Bifurcations, pattern formation and synchronization in a few RD systems and networks of RD systems

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Workshop

Partial differential equations describing far from equilibrium open systems

Outline

- Equation 1 (1d)
- Equation 3 (2d, patterns)
- Equation 4 (1d,2d, patterns)
- Networks (synchronization)

Equation 1

$$\begin{cases} u_t = & -u^3 + \alpha u - v + u_{xx}, (x,t) \in (0,1) \times (0,+\infty) \\ v_t = & u \end{cases} \quad (1)$$

with Neuman Boundary conditions

Remark In comparison with classical works of (for example M. Marion, R. Temam…), note that second equation of (1) has no dissipative term $-\delta v$.
Also, 0 belongs to the closure of eigenvalues of the linearized operator.

This brings some technical difficulties.

$$\begin{cases} u_t = & -u^3 + \alpha u - v + u_{xx}, (x,t) \in (0,1) \times (0,+\infty) \\ v_t = & u \end{cases} \tag{1}$$

We denote

$$\mathcal{H} = L^2(0,1) \times L^2(0,1)$$

Proposition 1. *For any initial conditions in $\mathcal{H}$, there exists a unique continuous function $(u,v)(t)$ solution of (1). Furthermore, $|u(t)|_{L^\infty}$ is bounded in the compacts sets of $(0,+\infty)$.*

Global Stability

$$\begin{cases} u_t = & -u^3 + \alpha u - v + u_{xx}, (x,t) \in (0,1) \times (0,+\infty) \\ v_t = & u \end{cases} \quad (1)$$

Proposition 2. *Assume $\alpha < 0$. For any initial conditions in $\mathcal{H}$, the solution $(u,v)(t)$ of (1) satisfies*

$$\lim_{t \to +\infty} ||(u,v)(t)||_{\mathcal{H}} = 0.$$

Proof(Main ideas)

- The main point is that:

$$\frac{d}{dt}||(u,v)(t)||^2_{\mathcal{H}} \leq 0$$

- Then we work on a Galerkin approximation. We gain compacity and apply the LaSalle's invariance principle.

- We conclude thanks to a uniform control between the Galerkin approximation and the solution in $\mathcal{H}$.

$\square$

Diffusionless ODE Equation

$$\begin{cases} u_t = & -u^3 + \alpha u - v \\ v_t = & u \end{cases} \quad (2)$$

Proposition 3. *The point* $(0,0)$ *is the unique stationary solution of equation* (2). *If* $\alpha \leq 0$ *all the trajectories converge towards* $(0,0)$. *If* $\alpha > 0$ *equation* (2) *admits a unique limit-cycle which attracts all the trajectories distinct from* $(0,0)$. *Furthermore, at* $u = 0$ *a supercritical Hopf bifurcation occurs.*

Cascade of Hopf Bifurcations

$$\begin{cases} u_t = & -u^3 + \alpha u - v + u_{xx}, (x,t) \in (0,1) \times (0,+\infty) \\ v_t = & u \end{cases} \quad (1)$$

Proposition 4. *The eigenvalues of the linearized operator associated to* (1) *write:*

$$\mu_k^{1,2} = 0.5\big(\lambda_k + \alpha \pm \sqrt{(\alpha + \lambda_k)^2 - 4}\big)$$

with

$$\lambda_k = -k^2\pi^2.$$

In particular, as α crosses λ_k from left to right the two complex conjugate eigenvalues $\mu_k^{1,2}$ cross the imaginary axis from left to right. Furthermore, for fixed α

$$\lim_{k\to+\infty} \mu_k^1 = -\infty, \lim_{k\to+\infty} \mu_k^2 = 0^-$$

A positively stable invariant set

$$\begin{cases} u_t = & -u^3 + \alpha u - v + u_{xx}, (x,t) \in (0,1) \times (0,+\infty) \\ v_t = & u \end{cases} \quad (1)$$

