Difficult homework(s)

Homework 1: Define the mapping $A : \mathbb{R}^{N\times d} \to \mathbb{R}^{N\times d}$ as

$$\xi \mapsto \arctan(1 + |\xi|^2)\xi \qquad \forall \xi \in \mathbb{R}^{N\times d},$$

where $|\xi|^2$ is given as

$$|\xi|^2 := \sum_{\nu=1}^{N} \sum_{i=1}^{d} (\xi_i^\nu)^2.$$

Consider the problem: $\Omega \subset \mathbb{R}^d$ and $u_0 \in W^{1,2}(\Omega; \mathbb{R}^N)$ and to find $u : \Omega \to \mathbb{R}^N$

N-equations.

$$-\operatorname{div} A(\nabla u) = 0 \quad \text{in } \Omega, \qquad u = u_0 \quad \text{on } \partial\Omega.$$

$u = (u_1, u_2, \dots, u_N)$

It is a system of N-partial differential equations, which in components can be written as

$$(1) \qquad -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\arctan\left(1 + \sum_{k=1}^{d}\sum_{\nu=1}^{N} (\partial_{x_k} u^\nu)^2\right) \partial_{x_i} u^\mu \right) = 0 \text{ in } \Omega,$$
$$u^\mu = u_0^\mu \text{ on } \partial\Omega$$

for $\mu = 1, \dots, N$.

- Define a notion of a weak solution and prove the existence and uniqueness of weak solution
- Show that $u \in W^{2,2}_{loc}(\Omega; \mathbb{R}^N)$. Hint: Prove that A is uniformly monotone operator from $\mathbb{R}^{N\times d} \to \mathbb{R}^{N\times d}$, i.e., that it satisfies for some c_1

  $$(A(\xi_1) - A(\xi_2)) \cdot (\cancel{x}_1^{\,\xi} - \xi_2) \geq c_1 |\xi_1 - \xi_2|^2.$$

  Then follow step by step the proof from winter semester for linear equations.
- Show **rigorously** that $|\nabla u|^2$ is a sub-solution to a certain elliptic problem with measurable coefficients. That is, show that $|\nabla u|^2 \in W^{1,1+\varepsilon}_{loc}(\Omega)$ for some $\varepsilon > 0$ and that there exists a bounded measurable elliptic matrix with components $b_{ij}(x)$ such that for all nonnegative $\varphi \in \mathcal{C}_0^\infty(\Omega)$ there holds

  $$\sum_{i,j=1}^{d} \int_\Omega b_{ij}(x) \partial_{x_i}(|\nabla u|^2) \partial_{x_j} \varphi \leq 0.$$

- **premium difficult homework:** Based on the property above, prove rigorously that $u \in W^{1,\infty}_{loc}(\Omega; \mathbb{R}^N)$

Homework 2: Consider the problem

$$\partial_t u - \Delta u - \Delta_p u = 0 \text{ in } (0,T) \times \Omega, \quad u = 0 \text{ on } (0,T) \times \partial\Omega, \quad u = u_0 \text{ in } \Omega$$

with some $p \in (1, 2]$ and $u_0 \in L^2(\Omega)$, where Δ_p is the p-Laplacian and $\Omega \subset \mathbb{R}^d$ is a Lipschitz set. Assume that $u \in L^2(0,T; W_0^{1,2}(\Omega))$ solves the above problem in the following sense: For all $\varphi \in \mathcal{C}_0^\infty((-\infty, T) \times \Omega)$ there holds (here $Q := (0,T) \times \Omega$)

$$\int_Q -u\partial_t \varphi + (\nabla u + |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi = \int_\Omega \varphi(0,x) u_0(x)\, dx$$

- Show that $\partial_t u \in L^2(0,T;(W_0^{1,2}(\Omega))^*)$ and that $u(0) = u_0$ and that for arbitrary $\psi \in W_0^{1,2}(\Omega)$ and almost all $t \in (0,T)$ we have
$$\langle \partial_t u(t), \psi \rangle + \int_\Omega (\nabla u(t,x) + |\nabla u(t,x)|^{p-2} \nabla u(t,x)) \cdot \nabla \psi(x)\, dx = 0$$
- Show that if $p \in (1,2)$ then for arbitrary $u_0 \in L^2(\Omega)$ there exists $\tau \in (0,\infty)$ such that $u(t) = 0$ for all $t \geq \tau$. Hint: You should prove that there is $\delta > 0$ such that (formally)
$$\frac{d}{dt}\|u\|_2^2 + \delta \|u\|_2^{2\lambda} \leq 0$$
for some$\lambda \in (0,1)$. Then you should apply your knowledge from the theory of ordinary differential equations (inequalities).
- Show that if $p = 2$ then for arbitrary $u_0 \neq 0$ and arbitrary $t \in (0,\infty)$ we have $u(t) \neq 0$. Hint: show that if $\{w_k\}_{k\in\mathbb{N}}$ are eigenfunctions of Laplace operator with eigenvalues λ_k then for every k and t there holds
$$\|u(t)\|_2^2 \geq e^{-2\lambda_k t} \|P^k u_0\|_2^2,$$
where P_k is the projection to the span of first k eigenvectors.

Additional hint to HW1: In the part of proving subsolution property. Here is just the **formal/non-rigorous** hint. But since you have that $u \in W^{2,2}_{loc}$ this formal procedure can be justified rigorously (= DO IT).

Apply $\frac{\partial}{\partial x_k}$ to the μ-th equation in (1). Then multiply the product by $\partial_{x_k} u^\mu$ and sum over $k = 1, \dots, d$ and $\mu = 1, \dots, N$ to get

$$\begin{aligned}
0 &= -\sum_{i,k=1}^{d} \sum_{\mu=1}^{N} \partial_{x_k} u^\mu \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_k} \left(\arctan\left(1 + \sum_{k=1}^{d}\sum_{\nu=1}^{N} (\partial_{x_k} u^\nu)^2\right) \partial_{x_i} u^\mu \right)\right) \\
&= -\sum_{i,k=1}^{d} \sum_{\mu=1}^{N} \frac{\partial}{\partial x_i} \left(\partial_{x_k} u^\mu \frac{\partial}{\partial x_k} \left(\arctan\left(1 + \sum_{k=1}^{d}\sum_{\nu=1}^{N} (\partial_{x_k} u^\nu)^2\right) \partial_{x_i} u^\mu \right)\right) \\
&\quad + \sum_{i,k=1}^{d} \sum_{\mu=1}^{N} \frac{\partial^2 u^\mu}{\partial_{x_k} \partial x_i} \frac{\partial}{\partial x_k} \left(\arctan\left(1 + \sum_{k=1}^{d}\sum_{\nu=1}^{N} (\partial_{x_k} u^\nu)^2\right) \partial_{x_i} u^\mu \right) =: I_1 + I_2.
\end{aligned}$$

Show that[1] $I_2 \geq 0$ and that I_1 can be rewritten as

$$I_1 = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{d} b_{ij}(x) \left(\sum_{\mu=1}^{N} |\nabla u^\mu|^2 \right)\right) =: -\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left(\sum_{i=1}^{d} b_{ij}(x) \left(|\nabla u|^2\right)\right)$$

with a proper elliptic matrix b_{ij}.

[1] In fact you should be able to get that $I_2 \geq c \sum_{\mu=1}^{N} |\nabla^2 u^\mu|^2$.