, \nu)) \cdot (\nu^n - \nu) \\
& \rightarrow \int (\bar{A} - A(u^n, \nu)) \cdot (\nu^n - \nu) \rightarrow 0 \quad \text{strongly in } L^1(\Omega)
\end{align*}
\]

\[
\forall \, \varepsilon > 0 \quad \exists \Omega \varepsilon, \lambda_1, \lambda_2 > 0 \quad \text{uniformly in } \Omega
\]

\[ u^n \rightarrow u \text{ uniformly} \]

\[ \Rightarrow \quad (A(u^n, \nu^n) - A(u, \nu)) \cdot (\nu^n - \nu) \text{ uniformly} \]

\[ \forall x \in \Omega \varepsilon \quad (A(u^n(x), \nu^n(x)) - A(u(x), \nu(x))) \cdot (\nu^n(x) - \nu(x)) \rightarrow 0 \quad \text{uniformly in } \Omega \varepsilon \]

Assume $\nu^n(x) \neq \nu(x)$. Then because $A$ is STRICTLY MONOTONE

\[ \lim (A(u^n(x), \nu^n(x)) - A(u(x), \nu(x))) \cdot (\nu^n(x) - \nu(x)) > 0 \quad \text{contradiction} \]

Example:

\[-\text{div} (\text{arctg} (1 + \text{sign}^2) \nu u) + \mu^{1,2} = f \quad \text{in } B_{1,0} \subseteq \mathbb{R}^3 \]

\[ \text{arctg}(1 + \text{sign}^2) \nu u \cdot n = 0 \quad \text{on } \partial B_{1,0} \]

Let $f = f_1 + f_2$, where $f_1 \in L^{\frac{24}{15}}(\Omega)$, $f_2 \in (W^{1,2}(\Omega))^*$

\[ \exists \, \nu \in W^{1,2}(\Omega) \cap L^{2,2}(\Omega) \quad \text{s.t.} \quad (\nu, f_1)_{W^{1,2}(\Omega)} + (\nu, f_2)_{W^{1,2}(\Omega)} = 0 \]

\[ A(u, \nu) = \text{arctg}(1 + \text{sign}^2) \nu u \]

\[ B(u, \nu) = \mu^{1,2} \]

A priori estimates

Set $\nu := \mu^{1,2}$

\[\begin{align*}
\|u\|_{W^{1,2}} + \|\nu\|_{W^{1,2}} & \leq \int \text{arctg}(1 + \text{sign}^2) |\nu u|^2 + \mu^{1,2} \\
& \leq \int (f_1 |\nu u|^2 + f_2 |\nu u|^2) \\
& \leq \epsilon (\|u\|_{W^{1,2}}^2 + C(\epsilon) \|f_1 \|_{W^{1,2}}^2 + C(\epsilon) \|f_2 \|_{W^{1,2}}^2) \\
& \leq 2 \epsilon (\|u\|_{W^{1,2}}^2 + \epsilon |\nu u|^2) + C(\epsilon) \left( \|f_1 \|_{W^{1,2}}^2 + \|f_2 \|_{W^{1,2}}^2 \right) \\
& \Rightarrow \|u\|_{W^{1,2}} + \|\nu\|_{W^{1,2}} \leq C (\|f_1 \|_{W^{1,2}} + \|f_2 \|_{W^{1,2}}) \]
\]
Galerkin \[ \{ u_i \}_{i=1}^m \] dense in \( W^{1,2}(\Omega) \cap L^{12n}(\Omega) \)
\[
\begin{align*}
\sum_{i=1}^n \alpha_i^n u_i + \int \mathcal{A}(\omega^n) u_i + B(u^n) u_i & = \int f_i u_i + \langle f_i, u_i \rangle \quad i = 1, \ldots, n \\
\sum_{i=1}^m u_i & \in W^{1,2} \\
\sum_{i=1}^m u_i & \in L^{12n} \\
A(\omega^n) & \rightarrow \overline{A} \; \text{ in } L^2 \\
B(u^n) & \rightarrow \overline{B} \; \text{ in } L^{12n} \\
\end{align*}
\]

since \( u \in W^{1,2}(\Omega) \cap L^{12n} \) it can be set \( \omega := u \)
\[
\lim \int A(\omega^n) u_i + B(u^n) u_i \rightarrow \int \overline{A} u_i + \overline{B} u_i 
\]

because \( \langle \xi_i, \eta_i \rangle - \langle \xi_2, \eta_2 \rangle = \langle \xi_i - \xi_2, \eta_i \rangle \) \( \langle B(u_1) - B(u_2) \rangle u_i = 0 \) (check at home)

we use Minty to get \( \overline{A} = \mathcal{A}(\omega) \) \( \overline{B} = B(\omega) \)

Example. Let \( W^{1,0}_0(\Omega) \) be equipped with \( \| u \|_{W^{1,0}} := \| \nabla u \|_p \) \( \Omega \) open, bounded, \( p \in (1, \infty) \)
Then \( \forall f \in (W^{1,0}_0(\Omega))^* \exists \ f \in L^p(\Omega; \mathbb{R}^d) \) s.t. \( \| f \|_{W^{1,0}} = \| f \|_{W^{1,0}'} \)
\[
\forall \xi \in C^\infty_c(\Omega) \; \langle f, \xi \rangle = -\int f \cdot \nabla \xi \quad \Rightarrow \quad \text{div} \ f = F \; \text{ in weak sense}
\]

use theorem \( \forall f \in (W^{1,0}_0(\Omega))^* \exists ! \ u \in W^{1,0}_0(\Omega) \)
\[
\sum_{i=1}^m \int \mathcal{A}(\omega^n) u_i + B(u^n) u_i = \langle f_i, u_i \rangle 
\]

\[
\| u \|_{W^{1,0}}^p \leq \| F \|_{(W^{1,0}_0)^*} \| u \|_{W^{1,0}_0} = \| F \|_{W^{1,0}'} \| u \|_{W^{1,0}}^p 
\]

Homework: \( \forall f \in (W^{1,0}_0(\Omega))^* \exists ! \ f \in L^p(\Omega; \mathbb{R}^d) \; \forall g \in L^p(\Omega): \text{div} \ f + g \)
\[
\| f \|_{W^{1,0}'} = \| f \|_{W^{1,0}}^p + \| g \|_{W^{1,0}}^p \; \text{then the representation is unique} \)
4. MINIMIZATION OF (CONVEX) FUNCTIONALS AND ITS RELATION TO MONOTONE OPERATOR THEORY

Given $F: \Omega \subset \mathbb{R}^d \rightarrow \mathbb{R}$, given $f: \Omega \rightarrow \mathbb{R}$

$$\min_{u \in \Omega} \int_{\Omega} F(u, \nabla u) \, dx$$

Assumptions: 1. $F$ is Carathéodory

2. $F(\xi \mathbf{1}) \geq c_1 |\xi|^p - c_2(x)$

3. $f \in W^{1,p}_0(\Omega)$

Theorem: Let 1-3. hold. Let $F$ be convex w.r.t. $\xi$. Then $\exists u \in W^{1,p}_0(\Omega)$

$$\int_{\Omega} F(u, \nabla u) \, dx \leq \int_{\Omega} F(u_0, \nabla u_0) \, dx$$

Proof:

Fundamental theorem in calculus of variations

$I$ $u \in W^{1,p}_0(\Omega)$

$$I : \inf_{u \in W^{1,p}_0(\Omega)} \int_{\Omega} F(u, \nabla u) \, dx - \int_{\Omega} F(u_0, \nabla u_0) \, dx$$

$\forall n \in \mathbb{N}$

$$\int_{\Omega} F(u_n, \nabla u_n) \, dx \leq \int_{\Omega} F(u_0, \nabla u_0) \, dx + 1 < \infty$$

Young, Poincaré

Reflexivity $\Rightarrow$ Existence of a subsequence $u_n \rightarrow u$ in $W^{1,p}_0(\Omega)$

Compact embedding $W^{1,p}_0(\Omega)$ in $L^p(\Omega)$

Theorem: Let $z \rightarrow z$ in $L^p(\Omega)$ and $x_n \rightarrow x$ in $L^p(\Omega)

Let $F(x, \xi)$ be Carathéodory and convex w.r.t. $\xi$:

$$F(x, \xi) \geq c \langle \nabla \rangle F(x, \xi)$$

and $F(x, 0) \equiv 0$ in $L^p(\Omega)$.

Then

$$\int_{\Omega} F(x, \nabla u) \, dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} F(x, \nabla u_n) \, dx$$

This property is called weak lower semicontinuity of convex functionals.

Use WLC:

$$I : \lim_{n \rightarrow \infty} \int_{\Omega} F(u_n, \nabla u_n) \, dx - \int_{\Omega} F(u, \nabla u) \, dx$$

Then $u$ is a minimizer.

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Proof: (only if $\frac{\partial F(x, \xi)}{\partial \xi} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is Carathéodory)

Lemma: Let $F: \mathbb{R}^n \rightarrow \mathbb{R}$ and $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be continuous, and $A(\xi) = \frac{\partial F(\xi)}{\partial \xi}$

Then 1. $F$ is (strictly) convex $\Rightarrow$ $A$ is (strictly) monotone

2. $F(\xi_1) - F(\xi_2) \geq A(\xi_2) : (\xi_1 - \xi_2)$ for $F$ convex
Proof of lemma: \(\mu, w \in \mathbb{R}^n\) arbitrary, define

\[
\psi_{t, w}(t) := F(u + t \cdot w) \quad \text{for} \quad t \in \mathbb{R}
\]

\[
\Rightarrow \quad \psi'_{t, w}(t) = \frac{dF(u + t \cdot w)}{dt} \quad \text{for} \quad t \in \mathbb{R}
\]

where \(\mu = A(u + t \cdot w) \cdot \nabla
\]

1. \(\Rightarrow\quad \text{if } \mu + 0 \text{ and } F \text{ strictly convex}
\]

\[
\psi_{w}(1) - \psi_{w}(0) \geq 0 \quad \text{strict } "\geq" \text{ if } \mu + 0 \text{ and } F \text{ strictly convex}
\]

\[
(A(u + \mu) - A(u)), \mu \geq 0 \quad \text{or } \mu > 0 \text{ if } \mu + 0
\]

\[
\eta := \eta - \mu \Rightarrow (A(u + \mu) - A(u)), (\eta - \mu) \geq 0 \quad \text{or } \mu > 0 \text{ if } \mu + \eta \text{ and } F \text{ strictly convex}
\]

2. \(\Rightarrow\quad \text{take } t_1 \geq t_2
\]

\[
\psi_{w}(t_1) - \psi_{w}(t_2) = (A(u + t_1 \cdot w) - A(u + t_2 \cdot w)) \cdot \nabla
\]

\[
= (A(u + t_1 \cdot w) - A(u + t_2 \cdot w)), (A(u + t_1 \cdot w) - A(u + t_2 \cdot w)) \geq 0 \quad \text{A strictly mon.}
\]

\[
F(u + t \cdot w) - F(u) = \int_t^1 \psi'_{w}(t) dt \geq \int_0^1 \psi'_{w}(t) dt = \psi'_{w}(t) - A(u) \cdot \nabla
\]

\[
\text{set } \eta := \eta - \mu \Rightarrow \int_0^1 \psi'_{w}(t) dt \text{ if } \eta + 0
\]

\[
F(u) - F(u) \geq (A(u), (\eta - \eta)) \quad (> 0 \text{ if } \eta + \mu)
\]

\[
\Rightarrow (2)
\]

Let \(\xi_1, \xi_2 \in \mathbb{R}^n\), \(\lambda \in (0,1)\), \(z := \lambda \xi_1 + (1-\lambda) \xi_2\)

\[
\Rightarrow \quad \text{we want } \quad F(z) \leq \lambda F(\xi_1) + (1-\lambda) F(\xi_2)
\]

\[
\text{use } (2) \text{ with } u := z, \eta := \xi_1 \Rightarrow F(\xi_1) - F(z) \geq (A(z), (\xi_1 - z)) \quad 1. \lambda \quad \{+ \}
\]

\[
\Rightarrow \quad (A(z), (\xi_1 - z)) \geq (A(z), (\xi_1 - \lambda \xi_1 + (1-\lambda) \xi_2))
\]

\[
= (A(z), (\xi_1 - \lambda \xi_1 + (1-\lambda) \xi_2)) + (1-\lambda) A(z), (\xi_2 - \lambda \xi_1 + (1-\lambda) \xi_2))
\]

\[
= (A(z), (\xi_1 - \lambda \xi_1 + (1-\lambda) \xi_2)) + (A(z), (\xi_2 - \xi_1)) = 0
\]

