Bifurcations, pattern formation and synchronization in a few RD systems and networks of RD systems

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Workshop

Partial differential equations describing far from equilibrium open systems
Outline

• Equation 1 (1d)
• Equation 3 (2d, patterns)
• Equation 4 (1d, 2d, patterns)
• Networks (synchronization)
Equation 1

\[
\begin{aligned}
\dot{u} &= -u^3 + \alpha u - v + u_{xx}, \quad (x, t) \in (0, 1) \times (0, +\infty) \\
\dot{v} &= u
\end{aligned}
\]  

with Neuman Boundary conditions
**Remark** In comparison with classical works of (for example M. Marion, R. Temam…), note that second equation of (1) has no dissipative term $-\delta v$.

Also, $0$ belongs to the closure of eigenvalues of the linearized operator.

This brings some technical difficulties.
\[
\begin{align*}
\left\{ \begin{array}{l}
 u_t &= -u^3 + \alpha u - v + u_{xx}, \quad (x, t) \in (0, 1) \times (0, +\infty) \\
v_t &= u
\end{array} \right.
\end{align*}
\]  

We denote
\[
\mathcal{H} = L^2(0, 1) \times L^2(0, 1)
\]

**Proposition 1.** For any initial conditions in \( \mathcal{H} \), there exists a unique continuous function \((u, v)(t)\) solution of (1). Furthermore, \( |u(t)|_{L^\infty} \) is bounded in the compacts sets of \((0, +\infty)\).
Global Stability

\[
\begin{align*}
    u_t &= -u^3 + \alpha u - v + u_{xx}, \quad (x, t) \in (0, 1) \times (0, +\infty) \\
    v_t &= u \\
\end{align*}
\]

(1)

**Proposition 2.** Assume \( \alpha < 0 \). For any initial conditions in \( \mathcal{H} \), the solution \((u, v)(t)\) of (1) satisfies

\[
\lim_{t \to +\infty} ||(u, v)(t)||_{\mathcal{H}} = 0.
\]
Proof (Main ideas)

• The main point is that:

$$\frac{d}{dt} \|(u, v)(t)\|_H^2 \leq 0$$

• Then we work on a Galerkin approximation. We gain compacity and apply the LaSalle’s invariance principle.

• We conclude thanks to a uniform control between the Galerkin approximation and the solution in $H$. 

$\square$
Proposition 3. The point $(0,0)$ is the unique stationary solution of equation (2). If $\alpha \leq 0$ all the trajectories converge towards $(0,0)$. If $\alpha > 0$ equation (2) admits a unique limit-cycle which attracts all the trajectories distinct from $(0,0)$. Furthermore, at $u = 0$ a supercritical Hopf bifurcation occurs.
Cascade of Hopf Bifurcations

\[
\begin{cases}
  u_t = -u^3 + \alpha u - v + u_{xx}, \quad (x, t) \in (0, 1) \times (0, +\infty) \\
  v_t = u
\end{cases}
\]

(1)

Proposition 4. The eigenvalues of the linearized operator associated to (1) write:

\[
\mu_k^{1,2} = 0.5(\lambda_k + \alpha^+ \sqrt{(\alpha + \lambda_k)^2 - 4})
\]

with

\[
\lambda_k = -k^2 \pi^2.
\]

In particular, as \(\alpha\) crosses \(\lambda_k\) from left to right the two complex conjugate eigenvalues \(\mu_k^{1,2}\) cross the imaginary axis from left to right. Furthermore, for fixed \(\alpha\)

\[
\lim_{k \to +\infty} \mu_k^1 = -\infty, \quad \lim_{k \to +\infty} \mu_k^2 = 0^-
\]
A positively stable invariant set

\[
\begin{align*}
  u_t &= -u^3 + \alpha u - v + u_{xx}, \quad (x, t) \in (0, 1) \times (0, +\infty) \\
  v_t &= u
\end{align*}
\]

(1)

Proposition 5. Assume \(0 \leq \alpha < \lambda_1\). Assume that initial conditions satisfy

\[ u(x, 0) = u(1 - x, 0) \text{ and } v(x, 0) = v(1 - x, 0). \]

Then, the solution \((u, v)(t)\) of (1) satisfies

\[ \lim_{t \to +\infty} \| (u, v)(t) \|_{\mathcal{H}} = 0. \]
Local Stability

\begin{equation}
\begin{aligned}
 u_t &= -u^3 + \alpha u - v + u_{xx}, \quad (x, t) \in (0, 1) \times (0, +\infty) \\
v_t &= u
\end{aligned}
\end{equation}

Theorem 1. For $0 < \alpha < \lambda_1$, there exists a sequence $(\mu_k)_{k \in \mathbb{N}}$ such that if 
$(u_k(0), v_k(0)) \in B(0, \mu_k)$
then 
$$\lim_{t \to +\infty} \| (u(t) - u_0(t), v(t) - v_0(t)) \|_{\mathcal{H}} = 0,$$
where $B(0, \mu_k) \subset \mathbb{R}^2$ is the ball of center $(0, 0)$ and radius $\mu_k$, and $u_k(t) = \int_0^1 u(x, t) \varphi_k(x) dx$, $v_k(t) = \int_0^1 v(x, t) \varphi_k(x) dx$, where $\varphi_0(x) = 1$ and $\varphi_k(x) = \sqrt{2} \cos(k\pi x)$ for $k > 1$. 

