Partial differential equations describing far-from-equilibrium open systems

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Keywords:

- continuum thermodynamics
- instability vs stability in the theory of PDEs/infinite dimensional dynamical systems
- small scales vs large scales vs **mesoscales**
- open systems - allow heat transfer and mass transfer (in general, nonhomogenous Dirichlet or Neumann data)
- rigorous analysis (weak/strong/classical solutions)
- description of the dissipative structures in simple geometries
- rigorous constructive methods for the description of the dissipative structures in general geometries
- general methods for general dynamical systems

Instabilities

Instability	Field
Kelvin–Helmholtz instability	Stability of shearing flow
Kruskal–Shafranov instability	Plasma physics
Peratt instability	Plasma physics
Plateau–Rayleigh instability	Stability of jets and drops
Rayleigh–Bénard instability	Natural convection, Rayleigh–Bénard convection
Rayleigh–Taylor instability	Instability created by density startification
Richtmyer–Meshkov instability	Plasma physics, Astrophysics
Saffman–Taylor instability	Flow in porous medium
Taylor–Caulfield instability	Stratified shear flows
Taylor–Couette instability	Flow in rotating cylinder
Tollmien–Schlichting instability	Wave instability in shearing flows
Velikhov instability	Plasma physics (non-equilibrium MHD)
Velikhov-Chandrasekhar instability	Stability of rotating fluid in magnetic field
Weibel instability	Plasma physics

Instability	Field
Benjamin–Feir instability	Surface gravity waves
Buneman instability	Plasma physics
Chandrasekhar–Donnelly instability	Taylor-Couette instability of Helium II
Cherenkov instability	Plasma physics
Chromo-Weibel instability	Plasma physics
Crow instability	Aerodynamics
Darrieus–Landau instability	Stability of propagating flame
Dean instability	Stability of flow in a curved pipe
D'Yakov–Kontorovich instability	Stability of a plane shock
Faraday instability	Vibrating fluid surfaces
Farley–Buneman instability	Plasma instability
Görtler instability	Stability of flow along a concave boundary layer
Holmboe instability	Stratified shear flows
Jeans instability	Stability of interstellar gas clouds

Pipe flow

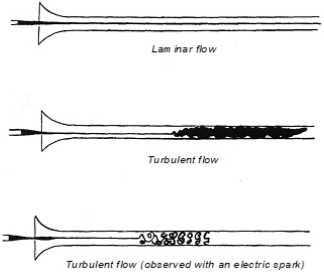

Figure: Reynolds experiment.

$$\boxed{\begin{aligned}\partial_t \boldsymbol{v} + \mathrm{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \frac{1}{Re}\Delta \boldsymbol{v} &= \nabla p,\\ \mathrm{div}\, \boldsymbol{v} &= 0\end{aligned}}$$

- $Re < 2300$ laminar flow
- $Re > 2900$ turbulent flow

Thermal convection

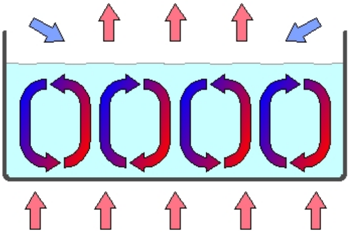

Figure: Rayleigh-Bénard convection

$$\operatorname{div} \boldsymbol{v} = 0,$$
$$\varrho_{ref} \partial_t \boldsymbol{v} + \varrho_{ref} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \mu \Delta \boldsymbol{v} + \varrho_{ref}(1 - \alpha(\theta - \theta_{ref}))\boldsymbol{b},$$
$$\varrho_{ref} c_p \partial_t \theta + \varrho_{ref} c_p \operatorname{div}(\boldsymbol{v}\theta) = \kappa \Delta \theta.$$

- Oberbeck–Boussinesq approximation
- The central idea is that the thermal expansivity of the fluid is neglected everywhere except in the buoyancy term in the balance of momentum
- appears when $\theta_{bot} \gg \theta_{top}$

Wrong/not proper models

- **thermal convection:** experimental data gives evidence to the so-called non-Oberbeck–Boussinesq effects, i.e., for large temperature gradient the Oberbeck–Boussinesq system does give adequate prediction
- **pipe flow:** the original Reynolds measurement is no longer valid, or more precisely, doing the same experiment at the same place nowdays gives much less critical Reynolds number; on the other hand for very long pipes and for experimental setting that is able to "almost" avoid disturbances coming from outside, the critical Reynolds number is much higher; Navier–Stokes system can be derived from Boltzmann equation for a fluctuation that is close to the uniform Maxwellian state, i.e., **close to the equilibrium**
- one should not blindly use Navier–Stokes or Oberbeck–Boussinesq systems in the **far-from-equilibrium settings**

Wrong/not proper boundary conditions - pipe flow

- usually the theory for dynamical systems is done for "conservative" boundary conditions
- **outflow:** how to chose outflow bc - Dirichlet/do nothing/Neumann/infinite pipe?
- **pipe with several outflows:** is there a non-heuristic way how to prescribe boundary conditions?
- **walls:** no-slips/partial slip/stick-slip/dynamical slip for velocity? Neumann vs Dirichlet for temperature?