Continuation of the proof of W-L-S

Step 2: We have \(F(x, z^0(x), \xi^0(x)) - F(x, z^0(x), \xi(x)) \geq A(x, z^0(x), \xi(x)) \text{ a.e. in } \Omega\)

\[
A(x, z^0(x)) = \frac{dF(x, z^0(x))}{d\xi^0(x)}
\]

\[
\forall \xi \in \Omega \quad 0 < \|\xi\| \leq \varepsilon \quad z^0 \to z \quad \text{uniformly in } \Omega
\]

\[
\Rightarrow \quad \int_{\Omega} F(z, z^0(x), \xi^0(x)) - F(z, z^0(x), \xi(x)) \geq \int_{\Omega} F(z, z^0(x), \xi^0(x)) - F(z, z^0(x), \xi(x)) \geq \int_{\Omega} F(z, z^0(x), \xi^0(x)) - F(z, z^0(x), \xi(x))
\]

\[
= \int_{\Omega} F(z, z^0(x), \xi^0(x)) - F(z, z^0(x), \xi(x)) \geq \int_{\Omega} A(z, z^0(x), \xi^0(x)) \cdot (\xi^0 - \xi) + \int_{\Omega} C(x)
\]

\[
\geq \int_{\Omega} F(z, z^0(x), \xi^0(x)) - F(z, z^0(x), \xi(x)) \geq \int_{\Omega} A(z, z^0(x), \xi^0(x)) \cdot (\xi^0 - \xi) + \int_{\Omega} C(x), \text{ take limit}
\]
\[
\liminf_{\varepsilon \to 0} F(z_1, z_2, \varepsilon) \geq \liminf_{\varepsilon \to 0} \left( \int_{\mathcal{S}_{\varepsilon}} F(z_1, z_2, \varepsilon) - c(x) \right) + \int_{\mathcal{S}_{\varepsilon}} A_1(z_1, \varepsilon, x) - (\varepsilon^2 x, \varepsilon) + c(x) \nabla x \cdot \varepsilon \nabla x \right)

\text{For } \varepsilon \to 0 \text{ uniformly, } A_0 \text{ bounded in } \mathcal{S}_{\varepsilon} \Rightarrow A_0(z_1, \varepsilon, x) \to A_0(z_1, x) \text{ in } L^1(\mathcal{S}_{\varepsilon})

\]

\[\int_{\mathcal{S}_{\varepsilon}} F(z_1, z_2, \varepsilon) - c(x) \geq \int_{\mathcal{S}_{\varepsilon}} A_1(z_1, \varepsilon, x) - (\varepsilon^2 x, \varepsilon) + c(x) \nabla x \cdot \varepsilon \nabla x \right) \text{ as } \varepsilon \to 0^+

\text{Example.} \quad F(u, \varepsilon) = a(u) |\varepsilon|^2 \quad \text{for } C^1(\mathbb{R}) \quad 0 < C_1 = a(s) \leq C_2 < \infty

\text{Minimize } \min_{u \in W^{1,2}(\Omega)} \int_{\Omega} F(u, \nabla u) \quad F \geq C_1 |\nabla u|^2 \quad F \text{ is convex w.r.t. } \varepsilon

\text{use theorem, } \exists u_0 \in W^{1,2}(\Omega) \quad \nabla u_0 \in L^2(\Omega) \quad \int_{\Omega} a(u) |\nabla u|^2 \leq \int_{\Omega} a(u_0) |\nabla u_0|^2.

\text{Set } u = u_0 + \varepsilon \phi \quad \phi \in C_c^{\infty}(\Omega) \quad \text{or } \phi \in W^{1,2}(\Omega) \cap L^\infty(\Omega)

\int_{\Omega} a(u) |\nabla u|^2 \leq \int_{\Omega} a(u_0 + \varepsilon \phi) (\varepsilon^2 |\nabla \phi|^2 + 2 \varepsilon \nabla u_0 \cdot \nabla \phi) - \int_{\Omega} \varepsilon \nabla \phi \cdot \nabla \phi + \frac{\varepsilon}{2} \int_{\Omega} a(u_0 + \varepsilon \phi) a(u) |\nabla u|^2

\text{Maximizing } \int_{\Omega} a(u_0 + \varepsilon \phi) a(u) |\nabla u|^2 \text{ in } \varepsilon \Rightarrow u \text{ solves in weak sense}

\text{Remark. Being a minimizer is much stronger than being a weak solution.}

\text{(In case that } F \text{ depends on } u\text{)}

\text{Example (minimization with constraint)}

\[\min_{u \in S} \int_{\Omega} \frac{|\nabla u|^2}{2} - fu \]

\[S = \{ u \mid u = 1 \text{ on } \partial \Omega, \quad u > 0 \text{ in } \Omega \}

\text{I = } \inf_{u \in S} \int_{\Omega} \frac{|\nabla u|^2}{2} - fu = \lim_{\varepsilon \to 0} \int_{\Omega} \frac{|\nabla u|^2}{2} - fu \quad u^n \text{ is a bounded sequence in } S \]

\[\text{\text{Minimize in } } W^{1,2}(\Omega), \quad u^n \text{ in } (H^1(\Omega))^2, \quad u^n \text{ in } (L^2(\Omega))^2 \}

\text{\text{Minimize in } } W^{1,2}(\Omega), \quad u^n \text{ in } (H^1(\Omega))^2, \quad u^n \text{ in } (L^2(\Omega))^2 \}

\Rightarrow u \in S \Rightarrow u \text{ is a minimizer
Uniqueness: \[ I = \sum_{i=1}^{n} \frac{1}{2} \left( \frac{2mu_i}{z} - f(u_i) \right) < \sum_{i=1}^{n} \frac{1}{2} - f(u) \leq I \]

\[ \varphi \in S \quad (1-\lambda)u + \lambda v \in S \quad \lambda \in (0,1) \]

Choose \( N = m + \varphi \) with \( \varphi \geq 0 \) and \( \varphi \in W_0^{1,2} \)

\[ \int_{\Omega} \varphi \varphi \leq \sum_{i=1}^{N} \Delta u \varphi \quad \varphi \geq 0 \quad \varphi \in C_0^\infty (\Omega) \quad \text{(formally)} \]

\[ \Rightarrow \varphi \leq -\Delta u \quad \text{in weak sense} \]

Assume that \( \Omega \in \mathcal{L} \) is open and \( u > E_0 \) in \( \Omega \)

Set \( \mu := m + \varphi \quad \varphi \in C_0^\infty (\Omega) \quad \varphi \geq 1 \) and \( \mu \geq E_0 \) (\( \varphi \) can be negative)

\[ \int_{\Omega} \varphi \varphi \leq \sum_{i=1}^{N} \Delta u \varphi \quad \varphi \in C_0^\infty (\Omega) \quad \varphi \geq 1 \]

\[ \Rightarrow \int_{\Omega} \varphi \varphi = \sum_{i=1}^{N} \Delta u \varphi \quad \Rightarrow \varphi \leq -\Delta u \in \Omega \]

Formally either \( \mu = 0 \) or \( \mu > E_0 \) \( \Rightarrow \) \( -\Delta u \leq \varphi \)

\[ \mu \geq 0 \quad \Rightarrow -\Delta u = \varphi \quad \text{in } \Omega \]

**MONOTONE OPERATOR THEORY (2) - IN CASE THE POTENTIAL EXISTS**

\[ -\text{div} A(u,\nabla u) + B(u,\nabla u) = f \quad \text{in } \Omega \]

\[ u = 0 \quad \text{on } \partial \Omega \]

The aim is: Is there some \( F(u,\nabla u) \) for which minimization of \( F(u,\nabla u) - f\mu \) gives a solution to (M02)?

Here \( A, B \) are Caratheodory.

**Lemma: Heuristic:**

\[ \Psi(t) = \int_{\Omega} F(u + t\varphi, \nabla u + t\nabla \varphi) - f(u + t\varphi) \]

\( \mu \) minimizer \( \Rightarrow \Psi(0) = 0 \)

\[ \Psi(0) = \int_{\Omega} \frac{dF}{dx}(u, \nabla u), \nabla \varphi + \frac{dF}{dx}(u, \nabla u) \nabla \varphi - f \varphi \]

We need at least \( A(u, \xi) = \frac{dF}{dx}(u, \xi) \) and \( B(u, \xi) = \frac{dF}{dx}(u, \xi) \)
Lemma: Let $A(u,\xi)$ and $B(u,\xi)$ be $C^1$. Then the following is equivalent:

1. $F$ such that $\frac{\partial F}{\partial \xi}(u,\xi) = A(u,\xi)$, $\frac{\partial F}{\partial u}(u,\xi) = B(u,\xi)$

2. $\forall i \neq j \frac{\partial A}{\partial \xi_i}(u,\xi) = \frac{\partial A}{\partial \xi_j}, \frac{\partial B}{\partial \xi_i}(u,\xi) = \frac{\partial A}{\partial u}(u,\xi)$

Proof:

2. is necessary: if $F$ exists then $\frac{\partial}{\partial \xi}(\frac{DF}{\partial \xi}) = \frac{\partial}{\partial \xi}(\frac{DF}{\partial u}) \Rightarrow \frac{\partial A}{\partial \xi} = \frac{\partial A}{\partial u} A_j$

2. is sufficient.

1. define $F(u,\xi) = \int_0^1 A(tu,\xi) \cdot \xi \, dt$ I want to show that $\frac{\partial F}{\partial \xi} = A, \frac{\partial F}{\partial u} = B$

$\frac{\partial F}{\partial \xi} = \int_0^1 \frac{\partial A}{\partial \xi}(tu,\xi) \cdot \xi \, dt \Rightarrow \int_0^1 \frac{\partial A}{\partial u}(tu,\xi) \cdot \xi \, dt$

$\frac{\partial F}{\partial u} = \int_0^1 \frac{\partial A}{\partial u}(tu,\xi) \cdot \xi \, dt$

Theorem: Let $A, B$ be Carathéodory and $\frac{\partial A}{\partial \xi} = \frac{\partial A}{\partial \xi}$ and $\frac{\partial B}{\partial \xi} = \frac{\partial A}{\partial u}$.

Let $A(w_0) = 0$, $A$ be monotone w.r.t. $\xi$ and $A(u,\xi) \leq C(1+|\xi|^p)^{-1}$, $A(u,\xi) \xi = c_1 |\xi|^p - c_2$, and $1B(u,\xi) \leq C(1+|\xi|^p)$ and $\text{cl u}^p$

Then $\forall f \in (W_0^p(\Omega))^* \exists u \in W_0^p(\Omega)$ such that

$\int_\Omega A(u,\xi) \, d\xi + B(u,\xi) \, d\xi = \langle f, \xi \rangle \quad \forall f \in W_0^p(\Omega) \cap L^\infty(\Omega)$

Proof: 1. we know $\exists F$ such that $\frac{\partial F}{\partial \xi} = B, \frac{\partial F}{\partial u} = A$

2. $A$ is monotone w.r.t $\xi$ $\Rightarrow F$ is convex w.r.t $\xi$

3. to check that $F$ is coercive, $F(u,\xi) = C_1 |\xi|^p - C_2$

assume 3. is true.

$\min_{u \in W_0^p(\Omega)} \int_\Omega F(u,\xi) \, d\xi = \langle f, \xi \rangle$ $\Rightarrow$ minimizing $F(u,\xi)$

$F(0) < \infty \Rightarrow$ infimum $< \infty$
Take \( \phi \in C^0_c(\Omega), \epsilon > 0 \)

\[
\int_{\Omega} F(\phi, u) - \langle f, u \rangle \leq \int_{\Omega} F(u + \epsilon \phi, u + \epsilon \phi) - \langle f, u + \epsilon \phi \rangle \quad \epsilon > 0
\]

\[
\langle f, u \rangle \leq \int_\Omega \frac{\partial F}{\partial u}(u, u) \phi + \frac{\partial F}{\partial \phi}(u, u) : \nabla \phi
\]

\[
\int_{\Omega} A(u, u) : \nabla \phi + B(u, u) \phi
\]

Proof of 3. We want \( F(u, \xi) \geq c_1 |\xi|^p - c_2 \)

\[
F(u, \xi) - F(0, 0) = F(u, \xi) - F(u, 0) + F(u, 0) - F(0, 0)
\]

\[
= \int_0^1 \frac{d}{dt} F(u, t\xi) + \frac{d}{dt} F(tu, 0) dt
\]

\[
= \int_0^1 A(u, t\xi) \cdot t\xi + \frac{1}{t} B(tu, 0) tu \, dt \geq \int_0^1 A(u, \xi) \cdot \xi \, dt
\]

\[
= \frac{1}{t} \int_0^1 A(u, t\xi) \cdot t\xi \, dt
\]

\[
= \frac{1}{t} \left( c_1 |\xi|^p - c_2 \right) \, dt \geq c_1 |\xi|^p - c_2
\]

Theorem. Let \( F, A, B \) be as in previous and satisfy the same assumption, in addition

\[
(Al_{\infty, \xi_1}) - A(u_2, \xi_2); \xi_1 - \xi_2 + (Bl_{\infty, \xi_1}) - B(u_2, \xi_2); u_1 - u_2 \geq 0.
\]

Then every weak solution is a minimizer.