Proposition 5. *Assume* $0 \leq \alpha < \lambda_1$. *Assume that initial conditions satisfy*

$$u(x,0) = u(1-x,0) \text{ and } v(x,0) = v(1-x,0).$$

Then, the solution $(u,v)(t)$ *of* (1) *satisfies*

$$\lim_{t \to +\infty} ||(u,v)(t)||_{\mathcal{H}} = 0.$$

Local Stability

$$\begin{cases} u_t = & -u^3 + \alpha u - v + u_{xx}, (x,t) \in (0,1) \times (0,+\infty) \\ v_t = & u \end{cases} \quad (1)$$

Theorem 1. *For* $0 < \alpha < \lambda_1$, *there exists a sequence* $(\mu_k)_{k\in}$ *such that if*

$$(u_k(0), v_k(0)) \in B(0, \mu_k)$$

then

$$\lim_{t\to+\infty} ||(u(t) - u_0(t), v(t) - v_0(t))||_{\mathcal{H}} = 0,$$

where $B(0,\mu_k) \subset \mathbb{R}^2$ *is the ball of center* $(0,0)$ *and radius* μ_k, *and* $u_k(t) = \int_0^1 u(x,t)\varphi_k(x)dx$, $v_k(t) = \int_0^1 v(x,t)\varphi_k(x)dx$, *where* $\varphi_0(x) = 1$ *and* $\varphi_k(x) = \sqrt{2}\cos(k\pi x)$ *for* $k > 1$.

BA, Qualitative Analysis of Reaction-Diffusion Systems in Neuroscience context.
https://arxiv.org/abs/1903.05754

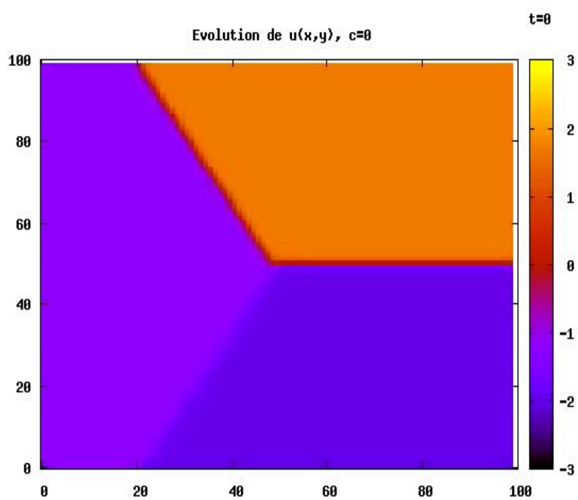
Equation 3

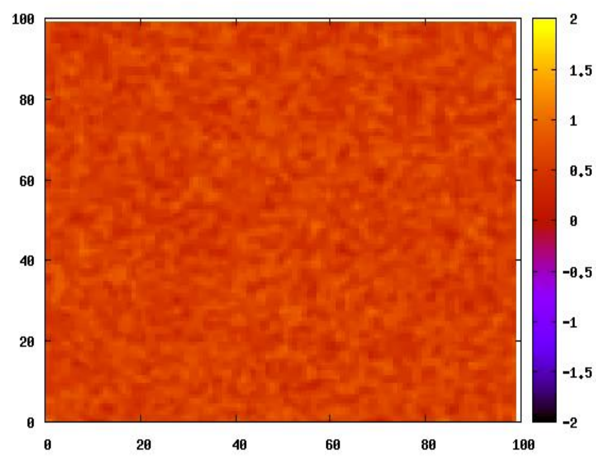
$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u \end{cases} \quad (3)$$

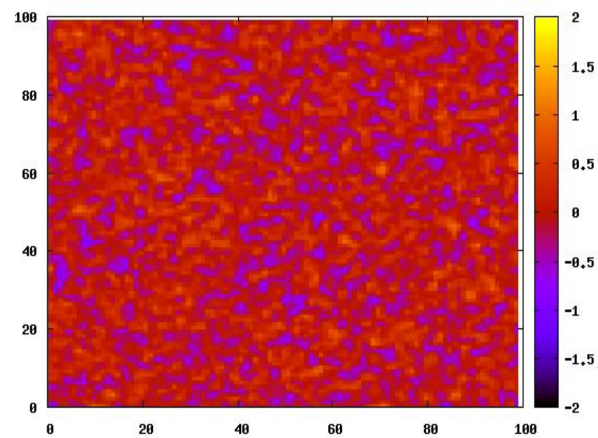
with $\Omega \subset \mathbb{R}^2$ an open regular bounded set and NBC.