Wrong/not proper concepts/methods

- **linearized stability:** prediction on small scales, also proper for long scales provided that the linearised operator is normal - not the case e.g. for pipe flow
  - **Taylor–Couette flow:** linearized stability leads to very good predictions - evenmore leads to very good description of the dynamical structures
  - **pipe flow:** the spectrum of linearized operator is complex, it is **numerically conjectured** that the spectrum is "far" from "crossing" zero, hence it predicts stability, but it is not the case
- **concept of (exponential) attractor:** perfect **theoretical** object for fluid flow with conservative boundary conditions but **for long scales**; due to disturbances it may not attract all solution at all; typically we do not know the structure (typical structures in the attractor); typically we just know its dynamics can be described by huge system of ODEs

Proposals - models

- **thermal convection:** to keep fluid thermally expansible and mechanically incompressible - i.e., the model "between" compressible Navier–Stokes equations and Oberbeck–Boussinesq system; study of asymptotic limits - one must start from proper pressure-temperature-density relation
- **pipe with several outflows:** choose boundary condition such that it minimize the entropy production
- **temperature** must be taken into account since for large Reynolds number there is significant heat transport

Proposals - models

- **thermal convection**

$$\alpha(\theta)(\partial_t\theta + \operatorname{div}(\theta \boldsymbol{v})) = \operatorname{div} \boldsymbol{v},$$

$$\varrho_{ref}\partial_t \boldsymbol{v} + \varrho_{ref} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla p + \mu\Delta \boldsymbol{v} + \frac{\mu}{3}\nabla(\operatorname{div} \boldsymbol{v}) + \varrho_{ref} \boldsymbol{b},$$

$$-\theta\alpha(\theta)(\partial_t p + \operatorname{div}(p\boldsymbol{v})) = \kappa\Delta\theta + 2\mu|\mathbf{D}_\delta|^2 - \frac{\varrho_{ref} c_p}{\alpha} \operatorname{div} \boldsymbol{v}$$

- **pipe flow** minimize the entropy production

$$\eta(\boldsymbol{v}) := \int \frac{|\mathbf{D}(\boldsymbol{v})|^2}{\theta}$$

over all solutions fulfilling Navier–Stokes–Fourier system with prescribed inflow and wall boundary conditions - choose proper geometry on the outflow!

Proposals - concepts

- **different topology:** instead of tracking $\boldsymbol{v}(t)$ point-wisely for t to track the dynamics of the whole trajectory $\{\boldsymbol{v}(\tau); \tau \in (t, t+\ell)\}$ - advantage: no regularity needed, concept of weak solution is sufficient - disadvantage: trajectories do not form a linear space
- proper **Lyapunov functionals** for open systems, e.g. based on relative energies/entropies, no regularity needed a priori
- **regularity theory** can help in justification of heuristic analysis

Proposals - concepts

- **numerically assisted proofs:** e.g., for linearization in pipe flow
- **finite dimensional approximations:** e.g.,

$$\partial_t \boldsymbol{x} = \mathsf{L}\boldsymbol{x} + \mathsf{N}(\boldsymbol{x}), \qquad \text{where} \qquad \mathsf{L} = \begin{bmatrix} -\frac{1}{Re} & 1 \\ 0 & -\frac{1}{Re} \end{bmatrix} \quad \text{and} \quad \mathsf{N}(\boldsymbol{x}) = x_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Here, Re denotes the analogue of the Reynolds number. The base solution is $\boldsymbol{x} = 0$, and the linear operator L is not normal and its eigenvalues are negative for all Re. The nonlinear term N fulfills $\mathsf{N}(\boldsymbol{x}) \cdot \boldsymbol{x} = 0$, which mimics the convective term in the Navier–Stokes setting.

  - if $Re < 2$, then the base solution $\boldsymbol{x} = 0$ is the only stationary solution.
  - if $Re > 2$, then there exist multiple stationary solutions, and the base state $\boldsymbol{x} = 0$ is still stable.

Goals

G1: qualitative analysis of relevant models and boundary conditions for far-from-equilibrium dynamics,

G2: rigorous description of dynamical systems on mesoscales,

G3: rigorous constructive methods for the description of the dissipative structures in the dynamical systems.

Goals in terms of pictures

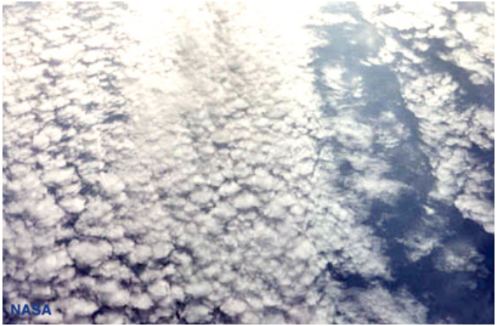

Figure: Rayleigh-Bénard convection

- Description of structures - why, when, how?
- Description of structures for weak solutions
- Rigorous qualitative justification of transition from stable to instable regimes