Proof. If \( F \) is convex w.r.t. all variables (thanks to \( L \)).

\[
\Rightarrow F(u_1, \xi_2) - F(u_1, \xi_1) \geq \frac{\partial F}{\partial \xi_1}(u_1, \xi_1) \cdot (\xi_2 - \xi_1) + \frac{\partial F}{\partial u_1}(u_1, \xi_1) \cdot (u_2 - u_1)
\]

\[
= A(u_1, \xi_1) \cdot (\xi_2 - \xi_1) + B(u_1, \xi_1) \cdot (u_2 - u_1)
\]

\[
\forall u_1, u_2 \in \mathbb{R}, \xi_1, \xi_2 \in \mathbb{R}^d
\]

\( \omega \in C^0_c(\Omega) \)

Set \( u_2 := \omega \quad u_1 := T_k(u) \quad \text{where} \quad T_k(u) = \text{sign } \min \{1, k\}

\[
\xi_2 := \omega \quad \xi_1 := \sigma T_k(u)
\]

\( u \) is a weak solution

\[
\int_\Omega F(\omega, u) - F(T_k(u), \sigma T_k(u)) \geq \int_\Omega A(T_k(u), \sigma T_k(u)) \cdot (\omega - \sigma T_k(u)) + B(T_k(u), \sigma T_k(u)) (\omega - T_k(u))
\]

\[
= \int_\Omega A(T_k(u), \sigma T_k(u)) \cdot \omega + B(T_k(u), \sigma T_k(u)) \sigma T_k(u) - \sigma \int_\Omega A(T_k(u), \sigma T_k(u)) \cdot \sigma T_k(u) + B(T_k(u), \sigma T_k(u)) T_k(u)
\]

\[
(\omega - T_k(u)) \geq \int_\Omega F(\omega, u) 
\]

\[
\int_\Omega \inf_{k \to \infty} \left[ \int_\Omega F(T_k(u), \sigma T_k(u)) + \int_\Omega A(T_k(u), \sigma T_k(u)) \cdot \omega + B(T_k(u), \sigma T_k(u)) \sigma T_k(u) 
\]

\[
- \inf_{k \to \infty} A(T_k(u), \sigma T_k(u)) \cdot \sigma T_k(u) + B(T_k(u), \sigma T_k(u)) T_k(u) \right]
\]

\[
\int_\Omega F(\omega, u) + \int_\Omega F(u, \omega) \geq \int_\Omega A(u, \omega) \cdot \nabla \phi + B(u, \omega) \phi
\]

\[
+ \liminf_{k \to \infty} \left[ - \int_\Omega A(u, \omega) \cdot \nabla \phi + B(u, \omega) \phi \right]
\]

\[
(\omega - u) \geq \int_\Omega F(\omega, u) + \int_\Omega F(u, \omega) + \liminf_{k \to \infty} \left[ - \int_\Omega A(u, \omega) \cdot \nabla \phi + B(u, \omega) \phi \right] + \liminf_{k \to \infty} \left[ - \int_\Omega A(u, \omega) \cdot \nabla \phi + B(u, \omega) \phi \right]
\]
\[ \begin{align*}
\frac{\partial}{\partial t} F(u, v_t) + \frac{\partial}{\partial x} f - \frac{\partial}{\partial y} f - \frac{\partial}{\partial z} f + \liminf_{k \to 0} \left[ \int_{W^{2, p}} \left( B(\text{sign}(u, 0) - B(u, v_t)) \text{sign}(u, 0) \right) \right] = 0
\end{align*} \]

**Monotone Operator (3) - Dual Approach**

Recall winter semester: A - elliptic symmetric matrix, I showed that (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (3).

1. \( u \) is a weak solution to \( -\text{div} A
\begin{bmatrix}
\begin{array}{c}
\n
\end{array}
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix} = f \quad \text{in } \Omega
\]

2. \( u \) is a minimizer \( u \in W^{1, 2}(\Omega), \ u = u_0 \text{ on } \partial \Omega, \ \forall v \in W^{1, 2}(\Omega), \int_{\Omega} u v = 0 \text{ on } \partial \Omega, \int_{\Omega} A \nabla u \cdot \nabla v = f v \text{ in } \Omega
\]

3. Dual formulation: \( \xi \in L^2(\Omega; \mathbb{R}^d) \) is a minimizer to a dual formulation and \( A u = \xi \)

\[
(\text{dual form:} \quad S = \{ \xi \in L^2(\Omega; \mathbb{R}^d), \forall \nu \in W_0^{1, 2}: \int_{\Omega} \xi \cdot \nabla \nu = \frac{1}{2} \int_{\Omega} \nabla \xi \cdot \nabla \nu \leq \min \{ \frac{1}{2} A \xi, \xi - \nu u_0, \xi \})
\]

**Theorem:** Let \( F: \mathbb{R}^d \to \mathbb{R}, A: \mathbb{R}^d \to \mathbb{R}^d, \frac{\partial F}{\partial \xi}(\xi) = A(\xi), F \) be strictly convex (\( A \) be strictly monotone) and \( A(0) = 0, F(0) = 0 \).

Let \( |F(\xi)| \leq c_1 (1 + |\xi|^p), \ |A(\xi)| \leq c_2 (1 + |\xi|^{p - 1}), \ F(\xi) \geq c_1 |\xi|^p - c_2, \ A(\xi) \xi \geq c_1 |\xi|^{p - 1} - c_2 \).

Let \( u_0 \in W^{1, 2}(\Omega), \ \xi \in L^p(\Omega), \text{ where } \int_{\Omega} \xi = 0 \text{ but } \int_{\partial \Omega} \xi > 0, \ \int_{\partial \Omega} \xi = 0 \text{ in } \Omega, \ \int_{\Omega} (f \psi(\xi))_\tau \psi \cdot \nabla \psi \)

Then the following is equivalent:

1. \( u \) is a weak solution to \( -\text{div} A
\begin{bmatrix}
\begin{array}{c}
\n
\end{array}
\end{bmatrix}
\begin{bmatrix}
\end{bmatrix} = f \quad \text{in } \Omega, \ u = u_0 \text{ on } \partial \Omega, \ A u = \xi \text{ on } \partial \Omega, \ u = \xi \text{ on } \partial \Omega, \int_{\Omega} A \nabla u \cdot \nabla v = \frac{1}{2} \int_{\Omega} \nabla \xi \cdot \nabla \nu \leq \frac{1}{2} \int_{\Omega} \nabla \xi \cdot \nabla \nu
\]

2. \( u \) is a minimizer to "primal formulation"

\[
(\Rightarrow) \quad V = \{ \nu \in W^{1, 2}(\Omega), \nu \text{ on } \partial \Omega \}, \ \nu \in V, \forall \nu \in V
\]

\[
\int_{\Omega} F(u) - \langle \nu, u \rangle - \int_{\Omega} \xi \nabla u = \int_{\Omega} F(w) - \langle \nu, w \rangle - \int_{\Omega} \xi \nabla w
\]

3. \( \xi = A u_0 \), \( \xi \) is a minimizer to "dual formulation"

\[
(\Rightarrow) \quad K = \{ \xi \in L^p(\Omega; \mathbb{R}^d), \forall \nu \in W_0^{1, 2}, \int_{\Omega} \xi \cdot \nabla \nu = \int_{\Omega} F^*(\xi) - \langle \nu, \xi \rangle - \int_{\Omega} \xi \nabla w \}
\]

\[\xi \in K, \forall \nu \in K : \int_{\Omega} F^*(\xi) - \langle \nu, \xi \rangle - \int_{\Omega} \xi \nabla w \leq \int_{\Omega} F^*(\xi) - \langle \nu, \xi \rangle - \int_{\Omega} \xi \nabla w \]

where \( F^*: \mathbb{R}^d \to \mathbb{R} \) is convex conjugate to \( F \)

**Remark:** \( F^*(\xi) + F(x) \geq \xi \cdot x \) - Young inequality

**Example:** \( F(\xi) = \frac{1}{p} |x|^p \), then \( F^*(\xi) = \frac{1}{p} |x|^p \),

\[
\left( \sup_{x \in \mathbb{R}} \left( \xi \cdot x - \frac{1}{p} |x|^p \right) \right) \geq \frac{p}{p} \xi \cdot x - \frac{1}{p} |x|^p = \frac{p}{p} \xi \cdot x - \frac{1}{p} |x|^p = \frac{p}{p} \xi \cdot x - \frac{1}{p} |x|^p
\]
Application to $p$-Laplacian

$$-\text{div} (|\nabla u|^{p-2} \nabla u) = -\Delta_p u = f \quad \text{in } \Omega \quad U(\xi) = |\xi|^p$$

$$M = M_0 \quad \text{on } \partial \Omega \quad F(\xi) = \frac{1}{p} |\xi|^p, \quad F^*(\xi) = \frac{1}{p} |\xi|^p$$

1. $M \in W^p$, $M = M_0 \quad \text{on } \partial \Omega, \forall \nabla \in W_0^p(\Omega). \quad \int_\Omega |\nabla u|^{p-2} \nabla u = \langle f, \nabla u \rangle$

2. $\forall u, v \in V \{ \nabla \in W_0^p(\Omega), \forall \nabla \in M_0 \quad \text{on } \partial \Omega \}, \quad \int_\Omega |\nabla u|^{p-2} \nabla u - \langle f, \nabla u \rangle \leq \int_\Omega |\nabla v|^{p-2} \nabla v - \langle f, \nabla v \rangle$

3. $K = \{ \xi \in L^p(\Omega, \mathbb{R}^d), \forall \xi \in W_0^p(\Omega) : \int_\Omega |\xi|^{p-2} \xi = \langle f, \nabla u \rangle \}$

$$\xi = A u, \quad \text{or } \xi \in K \quad \text{and } \eta \xi \in \mathbb{R} \int_\Omega |\xi|^{p-2} \xi - \langle f, \nabla u \rangle \leq \int_\Omega |\eta \xi|^{p-2} \eta \xi - \langle f, \nabla u \rangle$$

Proof.

$\Rightarrow$ $\Rightarrow$ 2

$\Rightarrow$ $\Rightarrow$ 2

$\Rightarrow$ $\Rightarrow$ 2 or 1

Properties of $F^*$:

(P1) $F^*(0) = 0$, $F^*$ is strictly convex

(P2) $\frac{\partial F^*(\xi)}{\partial \xi} = A^\ast(\xi)$, where $A^\ast$ is inverse to $A$

(P3) $|F^*(\xi)| \leq c (1 + |\xi|^p)$, $F^*(\xi) \geq C_1, 1|\xi|^p - C_2$

Dual formulation unique minimizer

$$\min_{\xi \in K} \int_\Omega F^*(\xi) \quad -\langle f, \nabla u \rangle \quad \text{for} \quad K = \{ \xi \in L^p(\Omega, \mathbb{R}^d), \forall \xi \in W_0^p(\Omega) : \int_\Omega |\nabla u|^{p-2} \nabla u = \langle f, \nabla u \rangle \}$

$$-\text{div} \xi = f \quad \text{in } \Omega, \quad \xi \in \mathbb{R} \quad \text{on } \partial \Omega$$

$F^*$ convex + (P3) $\Rightarrow$ $\exists$ a minimum

$F^*$ strictly convex $\Rightarrow$ $\exists$! minimizer

$\xi_1 + \xi_2$ - minimizers, $\xi = \frac{\xi_1 + \xi_2}{2}$ $\in K$
Define $\xi := A(\psi u)$ and show that $\xi$ is a minimizer

\[ a) \quad \xi \in K \quad \Rightarrow \quad \sum \xi \cdot \psi u = \sum A(\psi u) \cdot \psi u = \langle f_1 u \rangle + \mu \sum q \psi u \]

yes because $u$ is a weak solution

\[ b) \quad F^* \text{ is convex}, \quad \xi \in K \]

\[ (F^*(\xi) - F^*(\xi)) - (\sum \mu_0 \cdot (\xi - \sum \mu_0, \xi)) \leq \frac{\partial F^*}{\partial \xi}(\xi - \sum \mu_0, \xi) \]

\[ = A^*(\xi) = \sum \mu_0 \cdot (\xi - \sum \mu_0, \xi) \Rightarrow \sum (F^*(\xi) - \sum \mu_0, \xi) \leq \sum (\xi - \sum \mu_0, \xi) \cdot \psi (\mu - \mu_0) \]

\[ = \left\langle f_1 u, \mu - \mu_0 \right\rangle + \mu \sum q (\mu - \mu_0) - \left\langle f_1 u, \mu - \mu_0 \right\rangle - \mu \sum q (\mu - \mu_0) = 0 \quad \Rightarrow \quad \xi \text{ is a minimizer} \]

Proof of properties of $F^*$:

- **Convexity of $F^*$**: $\lambda \in [0, 1]$, $\xi_1, \xi_2 \in \mathbb{R}^d$

\[ F^*(\lambda \xi_1 + (1 - \lambda) \xi_2) = \sup_{z \in \mathbb{R}^d} \left\{ (\lambda \xi_1 + (1 - \lambda) \xi_2) \cdot z - F(z) \right\} \]

\[ = \sup_{z \in \mathbb{R}^d} \left\{ (\lambda \xi_1 \cdot z - \lambda F(z)) + ((1 - \lambda) \xi_2 \cdot z - (1 - \lambda) F(z)) \right\} \]

\[ \leq \lambda \sup_{z \in \mathbb{R}^d} \left\{ \xi_1 \cdot z - F(z) \right\} + (1 - \lambda) \sup_{z \in \mathbb{R}^d} \left\{ \xi_2 \cdot z - F(z) \right\} \]

\[ \Rightarrow F^*(\xi) + F(z) \geq \xi_1 \cdot z \quad \text{(from definition of $F^*$)} \]

\[ F^*(\xi) = \sup_{z \in \mathbb{R}^d} (\xi_1 \cdot z - F(z)) \quad \Rightarrow \quad \xi_1 \cdot z - F(z) \rightarrow -\infty \quad \text{as} \quad |z| \rightarrow \infty \quad \text{($p$-growth of $F$)} \]

\[ \Rightarrow \supremum \text{ is attained if} \quad \frac{\partial}{\partial z} (\xi_1 \cdot z - F(z)) = 0 \quad \Leftrightarrow \quad \xi_1 = \frac{\partial F}{\partial z} = A(z) \quad \Rightarrow \quad z = A^*(\xi) \]

\[ F^*(\xi) = \xi : A^*(\xi) - F(A^*(\xi)) \]

\[ F^*(A(z)) = A(z) \cdot z - F(z) \]