Patterns

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u \end{cases} \quad (3)$$







Invariant manifold

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u \end{cases} \quad (3)$$

Theorem 2. *Assume that we can divide the domain Ω into a partition*

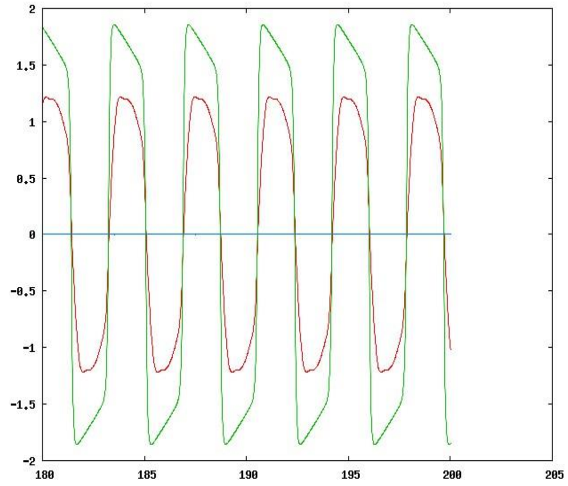
$$\Omega = \left(\cup_{i \in \{1,\dots,l\}} U_i \right) \cup \left(\cup_{i \in \{1,\dots,l\}} V_i \right)$$

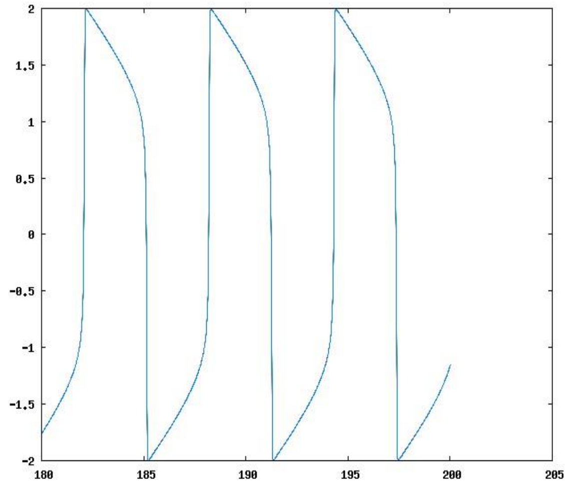
such that there exists a diffeomorphism ϕ that maps each U_i onto V_i with $\det J_\phi = 1$, *where J_ϕ denotes the Jacobian of ϕ, and initial conditions such that for all $x \in \cup_{i \in \{1,\dots,l\}} U_i$ and all $t \in \mathbb{R}^+$*

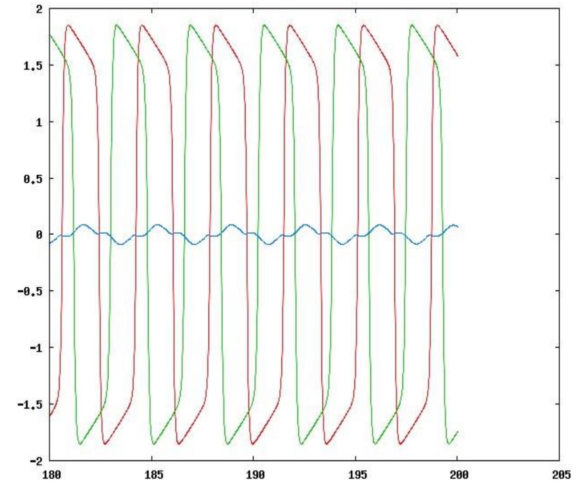
$$(u(\phi(x), t), v(\phi(x), t)) = -(u(x,t), v(x,t))$$

then the solution cannot converge to a non zero constant solution in space solution of the diffusion less system.

$$
\left\{
\begin{array}{rl}
\epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\
v_t = & u
\end{array}
\right. \quad (3)
$$