BA, Qualitative Analysis of Reaction-Diffusion Systems in Neuroscience context.
https://arxiv.org/abs/1903.05754
Equation 3

\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u
\end{align*}
\]  

with $\Omega \subset \mathbb{R}^2$ an open regular bounded set and NBC.
\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u
\end{align*}
\]
Invariant manifold

\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u
\end{align*}
\]

Theorem 2. Assume that we can divide the domain \( \Omega \) into a partition

\[
\Omega = \left( \bigcup_{i \in \{1, \ldots, l\}} U_i \right) \cup \left( \bigcup_{i \in \{1, \ldots, l\}} V_i \right)
\]

such that there exists a diffeomorphism \( \phi \) that maps each \( U_i \) onto \( V_i \) with \( \det J_\phi = 1 \), where \( J_\phi \) denotes the Jacobian of \( \phi \), and initial conditions such that for all \( x \in \bigcup_{i \in \{1, \ldots, l\}} U_i \) and all \( t \in \mathbb{R}^+ \)

\[
(u(\phi(x), t), v(\phi(x), t)) = -(u(x, t), v(x, t))
\]

then the solution cannot converge to a non-zero constant solution in space solution of the diffusion less system.
\[
\begin{cases}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u
\end{cases}
\] (3)
\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{align*}
\] (4)

\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v \\
v_t &= u - c
\end{align*}
\] (5)

$c = -1.5$

$c = 0$
Is system (4) able to generate oscillatory signal and to propagate it?

Idea: Oscillatory signal initiates at some point and propagates throughout excitatory tissue thanks to diffusion

\[
\begin{aligned}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{aligned}
\]
We consider $c(x)$ such that:

- $c(x)=0$ for $x$ close to the center (oscillatory dynamics for the ODE)
- $c(x)=c_0<-1$ otherwise (excitatory dynamics for the ODE)

\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{align*}
\]
\[
\begin{aligned}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{aligned}
\] (4)

1d

\(c_0 = -1.2\)  \hspace{1cm}  \(c_0 = -1.12\)  \hspace{1cm}  \(c_0 = -1.01\)
\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{align*}
\] (4)

1d

\[c_0 = -1.2\]  \quad \[c_0 = -1.12\]  \quad \[c_0 = -1.01\]
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{align*} (4)

\textbf{1d}

\begin{align*}
c(x) &\leq 0 \\
c(0) &= 0 \\
\lim_{p \to 0} c(x) &= 0 \\
\forall x \in (a, b), x \neq 0, c(x) \text{ is decreasing} \\
\forall x \in (a, b), x \neq 0 \lim_{p \to +\infty} c(x) &= -\infty
\end{align*}
\[
\begin{aligned}
&\epsilon u_t = -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
&v_t = u - c(x)
\end{aligned}
\]  

(4)

**Proposition 6.** The eigenvalues of the linearized operator associated to (4) write:

\[
\mu_{k}^{1,2} = \frac{1}{2\epsilon} \left( \lambda_{k}^+ - \sqrt{(\lambda_{k})^2 - 4\epsilon} \right)
\]

where

\[\lambda_{k}, k \in,\]

are the eigenvalues associated to the Sturm-Liouville equation

\[f'(\bar{u}) + u_{xx} = \lambda u\]

In particular, \(\lambda_{0} < 3\) and there exists a number \(p^*\) such that as \(p\) crosses \(p^*\) from right to left the two complex conjugate eigenvalues \(\mu_{0}^{1,2}\) cross the imaginary axis from left to right. Furthermore, for fixed \(p\)

\[
\lim_{k \to +\infty} \mu_{k}^1 = -\infty, \quad \lim_{k \to +\infty} \mu_{k}^2 = 0^-
\]
\[
\begin{cases}
\epsilon u_t = -u^3 + 3u - v + \Delta u, (x, t) \in \Omega \times (0, +\infty) \\
v_t = u - c(x)
\end{cases}
\]

Theorem 4. There exists $\delta > 0$ such that for $p > p^* - \delta$, there exists a sequence $(\mu_k)_{k \in }$ such that if
\[(u_k(0), v_k(0)) \in B(0, \mu_k)\]
then
\[\lim_{t \to +\infty} \| (u(t) - u_0(t), v(t) - v_0(t)) \|_{\mathcal{H}} = 0,\]
where $B(0, \mu_k) \subset \mathbb{R}^2$ is the ball of center $(0, 0)$ and radius $\mu_k$, and $u_k(t), v_k(t)$ denote the projection into the subspace associated with the $k$th eigenvalue.
\[
\begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x, t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{align*}
\]
\[
\left\{ \begin{align*}
\epsilon u_t &= -u^3 + 3u - v + \Delta u, \quad (x,t) \in \Omega \times (0, +\infty) \\
v_t &= u - c(x)
\end{align*} \right. 
\] (4)
Networks of FHN RD systems
Networks of FHN RD systems

\[
\begin{align*}
    u_{it} &= F(u_i, v_i) + Q \Delta u_i + \sum_{k=1}^{n} c_{ik} u_k, \quad i \in \{1, \ldots, n\} \\
    v_{it} &= -\sigma(x)v_i + \Phi(x, u_i),
\end{align*}
\]

Unidirectional line

Fully connected

Ring
Theorem 5 (Fully connected networks). We assume that the coupling terms $c_{ij}$ satisfy for all $i \neq j$ $c_{ij} > \frac{1}{n}$ then the network synchronizes in the norm $L^2$.


THANKS