Apply to Young inequality: $F^*(\xi) + F(z) - \xi_1 \cdot z \geq 0$, and $= 0 \quad \Leftrightarrow \quad z = A^*(\xi)$

\[ \Rightarrow \frac{\partial}{\partial z} (F^*(\xi) + F(z) - \xi_1 \cdot z) = 0 \quad \text{if} \quad z = A^*(\xi) \]

\[ \frac{\partial F^*}{\partial \xi}(\xi) = z = A^*(\xi) \]

\[ (F^*)^* = F \quad \text{for} \quad F \text{ convex and} \quad \frac{F(\xi)}{|\xi|} \rightarrow \infty \quad \text{as} \quad |\xi| \rightarrow \infty \]

\[ (F^*)^* \leq F \quad \text{for} \quad F \text{ non-convex and} \quad \frac{F(\xi)}{|\xi|} \rightarrow \infty \quad \text{as} \quad |\xi| \rightarrow \infty \quad \text{then} \quad (F^*)^* \text{ is convex} \]

Homework:

- $K := \{ \xi \in L^p(S) \cap \mathbb{R}^d, \quad \sum \xi \cdot \psi u = 0 \quad \forall \psi \in W_{0}^{1,p}(S) \}$

- $a_{ij}(x) \in L^\infty(S), \quad a_{ij}(x) \geq c_1, \quad x \in S, \quad a_{ij} = B_i(0) \in \mathbb{R}^2$

- Show that $\exists!$ minimizer to $\sum \frac{1}{p_i} \left| \xi_i \right|^2 + \iota \sum a_{ij}(x) \xi_i \xi_j$

Homework:

1. Prove that $\exists!$ minimizer to $\sum \frac{1}{p_i} \left| \xi_i \right|^2 + \iota \sum a_{ij}(x) \xi_i \xi_j$

2. Show that it is a dual formulation of some elliptic non-linear PDE with some data
PARABOLIC EQUATIONS (NONLINEAR VERSION)

\[ \partial_t u - \text{div} A(u, \nabla u) + B(u, \nabla u) = f \quad \text{in} \quad \Omega \times (0, T) \]
\[ u = u_0 \quad \text{on} \quad \partial \Omega \times (0, T) \]
\[ u(0) = u_0 \quad \text{in} \quad \Omega \]

Assumptions on \( A \) and \( B \) are the same as in the elliptic setting, i.e.

* \( A, B \) are Carathéodory
* \( |A(u, \xi)| + |B(u, \xi)| \leq C \left( 1 + |u|^{p-1} + |\xi|^{p-1} \right) \)
* \( \text{growth} \)
* \( A(u, \xi) \cdot \xi + B(u, \xi) \cdot u \geq C_1 |\xi|^{p-2} - C_2 (1 + |u|^{q-1}) \) with \( q > \max(2, p-1) \)
* \( \text{coercivity} \)

**Theorem:** Let \( \Omega \subset \mathbb{C}^n \), \( f \in L^p(0, T; W_0^{0,p}(\Omega)) \), \( u_0 \in L^p(\Omega) \), \( u_0 = 0 \).

Assume that \( A(u, \xi) \) is strictly monotone w.r.t. \( \xi \). Then \( \exists u \),
\[ u \in L^\infty(0, T; L^2(\Omega)) \cap L^p(0, T; W_0^{1,p}(\Omega)) \]
\[ \partial_t u = \text{div} \left( A(u, \nabla u) \right) \quad \text{in} \quad \Omega \times (0, T) \]
\[ A(u, \nabla u) \cdot \nabla + B(u, \nabla u) = f \quad \text{in} \quad \Omega \times (0, T) \]

**Gelfand triple:** \( V \hookrightarrow L^2 \hookrightarrow V^* \)

Note: \( \mu \in L^p(0, T; V) \), \( \partial_t \mu \in L^p(0, T; V^*) \) \( \Rightarrow \mu \in C([0, T]; L^2(\Omega)) \)

**Aubin–Lions lemma:** Let \( V_1 \subset V_2 \subset V_3 \) Banach spaces, \( V_1, V_2 \) reflexive, \( p \in (1, \infty) \).

Then the space \( \mathbb{U} := \{ \mu \in L^p(0, T; V_1) \mid \partial_t \mu \in L^p(0, T; V_3) \} \) is a \( L^p(0, T; V_3) \)

**Example of application:** \( u^n \) bounded in \( L^2(0, T; W_0^{1,2}) \) and \( \partial_t u^n \) bounded

in \( L^1(0, T; W_0^{1,0}) \), then \( u^n \rightharpoonup u \) in \( L^2(0, T; L^2) \) (for subsequence)

\[ V_1 = W_0^{1,2}(\Omega) \quad \Rightarrow \quad V_2 = L^2 \quad \text{and} \quad V_3 = (W_0^{1,0})^* \]

**Homework (up to 10% of exam):** Find/create difficult pde HW with nice

but tricky proof.

**Remark:** In the Aubin–Lions lemma, \( \partial_t u \in M(0, T; V_3) \) is enough.

**Proof (of A-L):**

**Energetic lemma:** Let \( V_1 \subset V_2 \subset V_3 \). Then \( \lambda \geq 0 \), \( \exists C > 0 \), \( \forall \mu \in V_1 \):
\[ \| \mu \|_{V_3} \leq \lambda \| \mu \|_{V_2} + C(\lambda) \| \mu \|_{V_2} \]

**Proof of EL:** By contradiction. \( \lambda > 0 \), \( \forall \mu \in V_1 \):
\[ \| \mu \|_{V_3} > \lambda \| \mu \|_{V_2} + C(\lambda) \| \mu \|_{V_2} \]

\[ M^n \neq 0 \quad \text{and} \quad n^n = \frac{M^n}{\| M^n \|_{V_2}} \]
\[ \lambda = \| M^n \|_{V_2} > C(\lambda) \| M^n \|_{V_2} + \lambda \| M^n \|_{V_2} \]

\[ \{ M^n \}_{n \in \mathbb{N}} \text{ is bounded in } V_1 \quad \Rightarrow \quad n^n \to 0 \text{ in } V_3 \]
Proof of A-L - continuation. Goal:
If $M\Sigma U$ is bounded $\iff \exists C^* \forall \mu \in M \int_0^{2T} \| \mu \|^2 + \| \partial_t \mu \|^2 \leq C^*$,
then $M$ is precompact in $L^p(0,2T; \mathcal{V}_2) \iff \forall \varepsilon \epsilon \{ \nu \}_{k=1}^N \forall \mu \in M \exists k=1, \ldots, N \int_0^{2T} \| \mu \|^2 + \| \partial_t \mu \|^2 \leq \varepsilon$

How to prove the goal:

1. Mollification w.r.t. $t$ and use of Arzelà-Ascoli
2. Mollification is "close" to origin
3. Combine it with Ehring

1. A-A for Banach valued functions, consequence: $\chi_{[0,T]} x \Rightarrow C^*(0,2T; \mathcal{X}) \Rightarrow C(0,T; \mathcal{X})$

How to prove the goal:

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Mollification w.r.t. $t$ and use of Arzelà-Ascoli

2. Mollification is "close" to origin

3. Combine it with Ehring

1. A-A for Banach valued functions, consequence: $\chi_{[0,T]} x \Rightarrow C^*(0,2T; \mathcal{X}) \Rightarrow C(0,T; \mathcal{X})$
L^\infty - estimate: \|u(t) - u_0\|_{L^\infty} \leq C^* \int_0^t \|\partial_tw(t,s)\|_{L^\infty} \psi_s(t) \, ds \, dt
\leq C^* \int_0^T \|\partial tw(t)\|_{L^\infty} dt \int_0^T \psi_s(t) \, dt \leq 2C^*

\text{for } t \in [0, T]: \int_0^T \|u(t) - u_0\|_{L^\infty} \, dt = \int_0^T \|u(t) - u_0\|_{L^p} \, dt \leq C_{p,c^*} \sup_{t \in [0,T]} \|u(t) - u_0\|_{L^p}
\leq (2C^*)p^{-1} 2 \delta c^* = (2C^*)p \delta

3. \mu \in M
\int_0^T \|u - u_0\|_{L^\infty} \psi_s(t) \, dt \leq \|\varepsilon\|_{L^p} \int_0^T \|u - u_0\|_{L^p} \psi_s(t) \, dt
\leq 2p \|\varepsilon\|_{L^p} + C(\varepsilon) \int_0^T \|u - u_0\|_{L^p} \psi_s(t) \, dt
\leq 2p \|\varepsilon\|_{L^p} + C(\varepsilon) \int_0^T \|u - u_0\|_{L^p} \psi_s(t) \, dt
\leq 2p \|\varepsilon\|_{L^p} + C(\varepsilon) \int_0^T \|u - u_0\|_{L^p} \psi_s(t) \, dt
\leq 2p \|\varepsilon\|_{L^p} + C(\varepsilon) \int_0^T \|u - u_0\|_{L^p} \psi_s(t) \, dt

\text{you give me } \varepsilon > 0, \text{ I choose } \delta > 0 \text{ so that } 2p \|\varepsilon\|_{L^p} = \frac{\varepsilon}{\delta}

\text{Then I choose } \delta > 0 \text{ so that } C(\varepsilon) \int_0^T \delta \, dt = \frac{\varepsilon}{\delta}

\text{Finally I choose } \varepsilon > 0 \text{ in } (\psi) \text{ so } C(\varepsilon) \varepsilon \leq \frac{\varepsilon}{\delta}

Now we have
\delta_t u - \text{div} A(u, \nabla u) + B(u, \nabla u) = f \quad \text{in } \Omega \times (0,T)
\mu = u_0 \quad \text{on } \partial \Omega \times (0,T)
\mu(0) = u_0 \quad \text{in } \Omega

A, B \text{ are } C^0 \text{Carathéodory}
\begin{align*}
|A(u, \xi)| + |B(u, \xi)| & \leq C(1 + |u|^p - |\xi|^p - |u|^p - |\xi|^p) \quad \text{growth} \\
A(u, \xi). \xi + B(u, \xi). u & \geq c_1 (|\xi|^p - c_2 (1 + |u|^2 + |u|^p - |\xi|^p) \quad \text{coercivity}
\end{align*}

1. A and B are monotone as whole operator
2. A is monotone (w.r.t. \xi) and B is linear w.r.t. \xi
3. A is strictly monotone (w.r.t. \xi)

Theorem: Let \Omega \subset C^0, \ A, B \text{ satisfy growth + coercivity and let at least one of:}
1. - 3. hold. Then \forall \psi \in L^p(\Omega) \quad \forall f \in L^p(0,T; L^p(\Omega)) \quad \exists u \in L^p(0,T; W_0^1(\Omega)) \cap C([0,T]; L^p(\Omega))
\text{and } \int_0^T \langle \delta_t u, \psi \rangle + \int_0^T A(u, \nabla u). \nabla \psi + B(u, \nabla u). \psi = \int_0^T f \, \psi \quad \text{and } u(0) = u_0.
Note: meaning of $<, >$: Gelfand triple $(W_0^p \cap L^1) \subseteq C(SL) \subseteq (W^p)^*$

$$W_0^p \text{ if } p \geq \frac{2d}{d+2} \quad (\Rightarrow W_0^p \subseteq L^2)$$

Proof (Rothe method): "$\Leftarrow$" approximate $u$ on by\n\[ \frac{u(t_k+\tau) - u(t_k)}{\tau} \]

Choose $n \in \mathbb{N}$, define $\tau = \frac{1}{n}$ and $t_k : t_0 = 0, t_k = t_k + \tau$. Then $t_n = T$.

If I know $u(t_k) = u_k$, I want to find $u(t_{k+1}) = u_{k+1}$.

1st explicit scheme - will never converge!!! Don't do that!

$$\frac{u_{k+1} - u_k}{\tau} = \text{div } A(u_k, v_{uk}) + B(u_k, v_{uk}) = f_k \quad (\text{approximation of } f)$$

2nd implicit scheme - good scheme. Rothe method:

$$\frac{u_{k+1} - u_k}{\tau} = \text{div } A(u_{k+1}, v_{uk+1}) + B(u_{k+1}, v_{uk+1}) = f_{k+1} \quad \mathcal{M}_0 = \mathcal{M}_0$$

1. For $\{u^n_{k+1}\}$, $f_k = \int_{t_{k-1}}^{t_k} f(t) \, dt$

$u_k \in L^2(SL)$ given

find $u_{k+1} \in W_0^p(SL) \cap L^2(SL)$:\n
$$\forall \omega \in V \left\{ \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} A(u_{k+1}, v_{uk+1}) \, dv_{uk+1} + B(u_{k+1}, v_{uk+1}) \, dv \right\} = \tau < f_{k+1}, \omega > + \int_{t_{k-1}}^{t_k} \omega \, du_k$$

Easy homework: prove existence of this solution, if $\tau \ll 1$.