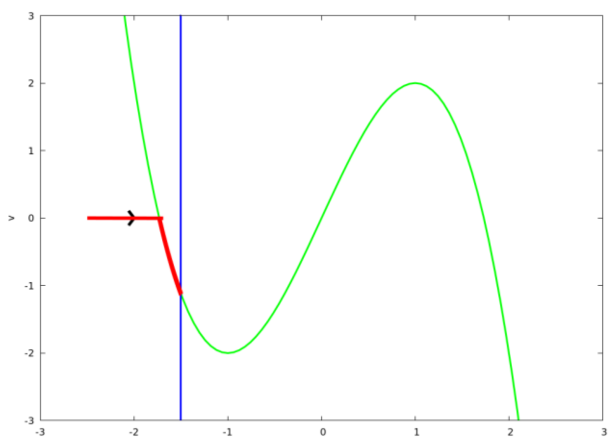




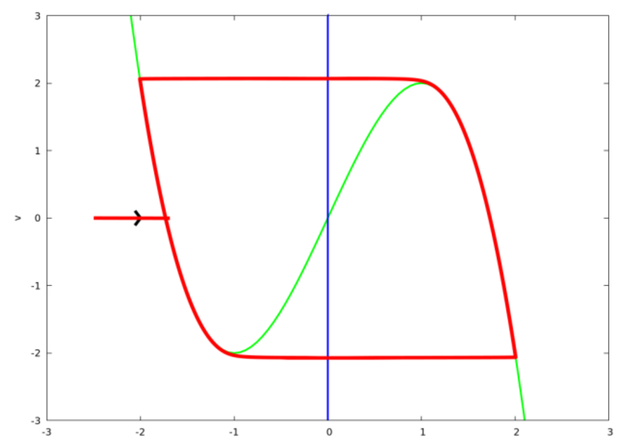
BA, M.A. Aziz-Alaoui, *Acta Biotheoretica, (2016)*

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{cases} \quad (4)$$

$$\boxed{\begin{cases} \epsilon u_t = & -u^3 + 3u - v \\ v_t = & u - c \end{cases} \quad (5)}$$



c=-1.5



c=0

$$
\left\{
\begin{array}{rl}
\epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\
v_t = & u - c(x)
\end{array}
\right. \qquad (4)
$$

Is system (4) able to generate oscillatory signal and to propagate it?

Idea: Oscillatory signal initiates at some point and propagates throughout excitatory tissue thanks to diffusion

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{cases} \quad (4)$$

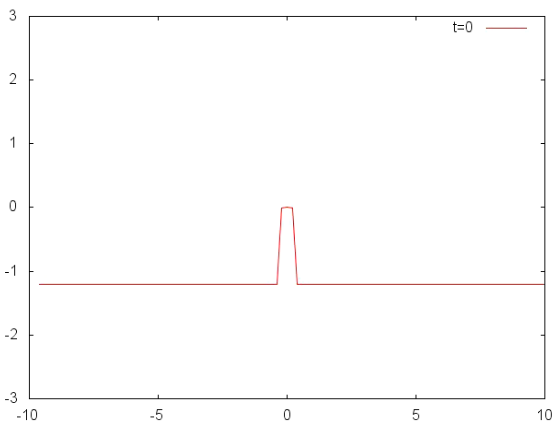
We consider *c(x)* such that:

- ***c(x)=0* for *x* close to the center (oscillatory dynamics for the ODE)**
- ***c(x)*=c_0<-1 otherwise (excitatory dynamics for the ODE)**

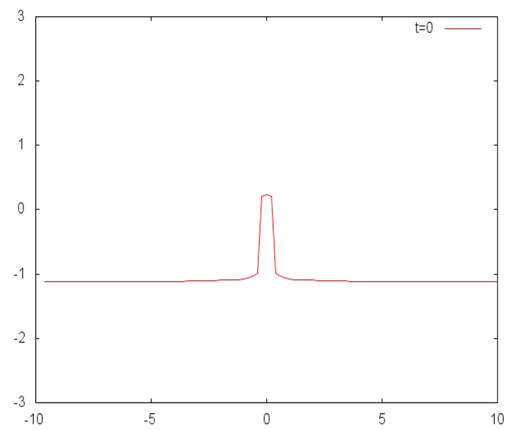
$$
\left\{ \begin{array}{rl} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{array} \right. \quad (4)
$$