2. Uniform estimate: (independent of $n$)

Set $W^n : = u_{k+1}$ in $(W-F(V_1, n))$

$$\alpha \int (u_{k+1}^2 - u_k u_{k+1}) \leq \tau \int \int A(u_{k+1}, v_{uk+1}) \cdot dv_{uk+1} + B(u_{k+1}, v_{uk+1}) \cdot dv \leq \tau \langle f_{k+1}, u_{k+1} \rangle$$

$$\alpha \int (u_{k+1}^2 - u_k u_{k+1}) + C_1 \tau \int \int \|u_{k+1}\|^p - C \tau \int \int \|u_k\|^p \leq C \|f_{k+1}\|^p + C \|u_{k+1}\|^p + C \|u_k\|^p$$

Poincaré, Young

$$\alpha \int (u_{k+1}^2 - u_k u_{k+1}) + C \tau \|u_{k+1}\|_{L^p}^p \leq C \|f_{k+1}\|^p + C \|u_{k+1}\|^2 + C \|u_k\|^p$$

$$\sum_{k=0}^{N-1} \int \int \|u_{k+1} - u_k u_{k+1}\|_{L^p}^p \leq C \|f_{k+1}\|_{L^p}^p + C \|u_{k+1}\|_{L^2}^p + C \|u_k\|_{L^p}^p$$

$$\frac{\|u_{k+1} - u_k u_{k+1}\|_{L^p}^p + C \tau \sum_{k=0}^{N-1} \|u_{k+1}\|_{L^p}^p \leq C \|f_{k+1}\|_{L^p}^p + C \|u_{k+1}\|_{L^2}^p + C \|u_k\|_{L^p}^p \leq C \|f_{k+1}\|_{L^p}^p + C \|u_{k+1}\|_{L^2}^p + C \|u_k\|_{L^p}^p$$

$$\|u_{k+1}\|_{L^2}^2 - \frac{\|u_0\|_{L^2}^2}{2} + C \tau \sum_{k=0}^{N-1} \|u_{k+1}\|_{L^2}^2 + C \tau \sum_{k=0}^{N-1} \|u_{k+1}\|_{L^2}^2 \leq C \|f_{k+1}\|_{L^p}^p + C \|u_{k+1}\|_{L^2}^2 + C \|u_k\|_{L^p}^p \leq C \|f_{k+1}\|_{L^p}^p + C \|u_{k+1}\|_{L^2}^2 + C \|u_k\|_{L^p}^p$$
3. Definition of $\mu^n, \bar{\mu}^n, f^n, A^n, B^n$

$\mu^n(t) := \mu_k$ \text{ where } $t \in (t_{k-1}, t_k)$

$f^n(t) := f_k$

$A^n(t) := A(\mu^n(t), \nabla u^n(t))$

$B^n(t) := B(\mu^n(t), \nabla u^n(t))$

$\bar{\mu}^n(t) := \frac{1}{T} \int (t - t_{k-1}) \mu_{k-1} + \frac{1}{T} \int (t_k - t) \mu_{k-1}$ \text{ for } $t \in (t_{k-1}, t_k)$

for all $t \in (0, T)$, \quad $\delta_t \bar{\mu}^n(t) = \frac{\mu_k - \mu_{k-1}}{T} - t \in (t_{k-1}, t_k)$

for all $t \in (0, T)$, \quad $\frac{\partial}{\partial t} \bar{\mu}^n(t) = \int_{\Omega} \frac{\partial}{\partial t} \bar{u}^n(t) \nabla \psi + \int_{\partial \Omega} A^n(t) \cdot \nabla u^n(t) \cdot \nabla \psi + B^n(t) \cdot \nabla u^n(t) \cdot \nabla \psi = \langle f^n(t), \psi \rangle$ \quad $\forall \psi \in V$

then, \quad $T \int_0^T \sum_{k \in \Omega} \mu_{k+1} - \mu_k \, dt = \int_0^T \sum_{k \in \Omega} \int_{\Omega} \mu_{k+1} \nabla u^n(t) \cdot \nabla \psi \, dx$

\text{Energy for } $k$

$\frac{1}{2} \int_{\Omega} \int_{\Omega} \left( \frac{\partial}{\partial t} \bar{u}^n(t) \right)^2 + c_0 \int_{\Omega} \int_{\Omega} \left( \nabla u^n(t) \right)^2 \, dx \leq c_0 \int_{\Omega} \int_{\Omega} \left( \nabla u^n(t) \right)^2 \, dx$

Gronwall's Lemma:

$\|u^n(t)\|_2^2 \leq \left( \|u_0\|_2^2 + \sum_{t=0}^T \|f^n(t)\|_2^2 + 1 \right) \leq c \left( \|u_0\|_2^2 + \sum_{t=0}^T \|f^n(t)\|_2^2 + 1 \right) \leq c(\text{data})$

$\sup_{t \in (0,T)} \|u^n(t)\|_2^2 + \sum_{t=0}^T \|u^n(t)\|_p^p \leq c(\text{data})$

$\sup_{t \in (0,T)} \|u^n(t)\|_2^2 + \sum_{t=0}^T \|B^n(t)\|_p^p \leq c(\text{data})$

$\delta_t \bar{u}^n(t) \|_2^2 \leq \left( \sup_{t \in (0,T)} \left( \delta_t \bar{u}^n(t) \right) \right) \leq \left( \sum_{t=0}^T \left( \|A^n(t)\|_p^p + \|B^n(t)\|_p^p + \|A^n(t)\|_p^p + \|B^n(t)\|_p^p \right) \right)$

$\sup_{t \in (0,T)} \delta_t \bar{u}^n(t) \|_2^2 \leq \left( \sum_{t=0}^T \left( \|A^n(t)\|_p^p + \|B^n(t)\|_p^p + \|A^n(t)\|_p^p + \|B^n(t)\|_p^p \right) \right)$

4. Existence of the limits

$\langle \delta_t \bar{\mu}^n(t), \psi \rangle + \sum_{t=0}^T \langle A^n(t) \nabla u^n(t) + B^n(t) \nabla u^n(t), \psi \rangle = \langle f^n(t), \psi \rangle$ \quad $\forall \psi \in V$ and a.a. $t \in (0, T)$

$\sup_{t \in (0,T)} \|u^n(t)\|_2^2 + \sum_{t=0}^T \|u^n(t)\|_p^p \leq c(\text{data})$

$\sum_{t=0}^T \|\delta_t \bar{\mu}^n(t)\|_2^2 \leq c(\text{data})$

$\bar{\mu}(0) = \mu_0.$

$\sum_{t=0}^T \delta_t \bar{\mu}^n(t) \cdot \mu_{k+1} - \mu_k - \sum_{t=0}^T A^n(t) \cdot \nabla u^n(t) + B^n \cdot \nabla u^n(t) = \int_{\Omega} \bar{f} \cdot \psi \, dx$

$\mu^n \rightarrow \mu$ \quad $\text{in } L^2(0, T; W_0^{1,2}(\Omega))$

$\mu^n \rightarrow \bar{\mu}$ \quad $\text{in } L^2(0, T; L^2(\Omega))$

$A^n \rightarrow A$ \quad $\text{in } L^2(0, T; L^2(\Omega))$

$\nabla u^n \rightarrow \nabla \bar{u}$ \quad $\text{in } L^2(0, T; L^2(\Omega))$

$\nabla \bar{u}^n \rightarrow \nabla \bar{u}$ \quad $\text{in } L^2(0, T; L^2(\Omega))$

$\delta_t \bar{u}^n \rightarrow \delta_t \bar{u}$ \quad $\text{in } L^2(0, T; W_0^{1,2}(\Omega))$

$\delta_t \bar{\mu}^n \rightarrow \delta_t \bar{u}$ \quad $\text{in } L^2(0, T; W_0^{1,2}(\Omega))$

$\varphi \in C^2_0(0, T)$

$\sum_{t=0}^T \langle \delta_t \bar{u}^n(t), \varphi \rangle + \sum_{t=0}^T \langle A^n(t) \nabla u^n(t) + B^n \nabla u^n(t), \varphi \rangle = \sum_{t=0}^T \langle f(t), \varphi \rangle$

$\sum_{t=0}^T \langle \delta_t \bar{\mu}^n(t), \varphi \rangle + \sum_{t=0}^T \langle A^n(t) \nabla u^n(t) + B^n \nabla u^n(t), \varphi \rangle = \sum_{t=0}^T \langle f(t), \varphi \rangle$

$\varphi$ \quad $\text{arbitrary}$
5. Identification of $\hat{M}$

1. $\hat{M} = \mu$

Recall $\sum_{k=0}^{P-1} \sum_{n=0}^{M-1} (u_{kn} - u_{kn-1})^2 \leq c(data)$, 

$\mu^*(\cdot) = \mu_{kn}$, $\hat{M}^*(\cdot) = \frac{tk-t}{k} - \frac{tk-t}{\epsilon} - \mu_{kn}$

$\hat{M}^*(t_{k+1}) = \mu_{kn} (\frac{tk-t}{\epsilon}) - \frac{tk-t}{\epsilon} \mu_{kn-1}$

$\hat{M}^*(t_{k+1}) = (\mu_{kn} - \mu_{kn-1}) (\frac{tk-t}{\epsilon}) + \mu_{kn-1} \frac{tk-t}{\epsilon}$

$\|\hat{M}^*(t_{k+1}) - \hat{M}^*(t_{k})\|^2_n \leq \sum_{k=0}^{P-1} \sum_{n=0}^{M-1} (u_{kn} - u_{kn-1})^2 (\frac{tk-t}{\epsilon})^2$ for $t \in (t_{k-1}, t_k)$

$\sum_{k=0}^{P-1} \sum_{n=0}^{M-1} \|\hat{M}^*(t_{k+1}) - \hat{M}^*(t_{k})\|^2_n \leq \sum_{k=0}^{P-1} \sum_{n=0}^{M-1} \|\mu_{kn} - \mu_{kn-1}\|^2_n (\frac{tk-t}{\epsilon})^2$

$\Rightarrow \mu = \hat{M}$.

2. $\hat{M}^* \rightarrow \mu$ in $L'(0, T; L'(\Sigma))$

Apply Aubin-Lions to $\hat{M}$: $V_1 = W_0^{1, \infty}$, $V_2 = L^1$, $V_3 = (W_0^{1, \infty} (\Sigma))^*$

$\hat{M}^* \rightarrow \mu$ in $L'(0, T; L'(\Sigma))$

Then

$\sum_{n=0}^{M-1} \|\hat{M}^* - \mu\|_{L^1_n}^2 \leq \sum_{n=0}^{M-1} \|\hat{M}^* - \mu\|_{L^1_n}^2 + \sum_{n=0}^{M-1} \|\hat{M}^* - \mu\|_{L^1_n}^2 \Rightarrow 0$

3. Show that $\limsup_{n \rightarrow \infty} \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast = \sum_{n=0}^{M-1} \langle A, d\mu^\ast \rangle + B^\ast \mu^\ast$

$\sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast = \sum_{n=0}^{M-1} \langle A, d\mu^\ast \rangle + B^\ast \mu^\ast$

$\sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast = \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast$

$\limsup_{n \rightarrow \infty} \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast \leq \limsup_{n \rightarrow \infty} \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast$

$\liminf_{n \rightarrow \infty} \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast \geq \liminf_{n \rightarrow \infty} \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast$

Why $\liminf_{n \rightarrow \infty} \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast \geq \sum_{n=0}^{M-1} \langle A, d\mu_n^\ast \rangle + B^\ast \mu_n^\ast$?

if $\hat{M}^*(T) \rightarrow M(T)$ in $L'(\Sigma)$ it is clear from the LSC of the norm.

we know $\|\hat{M}^*(T)\|^2 \leq c(data)$, $\Rightarrow \hat{M}^*(T) \rightarrow M$ in $L'(\Sigma)$

$\hat{M}^*(T) = \hat{M}_0 + \int_0^T d\hat{M}^*$, $M \in V$
Test the limit equation by $u$!

\[ \int_{\Omega} \langle \partial_t u, u \rangle + \int_{\Omega} \left( - \bar{A} \cdot \nabla u + \bar{B} u \right) = \int_{\Omega} \langle f, u \rangle + \frac{\|u_0\|^2}{2} - \frac{\|u(0)\|^2}{2} \geq \limsup \int_{\Omega} A u \cdot \nabla u + B u u \]

**Monotonicity**

Assume $A, B \in L^p(0,T; L^p(\Omega))$ for some $p > 1$. For $u_n \in H^1(\Omega)$ weak limit + limsup:

\[ \|u_n - u_m\|^2 \leq \limsup \int_{\Omega} \left( \bar{A} - A(u_m, \nabla u_m) \right) \cdot \left( \nabla u_n - \nabla u_m \right) + \left( \bar{B} - B(u_m, \nabla u_m) \right) \cdot \left( u_n - u_m \right) \]

So $\|u_n - u_m\|^2 \to 0$ as $n \to \infty$.