1d

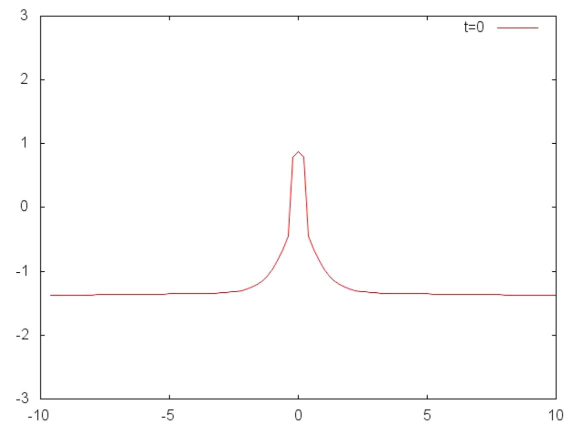
$c_{0=} - 1.2$



$c_{0=} - 1.12$



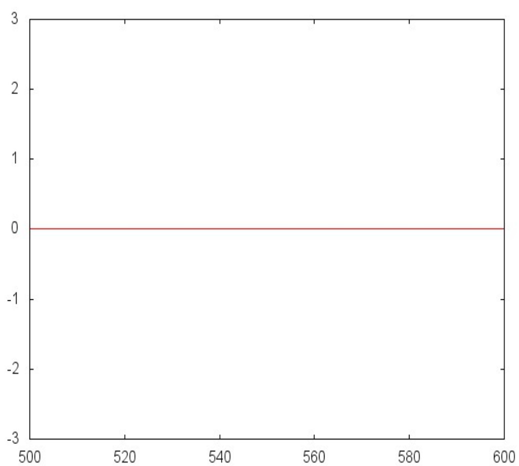
$c_{0=} - 1.01$



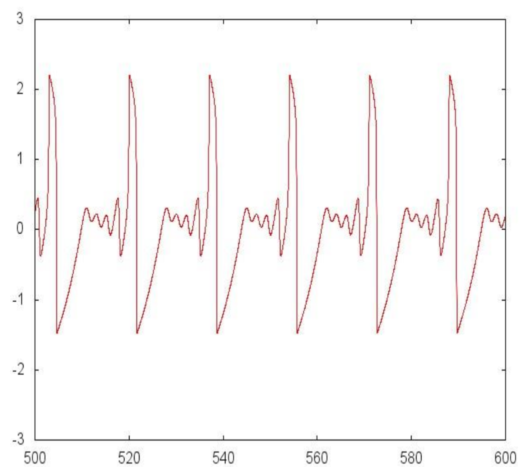
$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{cases} \quad (4)$$

1d

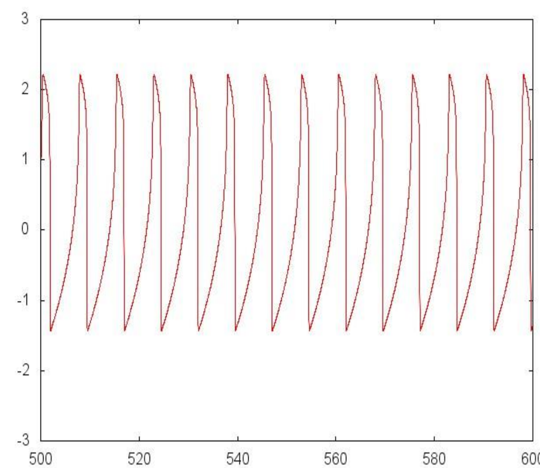
$c_{0=} - 1.2$



$c_{0=} - 1.12$



$c_{0=} - 1.01$



$$
\left\{\begin{array}{rl} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0,+\infty) \\ v_t = & u - c(x) \end{array}\right. \quad (4)
$$