**Maximum principle for parabolic equation**

**Problem:**

\[ \partial_t u - A u = f \geq 0 \quad \text{in} \quad (0, T) \times \Omega \]

\[ u = 0 \quad \text{on} \quad (0, T) \times \partial \Omega \]

\[ u(0) = u_0 \quad \text{in} \quad \Omega \]

\[ m = \text{ess \ min} \left( u, u_0(x) \right) \Rightarrow m \]

Test by $(u-m) - = \min(0, u-m) \in L^2(0, T; W^{1,2}_0(\Omega))$:

\[ \int_0^T \langle \partial_t u, (u-m) - \rangle + \int_{\Omega} \left[ \bar{A} \cdot \nabla u \cdot u + (u-m)^2 \right] \leq \int_0^T \|f(u-m) - \|_0 \leq 0 \]

\[ \int_0^T \left[ 1^2 (u-m) \right] \leq 0 \]
\[ S \langle \partial_m u (u-m) \rangle = \frac{1}{2} \int_0^\infty \int_\Omega S (\nabla u (u-m)^2) - \frac{1}{2} \int_\Omega S (\nabla u (u-m)^2) \] 
\[ = \frac{1}{2} \int_\Omega S (\nabla u (u-m)^2) - \frac{1}{2} \int_\Omega S (\nabla u (u-m)^2) \leq 0 \]
\[ \Rightarrow \int_\Omega S (\nabla (u-m)^2) = 0 \]
\[ \Rightarrow (u-m)_- = 0 \Rightarrow u \geq m \]

Now, in general:

\[ \partial_t u - \text{div} \, A(u, \nabla u) = f \text{ in } (0, \infty) \times \Omega \]
\[ u = u_0 \text{ on } (0, \infty) \times \partial \Omega \]
\[ u(0) = u_0 \text{ in } \Omega \]

\[ |A(u, \xi)| \leq C (1 + |\xi|^2), \quad A(u, \xi), \xi \geq 0 \]

Let \( u \in L^p((0, \infty), W^{1, p}(\Omega)), \partial_t u \in L^p((0, \infty), W^{1, q}(\Omega)), u_0 \in W^{1, q}(\Omega) \) be a solution, where \( V = W^{1, q}_{0} \cap L^\infty \), \( f \geq 0 \).

Set \( m = \min \{ \text{essinf}_{x \in (0, \infty)} u_0(x), \text{essinf}_{x \in (0, \infty)} u_0(x) \} \), then \( u \geq m \) a.e. in \( (0, \infty) \times \Omega \).

Proof: \( WF: \) \( \langle \partial_m u, w \rangle + \int_\Omega A(u, \nabla u) \cdot \nabla w \geq \int_\Omega f \cdot w \) for \( w \in V \).

Set \( w = (u(t) - m)_- := \min (0, u(t) - m)_- \).

Since \( u \geq u_0 \geq m \) on \( (0, \infty) \) and \( u(t) \in L^p(\Omega) \cap W^{1, q}(\Omega) \) \( \Rightarrow w \in V \).

\[ \Rightarrow \langle \partial_m u (u-m)_- \rangle + \int_\Omega A(u, \nabla u) \cdot \nabla (u-m)_- \leq \int_\Omega f (u-m)_- \leq 0 \]
\[ = A(u, \nabla u) \cdot \nabla (u)_- \chi_{\{u \leq m\}} \geq 0 \]

\[ \Rightarrow \langle \partial_m u (u-m)_- \rangle \leq 0 \Rightarrow \int_0^t \langle \partial_m u (u-m)_- \rangle \leq 0 \quad (\ast) \]

Lemma: Let \( g \in C^{0,1}, \quad g(m) = 0, \quad G := g', \text{ then } \int_0^t \langle \partial_m u \rangle \langle G(u) \rangle = \int_0^t \langle G(u(t)) - G(u(0)) \rangle \).

In \( (\ast) \) use lemma with \( G(s) := \frac{(s-m)^2}{2} \).

\[ \int_\Omega (u(t) - m)^2 = \int_\Omega (u_0 - m)^2 = 0 \Rightarrow u \geq m \text{ a.e.} \]

Proof of lemma:
\[ u^e := \frac{1}{t} \int_0^t u \]
\[ \partial_t u^e \rightarrow \partial_t u \text{ in } L^p((0, \infty), W^{1, q})(\Omega) \]
\[ u^e \rightarrow u \text{ in } L^p((0, \infty), W^{1, q})(\Omega) \]
\[ C(0, \infty, L^q(\Omega)) \]
\[ \int_0^t \langle \partial_t u^e \rangle \langle G(u^e) \rangle = \int_0^t \langle \partial_t u^e \rangle \langle G(u^e) \rangle = \int_0^t \langle \partial_t u^e \rangle \langle G(u^e) \rangle = \lim_{\varepsilon \rightarrow 0} \int_0^t \langle \partial_t u \rangle \langle G(u) \rangle = \int_0^t \langle \partial_t u \rangle \langle G(u) \rangle = \int_0^t \langle G(u(t)) - G(u(0)) \rangle. \]
semigroup theory

Introduction. Exponential function, linear operator.

For \( a \in \mathbb{R} \), \( e^a \) has several definitions:

\[
e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!} = \lim_{n \to \infty} (1 + \frac{a}{n})^n
\]

Important property: \( f(x) = e^x \), \( f(x+y) = f(x)f(y) \) \( \implies \) continuity \( \quad (e) \)

For \( A \in \mathbb{R}^{d \times d} \), a matrix or for \( A \) a bounded linear operator we can define

\[
e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}
\]

thanks to the fact that the series is convergent

\[
\|e^A\| \leq \sum_{k=0}^{\infty} \frac{\|A^k\|}{k!} \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|} < \infty
\]

Our goal is to study equations of type

\[
\begin{align*}
\mu'(t) &= A\mu(t) \quad t > 0 \\
\mu(0) &= \mu_0
\end{align*}
\]

where \( \mu_0 \in X \) - a real Banach space and \( A : D(A) \subset X \to X \) is a linear (possibly unbounded) operator, which is independent of \( t \), \( D(A) \) is linear subspace of \( X \).

We study existence and uniqueness of a solution \( \mu : [0, \infty) \to X \).

For bounded operator, \( \mu(t) = e^{tA} \mu_0 \). What if \( A \) is unbounded?

Reminder:

Definition: A linear operator \( L : X \to X \) is bounded whenever

\[ \exists M > 0 \quad \forall x \in X : \|Lx\| \leq M \|x\| \]

Lemma: Let \( L \) be linear operator, then the following is equivalent:

1. \( L \) is bounded
2. \( L \) is continuous
3. \( L \) is continuous at \( 0 \).

Examples.
1. \( \Delta : W^{2,2} \to L^2 \) is bounded, \( \|\Delta u\|_{L^2} \leq \left( \sum_{k=2}^{\infty} \left\| \frac{\partial^k u}{\partial x^k} \right\|_{L^2}^2 \right)^{1/2} \leq \left( \sum_{k=2}^{\infty} \|\Delta u\|_{L^2}^k \right)^{1/2} \)
2. \( \Delta : W^{1,2} \to W^{2,2} \) is unbounded.

Contradiction. If bounded, \( \exists M > 0 \) s.t. \( \|\Delta u\|_{L^2} \leq M \|u\|_{L^2} \forall u \in W^{2,2} \)

For sure \( \exists u \in W^{2,2} \) s.t. \( \Delta u \notin W^{1,2} \).

Note: In \((1)\), operator \( A \) acts from \( X \) to \( X \), i.e. for \( A = \Delta \) the Banach space \( X \) must include smooth functions for \( A \) to be bounded.

Instead: we study \((1)\) for unbounded operators.
Semigroup.

**Notation:** \( \mathcal{L}(X) := \{ L: X \to X \text{ is linear bounded operator} \} \) is a Banach space

\[
\| L \|_{\mathcal{L}(X)} = \sup_{x \in X, \| x \|_X = 1} \| Lx \|_X
\]

Unbounded operator is the couple \( (A, D(A)) \), where the domain of \( A \), \( D(A) \subset X \) is a subspace, \( A: D(A) \to X \) is linear.

**Definition:** The function \( S(t): [0, \infty) \to \mathcal{L}(X) \) is called a semigroup, iff
1. \( S(0) \) is identity, \( S(0)x = x \quad \forall x \in X \)
2. \( S(t)S(s) = S(t+s) \quad \forall t, s \geq 0 \)

If moreover 3. \( S(t)x \to x \) as \( t \to 0^+ \quad \forall x \in X \), we call \( S(t) \) a \( C_0 \)-semigroup.

**Remarks:** Due to property (e) of exponential, a \( C_0 \)-semigroup is a suitable candidate for generalized exponential via relation "\( S(t) = e^{tA} \)."

Stronger assumption 3'. \( \| S(t) - I \|_{\mathcal{L}(X)} \to 0 \) as \( t \to 0^+ \) (uniform continuity) implies \( S(t) = e^{tA} \) for some linear continuous operator \( A \).

**Lemma 1.** Let \( S(t) \) be a \( C_0 \)-semigroup in \( X \). Then
1. \( \exists M \geq 1, \omega \geq 0 \) s.t. \( \| S(t) \|_{\mathcal{L}(X)} \leq M e^{\omega t} \quad \forall t \geq 0 \).
2. \( t \mapsto S(t)x \) is continuous mapping from \( [0, \infty) \) to \( X \) \( \forall x \in X \) fixed.

**Proof:**
1. We claim that \( \exists M \geq 1, \omega \geq 0 \) s.t. \( \| S(t) \|_{\mathcal{L}(X)} \leq M e^{\omega t} \quad \forall t \in [0, \infty) \).
   
   If not, then \( \exists \{ t_n \}, t_n \to 0^+ \) s.t. \( \| S(t_n) \|_{\mathcal{L}(X)} \to \infty \).
   
   But due to 3., from definition of the semigroup: \( S(t_0)x \to x \quad \forall x \in X \) \( \Rightarrow \| S(t_0) \|_{\mathcal{L}(X)} \) is bounded. \( \forall n \in \mathbb{N} \) (Uniform boundedness principle) \( \quad \forall x \in X \).

   \( FA: \| S(t_n) \|_{\mathcal{L}(X)} \) is bounded. \( \forall n \in \mathbb{N} \) (Uniform boundedness principle) \( \quad \forall x \in X \).

   **Set:** \( \omega = \frac{4}{3} \ln M \) (i.e. \( M = e^{\omega \delta} \)), for every \( t \geq 0 \), \( \exists n \in \mathbb{N}, \delta \in [0, \delta) \) s.t. \( t = n \delta + \epsilon \).

   \[
   \| S(t) \|_{\mathcal{L}(X)} = \| S(\delta) \cdots S(\delta)S(\epsilon) \|_{\mathcal{L}(X)} \leq \| S(\delta) \|_{\mathcal{L}(X)}^n \| S(\epsilon) \|_{\mathcal{L}(X)} \leq M^n M = e^{\omega n}M \leq M e^{\omega t}.
   \]

2. **Continuity:** in \( 0^+ \) we have from 3. of def. Let \( t > 0 \).

   Continuity from the right: \( \lim_{h \to 0^+} S(t+h)x = \lim_{h \to 0^+} S(t)S(h)x = S(t)x \)

   From the left (wlog \( h < t \)):

   \[
   \| S(t-h)x - S(t)x \|_X = \| S(t-h)(x - S(h)x) \|_X \leq \lim_{h \to 0^+} \| S(t-h) \|_X \| x - S(h)x \|_X \| X = 0.
   \]

   \[
   \| S(t-h) \|_X \leq Me^{\sigma(t-h)} \to Me^{\omega t} \quad \text{as} \quad h \to 0^+
   \]

   \[
   \| x - S(h)x \|_X \to 0 \quad \text{due to 3}.
   \]
Definition (Generator of a semigroup). An unbounded operator \((A, D(A))\) is called a generator of a semigroup \(S(t)\) iff

\[
Ax = \lim_{h \to 0^+} \frac{S(h)x - x}{h}
\]

\[D(A) = \{ x \in X : \lim_{h \to 0^+} \frac{S(h)x - x}{h} \text{ exists in } X \} \]

Remark: \(A\) is linear \((S(t) \in \mathcal{L}(X))\) and \(D(A) \subseteq X\) is linear subspace.

Theorem 1 (properties of generator). Let \((A, D(A))\) be a generator of \(S(t)\) a \(c_0\)-semigroup in \(X\). Then:

1. \(x \in D(A) \Rightarrow S(t)x \in D(A) \quad \forall t \geq 0\)
2. \(x \in D(A) \Rightarrow AS(t)x = S(t)Ax = \frac{d}{dt} S(t)x \quad \forall t \geq 0 \quad \text{(in } t = 0^+)\)
3. \(x \in X, t \geq 0 \Rightarrow x_t := \int_0^t S(s)x \, ds \in D(A), \quad A(x_t) = S(t)x - x\).