1d

$$
c(x) \leq 0
$$

$$
c(0) = 0
$$

$$
\lim_{p \to 0} c(x) = 0
$$

$$
\forall x \in (a,b), x \neq 0, c(x) \text{ is decreasing}
$$

$$
\forall x \in (a,b), x \neq 0 \lim_{p \to +\infty} c(x) = -\infty
$$

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{cases} \quad (4)$$

Proposition 6. *The eigenvalues of the linearized operator associated to* (4) *write:*

$$\mu_k^{1,2} = \frac{1}{2\epsilon}\left(\lambda_k^+ - \sqrt{(\lambda_k)^2 - 4\epsilon}\right)$$

where

$$\lambda_k, k \in,$$

are the eigenvalues associated to the Sturm-Liouville equation

$$f'(\bar{u}) + u_{xx} = \lambda u$$

In particular, $\lambda_0 < 3$ *and there exists a number* p^* *such that as* p *crosses* p^* *from right to left the two complex conjugate eigenvalues* $\mu_0^{1,2}$ *cross the imaginary axis from left to right. Furthermore, for fixed* p

$$\lim_{k \to +\infty} \mu_k^1 = -\infty, \lim_{k \to +\infty} \mu_k^2 = 0^-$$

BA, *IJBC, (2016)*

$$\boxed{\left\{\begin{array}{rl} \epsilon u_t = & -u^3+3u-v+\Delta u, (x,t)\in\Omega\times(0,+\infty) \\ v_t = & u-c(x) \end{array}\right. \quad (4)}$$

Theorem 4. *There exists $\delta > 0$ such that for $p > p^* - \delta$, there exists a sequence $(\mu_k)_{k\in}$ such that if*

$$(u_k(0), v_k(0)) \in B(0, \mu_k)$$

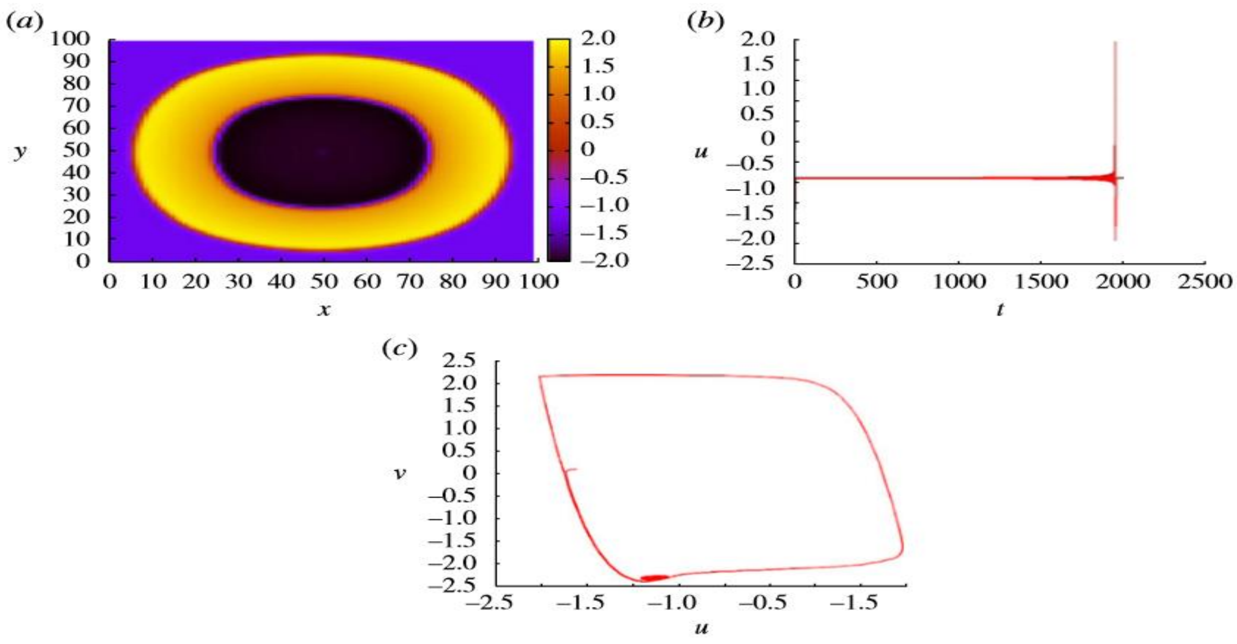
then

$$\lim_{t\to+\infty} ||(u(t)-u_0(t), v(t)-v_0(t))||_{\mathcal{H}} = 0,$$

where $B(0,\mu_k) \subset \mathbb{R}^2$ is the ball of center $(0,0)$ and radius μ_k, and $u_k(t), v_k(t)$ denote the projection into the subspace associated with the kth eigenvalue.

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{cases} \quad (4)$$

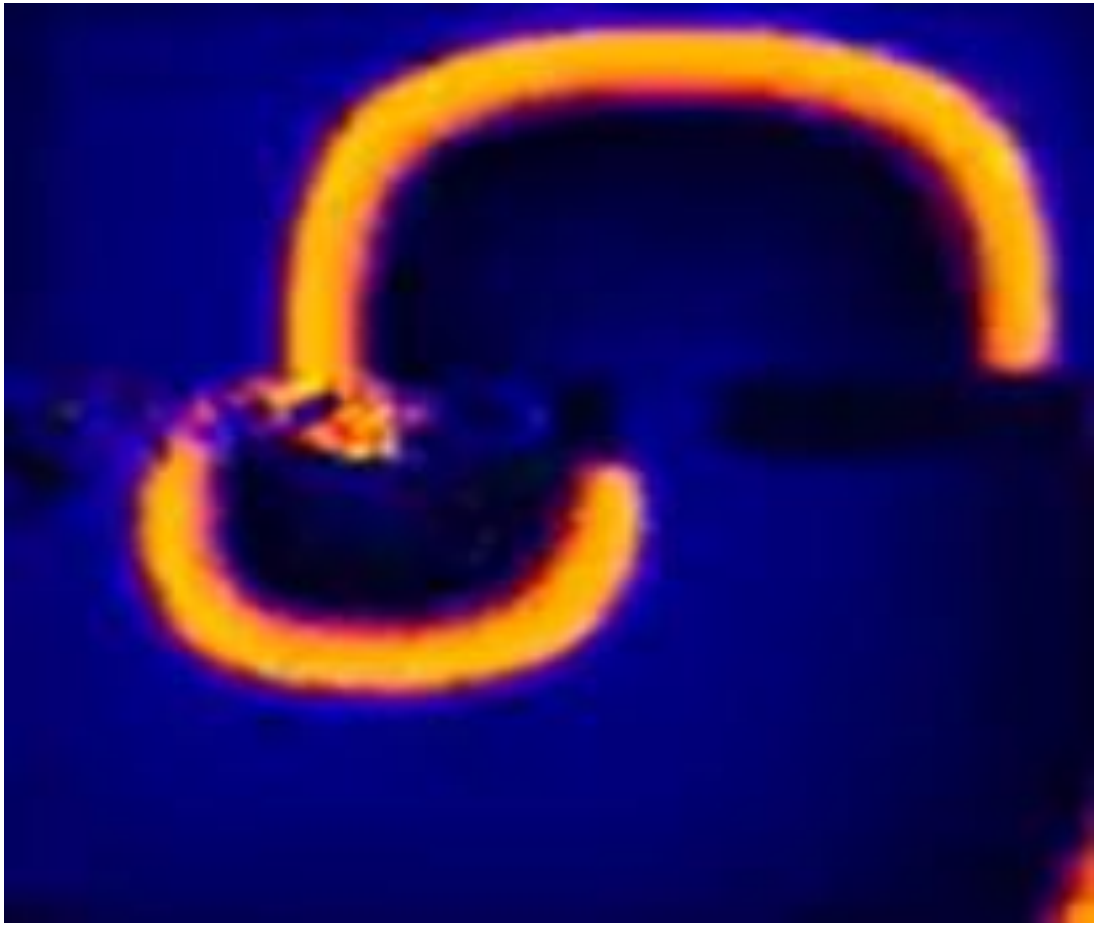
2d



BA, J-P Francoise, *PRSTA, (2009)*

$$\begin{cases} \epsilon u_t = & -u^3 + 3u - v + \Delta u, (x,t) \in \Omega \times (0, +\infty) \\ v_t = & u - c(x) \end{cases} \quad (4)$$