Proof. 1. \(x \in D(A), \quad t \geq 0 \) given

\[
\lim_{s \to 0^+} \frac{S(s)x - S(t)x}{s} = \lim_{s \to 0^+} \frac{S(s)x - S(t)x}{s} = \lim_{s \to 0^+} \frac{S(s)x - x}{s} = S(t)Ax
\]

\[
\Rightarrow AS(t)x = S(t)Ax
\]

\[
= \frac{d}{dt} S(t)x = S(t)Ax \quad \forall t \geq 0 \text{ from the right}
\]

For \(t > 0\) from the left:

\[
\lim_{h \to 0^+} \frac{S(t-h)x - S(t)x}{-h} - S(t)Ax = \lim_{h \to 0^+} S(t-h)\left( \frac{x - S(h)x}{-h} - S(t)Ax \right) = 0
\]

3. \(S(h)x^t - x^t = S(h) \int_0^t S(s)x \, ds - \int_0^t S(s)x \, ds \quad S(h) \in \mathcal{X}(X) + \mathcal{X}(X) + \mathcal{X}(X)
\]

\[
= \int_0^t S(s+h)x \, ds - \int_0^t S(s)x \, ds \\
= \frac{1}{2} \int_{t+h}^{2t} S(s)x \, ds - \int_0^h S(s)x \, ds \\
= \frac{1}{2} \int_{t+h}^{2t} S(s)x \, ds
\]

\(L1, 2. \quad (t \mapsto S(t)x \text{ cont.})\)

\[
\lim_{h \to 0^+} \frac{1}{h} (S(h)x^t - x^t) = \lim_{h \to 0^+} \left( \frac{1}{h} \int_{t+h}^{2t} S(s)x \, ds - \frac{1}{h} \int_0^h S(s)x \, ds \right) = S(t)x - S(0)x = S(t)x - x
\]

Remark: \(\mu : t \mapsto \mu(t) = S(t)x_0\) is a classical solution to \(\frac{d}{dt} \mu(t) = Ax(t), \quad \mu(0) = x_0\) if \(\mu \in D(A)\).

Definition (closed operator). We say that an unbounded operator \((A, D(A))\) is closed iff \(\mu_n \in D(A), \mu_n \to \mu, A\mu_n \to A\mu \Rightarrow \mu \in D(A)\) and \(A\mu = 0\).

Remark: unboundedness & closedness is natural for derivative operators, e.g.: \(A: \mu(t) \mapsto \frac{d}{dt} \mu(t), \ D(A) = W^1(\Omega, X), \ \mu_n(t) \in W^1(\Omega, X), \ \mu_n(t) \to \mu(t) \text{ in } L^1(\Omega, X), \)

\(\frac{d}{dt} \mu_n(t) \to g(t) \text{ in } L^1(\Omega, X) \Rightarrow \mu(t) \in W^1(\Omega, X) \text{ and } \frac{d}{dt} \mu(t) = g(t).\)
Theorem 2 (density and closedness of a generator): Let $(A,D(A))$ be a generator of a $C_0$-semigroup $S(t)$ in $X$. Then $D(A)$ is dense in $X$ and $(A,D(A))$ is closed.

Proof. Density. For any $x \in X$ and $t \geq 0$, $x^t \in D(A)$ ($T_{1,3}$) and

$$X = \lim_{t \to 0^+} x^t (\lim_{t \to 0^+} \frac{d}{dt} S(t) x ds = S(0) x = x)$$

Closedness. Let $x_n \in D(A)$, $x_n \to x$, $Ax_n \to y$.

We want to show that $x \in D(A)$, $Ax = y$.

Claim: $S(t) x = \phi$. Indeed, $\frac{d}{dt} S(t) x = S(t) Ax$ (and RHS is continuous in $t$).

Newton-Leibniz:

$$S(h) x_n - S(0) x_n = \int_0^h \frac{d}{ds} S(s) x_n ds = \int_0^h S(s) A x_n ds \quad n \to 0$$

$$\frac{d}{dh} (S(h) x - x) = \int_0^h S(s) y ds, \quad n \to 0$$

$$Ax = S(0) y = y.$$ 

Remark: by closedness of $A + T_{1,3}$.

Lemma 2 (uniqueness of a semigroup): Let $S(t)$, $\tilde{S}(t)$ be semigroups with the same generator. Then $S(t) = \tilde{S}(t)$ $\forall t > 0$.

Proof. Define $y(t) = S(t-t) \tilde{S}(t) x$, $x \in D(A)$ and $t > 0$ fixed.

$y(t) = \int_0^t \frac{d}{ds} (S(t-s) \tilde{S}(s) x) ds = S(t-t) A \tilde{S}(t) x + S(t-t) A S(t) x = 0 \quad \forall x \in D(A)$

$D(A)$ is dense in $X$ ($T2$) and $T > 0$ arbitrary $\Rightarrow S(x) = \tilde{S}(t) \forall t > 0$.

Definition (resolvent): Let $(A,D(A))$ be unbounded operator. We define

resolvent set $\rho(A) = \{ \lambda \in \mathbb{R} \mid \lambda I - A : D(A) \to X \text{ is one-to-one and onto} \}$

resolvent $R(\lambda A) = (\lambda I - A)^{-1} : X \to D(A)$, $\lambda \in \rho(A)$.

Remark: $(A,D(A))$ closed $\Rightarrow \lambda I - A : D(A) \to X$ continuous $\Rightarrow R(\lambda A) \in \mathbb{K}(X)$
Lemma 3 (properties of resolvent operators): Let \((A, \text{Dom}(A))\) be a generator of a \(C_0\)-semigroup \(S(t)\), let \(\|S(t)\|_{\text{op}} \leq M e^{\omega t}\). It holds:

1. \(A R(\lambda, A) x = \lambda R(\lambda, A) x - x \quad \forall x \in X\)

2. \(R(\lambda, A) A x = \lambda R(\lambda, A) x - x \quad \forall x \in \text{Dom}(A)\)

3. \(R(\lambda, A) x - R(\lambda, A) x = \frac{1}{\lambda - \lambda} R(\lambda, A) (\lambda x) x \quad \forall x \in X, \quad R(\lambda, A) R(\lambda, A) = R(\lambda, A) R(\lambda, A)\)

4. \(\forall \lambda > 0 : \lambda \in \rho(\lambda, A), \quad R(\lambda, A) = \int_0^\infty e^{\lambda t} S(t) x \, dt, \quad \|R(\lambda, A)\|_{\text{op}} \leq \frac{M}{\lambda - \omega}\).

Proof: 1. \(A R(\lambda, A) x = [(\lambda - \lambda I) + \lambda I] R(\lambda, A) x = \lambda R(\lambda, A) x - x, \quad 2. - the same

3. LHS: \((\lambda I - A)(\mu I - A)(R(\lambda, A) - R(\mu, A)) x\)

\[= (\lambda I - A) (\mu I - A) R(\lambda, A) x = (\lambda I - A) (\mu I - A) R(\lambda, A) x = (\lambda - \lambda)(\lambda I - A) R(\lambda, A) x = (\mu - \lambda)(\lambda I - A) R(\lambda, A) x\]

RHS: \((\lambda I - A)(\mu I - A) R(\lambda, A) R(\mu, A) = (\lambda I - A)(\mu I - A) R(\lambda, A) R(\lambda, A) x = (\mu - \lambda)(\lambda I - A) R(\lambda, A) x\)

LHS: \((\mu I - A)(\lambda I - A) R(\lambda, A) R(\mu, A) = I\)

RHS: \((\mu I - A)(\lambda I - A) R(\mu, A) R(\lambda, A) = (\mu I - A)(\lambda I - A) R(\mu, A) R(\lambda, A) R(\lambda, A) x = (\mu - \lambda)(\lambda I - A) R(\lambda, A) x\)

4. \(S(t)\) gen. by \((A, \text{Dom}(A)) \iff \hat{S}(t) = e^{\omega t} S(t)\) gen. by \((\hat{A}, \text{Dom}(\hat{A}))\), \(\hat{A} = A - \omega I, \text{Dom}(\hat{A}) = \text{Dom}(A)\),

\[R(\lambda, \hat{A}) = R(\lambda + \omega, A) \Rightarrow \text{WLOG } \omega = 0.

Then \(\|S(t)\|_{\text{op}} \leq M, \lambda > 0\), denote \(\tilde{R} x = \int_0^\infty e^{\lambda t} S(t) x \, dt\) (well-defined)

\[\|\tilde{R} x\|_{\text{op}} < \int_0^\infty e^{\lambda t} M 11 x \, dt = \frac{M}{\lambda} 11 x \Rightarrow \tilde{R} x \in X, \quad \|\tilde{R} x\|_{\text{op}} \leq \frac{M}{\lambda}\]

\[\tilde{R} x \in \text{Dom}(A)\]: let \(\lambda > 0\) and \(x \in X\):

\[S(\lambda) \tilde{R} x - \tilde{R} x = \int_0^\infty e^{\lambda t} [S(\lambda t + h) x - S(t) x] \, dt\]

\[= h \int_0^\infty e^{\lambda t} S(t) x \, dt - \int_0^\infty e^{\lambda t} S(t) x \, dt\]

\[= e^{\lambda h} (\int_0^\infty e^{\lambda t} S(t) x \, dt - \int_0^h e^{\lambda t} S(t) x \, dt) - \int_0^\infty e^{\lambda t} S(t) x \, dt\]

\[= e^{\lambda h} (\int_0^h e^{\lambda t} S(t) x \, dt - \int_0^\infty e^{\lambda t} S(t) x \, dt\]

\[= \lim_{h \to 0^+} \frac{1}{h} \left( S(\lambda) x - \tilde{R} x \right) = \lim_{h \to 0^+} \left( \frac{e^{\lambda h} - 1}{h} \right) \int_0^\infty e^{\lambda t} S(t) x \, dt - \lim_{h \to 0^+} \frac{1}{h} \int_0^h e^{\lambda t} S(t) x \, dt\]

\[\Rightarrow x = \lambda \tilde{R} x - A \tilde{R} x = (\lambda I - A) \tilde{R} x \quad \forall x \in X\]

For \(x \in \text{Dom}(A)\), \(A \tilde{R} x = A \int_0^\infty e^{\lambda t} S(t) x \, dt = \int_0^\infty e^{\lambda t} S(t) A x \, dt = \tilde{R} A x\)

\[\Rightarrow A \tilde{R} x \xrightarrow{x \in X} \tilde{R} A x \Rightarrow x = \tilde{R} (\lambda I - A) x, \quad x \in \text{Dom}(A)\]

\(\lambda I - A\) is one-to-one and onto
A > 0 was arbitrary \( \Rightarrow \) \( p(A) = (0, \infty) \) and \( R = (\lambda I - A)^{-1} = R(\lambda, A) \).

**Definition (semigroup of contractions):** We say that \( S(t) \) is contraction semigroup if \( \| S(t) \|_{\mathcal{L}(X)} \leq 1 \ \forall t \geq 0 \).

**Theorem 3 (Hille-Yosida (for contractions)):** For \( (A, D(A)) \) an unbounded operator, the following is equivalent:

1. \( \exists \) co-semigroup of contractions generated by \( (A, D(A)) \)
2. \( (A, D(A)) \) is closed, \( D(A) \) is dense in \( X \) if \( (0, \infty) \subset p(A) \) and \( \| R(\lambda, A) \|_{\mathcal{L}(X)} \leq \frac{A}{\lambda} \).