2d

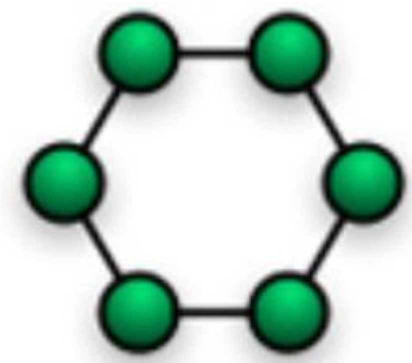


Networks of FHN RD systems

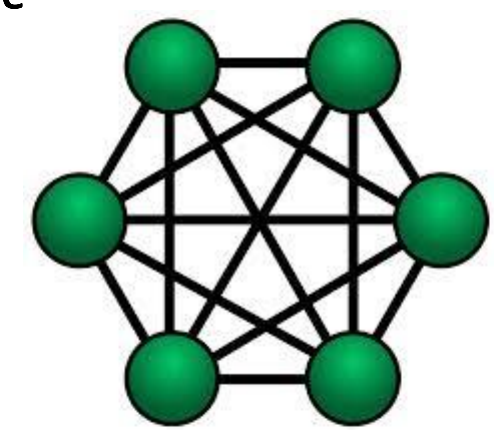
Networks of FHN RD systems

$$\begin{cases} u_{it} = F(u_i, v_i) + Q\Delta u_i + \sum_{k=1}^{n} c_{ik} u_k, & i \in \{1, \ldots, n\} \\ v_{it} = -\sigma(x) v_i + \Phi(x, u_i), & \end{cases}$$

Unidirectionnal line



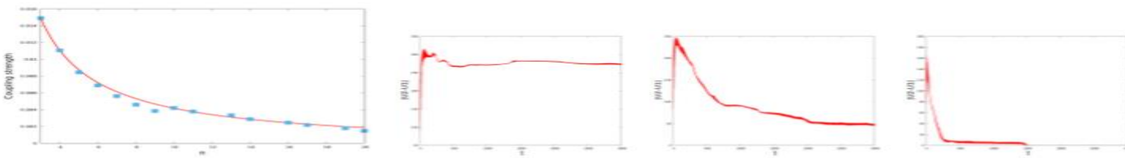
Ring

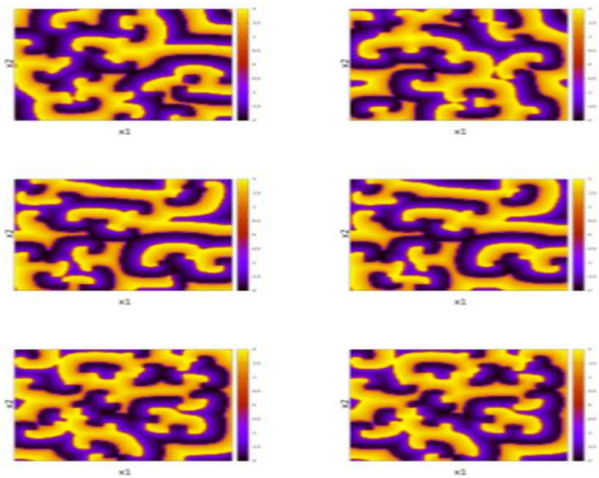


Fully connected

Networks

Theorem 5 (Fully connected networks). *We assume that the coupling terms c_{ij} satisfy for all $i \neq j$ $c_{ij} > \frac{1}{n}$ then the network synchronizes in the norm L^2.*





BA, M.A. Aziz-Alaoui, V.L.E. Phan, *IMA JAM, (2019)*

BA, M.A. Aziz-Alaoui, *ESAIM Proc, (2013)*

BA, M.A. Aziz-Alaoui, *CAMWA, (2012)*

THANKS