**Proof.** "1. \( \Rightarrow 2." \) already proven by T2 (closed, dense) and L3 (resolvent)

"2. \( \Rightarrow 1." \) Yosida's approximation: \( A_n = n A R(n, A), \ n \in \mathbb{N} \)

Strategy: \( An x \rightarrow Ax \) as \( n \rightarrow \infty \), \( S(t) = \lim_{n \rightarrow \infty} e^{tA_n} \ (An \in \mathcal{L}(X)) \)

**Step 1. Properties of \( An \).**
\[
A_n = n A R(n, A) = n^2 R(n, A) - n I \in \mathcal{L}(X) \ (n \in \mathbb{N})
\]

\[\| n R(n, A) x \|_X = \| R(n, A) Ax \|_X \leq \| R(n, A) \|_{\mathcal{L}(X)} \| Ax \|_X \leq \frac{A}{n} \| Ax \|_X \rightarrow 0 \ \text{as} \ n \rightarrow \infty \]

\[\Rightarrow \| n R(n, A) x \|_{\mathcal{L}(X)} \rightarrow 0 \ \forall x \in D(A), \ D(A) \text{ dense in } X \ \& \ \| \text{Im} R(n, A) \|_{\mathcal{L}(X)} \leq 1 \ \Rightarrow \ \forall x \in X \]

\[A_n x = n A R(n, A) x = n R(n, A) Ax \rightarrow Ax \ \forall x \in D(A) \]

**Step 2. Approximation of the semigroup \( S(t) \) by \( S_n(t) \).**

\[S_n(t) := e^{tA_n} = \sum_{k=0}^{\infty} \frac{(tA_n)^k}{k!} A_n \in \mathcal{L}(X) \ (An \in \mathcal{L}(X))
\]

\[e^{tA_n} = e^{-ntI + n^2tR(n, A)} = e^{-nt} \cdot e^{n^2tR(n, A)}
\]

\[\| e^{n^2tR(n, A)} \|_{\mathcal{L}(X)} = \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \| R(n, A) \|_{\mathcal{L}(X)} \leq \sum_{k=0}^{\infty} \frac{(nt)^k}{k!} \| R(n, A) \|_{\mathcal{L}(X)}^k \leq e^{nt}
\]

\[\Rightarrow \| S_n(t) \|_{\mathcal{L}(X)} \leq e^{nt} \cdot e^{nt} = 1 \ \Rightarrow \ S_n(t) \text{ are contractions.}
\]

**Step 3. Existence of the limit**

Let \( x \in D(A) \) and \( t > 0 \).

\[S_n(t) x - S_n(t) x = \int_0^t \frac{d}{ds} [S_m(t-s) S_n(s) x] ds = \int_0^t \frac{d}{ds} [e^{(t-s)A_n} e^{sA_n} x] ds
\]

\[= \int_0^t (-A_n e^{(t-s)A_n} e^{sA_n} x + A_n e^{(t-s)A_n} e^{sA_n} x) ds
\]

\[(R(n, A) R(m, A) = R(m, A) R(n, A) \Rightarrow A_n A_m = A_m A_n) \Rightarrow A_n S_m(t) = S_m(t) A_n \ \forall t > 0)
\]

\[= \int_0^t e^{(t-s)A_n} e^{sA_n} (A_n x - A_m x) ds
\]

\[\Rightarrow \| S_n(t) x - S_n(t) x \|_{\mathcal{L}(X)} \leq \int_0^t \| S_m(t-s) S_n(s) \|_{\mathcal{L}(X)} \| A_n x - A_m x \|_X ds \leq t \| A_n x - A_m x \|_X \rightarrow 0
\]

\[\Rightarrow \{S_n(t)\} \text{ satisfy BC condition uniformly w.r.t. } t \in [0, T] \Rightarrow \exists \text{ limit } x \in D(A)
\]
Step 4. Check that $S(t)$ is generated by $(A, D(A))$.

Denote the generator of $S(t)$ by $(A, D(A))$ and let $x \in D(A)$. $x \in D(A)$

$$S_n(t) x - X = \int_0^t S(s) x \, ds = \int_0^t S(s) A x \, ds$$

Now, for any $\lambda > 0$, $\lambda \in \mathcal{P}(A)$ (by assumption) and $\lambda \in \mathcal{P}(\tilde{A})$ (by $L3, 4$).

Therefore, $\lambda I - \tilde{A}$ is one-to-one and onto.

$$\Rightarrow D(\tilde{A}) = D(A) \Rightarrow D(\tilde{A}) = D(A) \Rightarrow \tilde{A} = A \Rightarrow \tilde{A} = A$$

Theorem (generalized H-Y): $(A, D(A))$ generates a semigroup, satisfying $\|S(t)\|_{L(L)} \leq e^{\lambda t}$

$(A, D(A))$ is closed, densely defined, and $\forall \lambda > 0$, $\lambda \in \mathcal{P}(A)$ and $\|R^\lambda(A)\|_{L(L)} \leq \frac{M}{(\lambda - 1)^n} \#(\mathcal{N})$.

Application of the semigroup theory in the PDEs

$\L_2 \mu - \Delta \mu = 0$ in $(0, T) \times \Omega$

$\mu(0) = \mu_0$ in $\Omega_1$, $\mu = 0$ on $(0, T) \times \partial \Omega$

$A = \Delta$ in $\Omega_1$, $A : D(A) \to X$

$D(A) = W^{2, p}(\Omega_1) \cap \nabla W^{2, p}(\Omega_1)$, $X = L^2(\Omega_1)$

1. Set $S(t) \mu_0 = \mu(t); S(t)$ is a semigroup

2. $\Delta \mu = \mu$, $\Delta \mu = D(A)$ gives a semigroup $S$.

4. $\lambda = 0$, $(\lambda I - A) : W^{2, p}(\Omega) \cap \nabla W^{2, p}(\Omega) \to L^2(\Omega)$

5. $\mu = R \lambda$, $\Delta \mu = \lambda \mu$

6. $\lambda \mu = \Lambda$, $\mu = \frac{1}{\lambda - 1} \mu_0$
Semigroup for wave equation

\[ \begin{align*}
\delta_{tt} u - \Delta u &= 0 \quad \text{in } (0,T) \times \Omega \\
\mu(0) &= \mu_0 \quad \text{in } \Omega \\
\delta_t u(0) &= N_0 \quad \text{in } \Omega \\
\mu &= 0 \quad \text{on } \partial \Omega \times (0,T)
\end{align*} \]

\[\begin{align*}
\partial_t u &= n \\
\partial_t \nu &= \Delta u \\
A(u) : (\mu, \nu) &\mapsto (\nabla \mu, \Delta \mu)
\end{align*} \]

\[\begin{align*}
\partial_t u &= \lambda (u) \\
X &= \{ u = (\mu, \nu) \mid \mu \in W_{0,0}^2(\Omega) \land \nabla \in L^2(\Omega) \} = W_{0,0}^2(\Omega) \times L^2(\Omega) \\
D(A) \subseteq X \\
D(A) &= \{ \lambda \in L^2(\Omega) \cap W_0^{2,2}(\Omega) \} \times L^2(\Omega)
\end{align*} \]

Use of Hille-Yosida Theorem:

1. \( A \) is closed - the same as for parabolic equation

2. estimates for resolvent (different)

   \[ \forall \lambda \in \mathbb{R} \quad (\lambda I - A) \text{ is onto and invertible} \]

   \[ \begin{align*}
   R_\lambda := (\lambda I - A)^{-1} \\
   \| R_\lambda \| \leq \frac{m}{\| \lambda \|} \\
   (\lambda I - A)(U) = (\lambda \mu - n, \lambda \nu - \Delta \mu)
   \end{align*} \]

   \[ \forall (f_1, f_2) \in X \quad \lambda \mu - n = f_1 \in W_{0,0}^2, \lambda \nu - \Delta \mu = f_2 \in L^2(\Omega) \]

   \[ \begin{align*}
   \lambda \mu - n &= f_1 \\
\Rightarrow \quad \lambda^2 \mu - \Delta \mu &= f_2 + \lambda f_1 \quad \text{in } \Omega \\
\mu &= 0 \quad \text{on } \partial \Omega \quad \{ \Rightarrow \exists! \mu \in W_{0,0}^2(\Omega) \}
\end{align*} \]

Now, we want

\[ \| u \|^2 \leq \frac{\| \mu \|^2}{\lambda^2} \]

Equations:

\[ \begin{align*}
\lambda \mu - n &= f_1 \\
\lambda \nu - \Delta \mu &= f_2
\end{align*} \]

\[ \begin{align*}
\text{multiply by } \lambda \mu \quad & S \\
\text{test by } u \lambda
\end{align*} \]

\[ \begin{align*}
\lambda \mu u_t - \mu \cdot u &= S f_1, u \\
\lambda \mu \cdot u - \lambda^2 u &= S f_1, \nu \\
\lambda \mu u + \lambda \Delta u &= S \mu f_2, u \\
\frac{1}{2} \lambda^2 S f_2, u - \lambda S f_1, \nu \\
S f_2, u &= \frac{1}{4} \lambda (\| \mu \|^2 + \| \nabla \mu \|^2) + \frac{\lambda}{4} \| f_2 \|^2$
\]
Semigroup when you cannot use regularity

\( \partial_t u + Lu = 0 \quad \text{in} \quad (0,T) \times \Omega \)

\( Lu = -\text{div} \; A(x) \nabla u + b \cdot \nabla u + cu \)

\( u = 0 \quad \text{on} \quad (0,T) \times \partial \Omega \)

\( u(0) = u_0 \)

\( \Omega = L^2(\Omega) \)

\( D(A) = \{ u \in W_0^1(\Omega); \; \| L u \|_2 = \sup_{0 \leq \varphi \leq 1} \int \nabla u \cdot \nabla \varphi + b \cdot \nabla u \varphi + cu \varphi < \infty \} \)

The density of \( D(A) \) in \( \Omega \) is clear.

Comment: \( D(A) \) can be equivalently defined as

\( D(A) = \{ u \in W_0^1(\Omega); \; \text{for } f \in L^2(\Omega) \text{ solving } \int \nabla u \cdot \nabla \varphi + b \cdot \nabla u \varphi + cu \varphi = \int f \varphi \text{ for } \varphi \in W_0^1(\Omega) \} \)

\( A \) is closed: \( u_n \rightarrow u \; \text{in} \; L^2(\Omega) \), \( \| L u_n \|_2 \rightarrow \| Lu \|_2 \)

It means:

\( \int A u_n \cdot v_n - b \cdot \nabla u_n \cdot \nabla v_n + cu_n v_n = \int f v_n \; \text{for } v_n \in W_0^1(\Omega) \)

\( \int A u_n \cdot v_n - b \cdot \nabla u_n \cdot \nabla v_n + cu_n v_n \rightarrow \int f v_n \; \text{for } v_n \in W_0^1(\Omega) \)

Proof. 1. Show that \( u \in W_0^1(\Omega) \)

we know \( u_n \rightarrow u \; \text{in} \; L^2(\Omega) \)

\( \| u_n \|_2 \leq c \) \( \text{test by } u_n \)

\( \int A u_n \cdot u_n = \int f u_n - b \cdot \nabla u_n \cdot \nabla u_n - c u_n^2 \)

\( \leq \| f \|_2 \| u_n \|_2 + \| b \|_\infty \| \nabla u_n \|_2 \| u_n \|_2 + \| c \|_\infty \| u_n \|_2^2 \leq c (1 + \| u_n \|_2) \)

ellipticity: \( c_1 \| u_n \|_2^2 \leq c (1 + \| u_n \|_2 \leq \| u_n \|_2^2 + c \) \rightarrow \| u_n \|_2^2 \leq c \) constant

for a subsequence \( u_n \rightarrow u \; \text{in} \; W_0^1(\Omega) \)

\( a \) \( \exists \lambda > 0 \) \( (\lambda I - A) \) is invertible and onto \( \lambda R \| u \| = \| (\lambda I - A)^{-1} u \| \leq \lambda R \| u \|

\( \Rightarrow \exists \lambda > 0 \; \lambda \in L^2(\Omega) \Rightarrow \| \lambda u \| + \int A \nabla u \cdot \nabla \varphi + b \nabla u \varphi + cu \varphi = \int f \varphi \; \text{for } \varphi \in W_0^1(\Omega) \)

and \( \| u \|_2 \leq \frac{c_1}{\lambda} \| R \|_2 \)

In winter semester, we did: \( \exists \lambda > 0 \; \lambda \in L^2(\Omega) \Rightarrow \| \lambda u \| + \int A \nabla u \cdot \nabla \varphi + b \nabla u \varphi + cu \varphi = \int f \varphi \; \text{for } \varphi \in W_0^1(\Omega) \)

\( \lambda I - \lambda^* I + \lambda^* I + Lu = f \)
\[ \delta_t u - \Delta u = 0 \quad \text{in} \quad (0,T) \times \Omega \]
\[ u(0) = u_0 \quad \text{in} \quad \Omega \]
\[ u \in C^2(\Omega) \Rightarrow \exists u \in L^2(\Omega) \cap W^{1,2}(\Omega) \]
\[ u \in C(\Omega) \cap L^p(\Omega) \quad \text{will be done by semigroup} \]
\[ 2 < p < \infty \quad \Rightarrow \quad u \in L^p(\Omega) \quad \Rightarrow \quad A u = \Delta u \]
\[ B(A) = \{ W^{1,2}_0(\Omega) \subseteq u, \Delta u \in L^p(\Omega) \} \]
\[ \text{estimate for resolvent} \]
\[ \lambda - A u = f \quad \text{in} \quad \Omega \]
\[ u = 0 \quad \text{on} \quad \partial \Omega \]
\[ \exists ! \quad u \in W^{1,2}_0 \]

I multiply by \[ L(t) \]

\[ \int_0^T u(t) \, dt \]

methods: \[ \nu \cdot \nu (1 \circ u) = 0 \]

method: \[ \nu \cdot \nu (1 \circ u) = 0 \]

correction: test by \[ \nu \cdot \nu (1 \circ u) = 0 \]

\[ \text{Semigroup with right hand side} \]

\[ \delta_t u - \Delta u = f \quad \text{in} \quad (0,T) \times \Omega \]
\[ u = 0 \quad \text{in} \quad (0,T) \times \partial \Omega \]
\[ u(0) = u_0 \quad \text{in} \quad \Omega \]

\[ \text{Set:} \quad u(t) = S(t) u_0 + \int_0^t S(t - \tau) f(\tau) \, d\tau \]

compute \[ \delta_t u(t) = \delta_t (S(t) u_0) + \int_0^t \delta_t (S(t - \tau) f(\tau)) \, d\tau \]

\[ = \delta_t (S(t) u_0) + S(t) f(t) + \int_0^t \delta_t (S(t - \tau) f(\tau)) \, d\tau \]

\[ = A S(t) u_0 + \int_0^t A S(t - \tau) f(\tau) \, d\tau + f(t) \]

\[ = f(t) + A (S(t) u_0 + \int_0^t S(t - \tau) f(\tau) \, d\tau) - f(t) + A u(t) \]