Bifurcation of symmetric domain walls for the Bénard-Rayleigh convection problem

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Online workshop, Prague, 21-25 September 2020

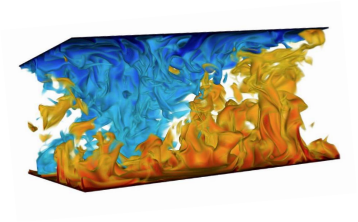
Bénard-Rayleigh problem

Bénard-Rayleigh problem

or . . .

Rayleigh-Bénard problem

Bénard-Rayleigh convection



[Erwin P. van der Poel
& Rodolfo Ostilla Mónico]

- viscous fluid layer between two horizontal plates
- heated from below and cooled from above

- *classical problem in hydrodynamics*
- *extensively studied . . . (experiments, analysis)*

Governing equations

three-dimensional convection

$$
\begin{aligned}
\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} + \nabla p &= \mathcal{P}(\theta e_z + \Delta \boldsymbol{V}) \\
\nabla \cdot \boldsymbol{V} &= 0 \\
\frac{\partial \theta}{\partial t} + \boldsymbol{V} \cdot \nabla \theta &= \Delta \theta + \mathcal{R}(\boldsymbol{V} \cdot e_z)
\end{aligned}
$$

- velocity field $\boldsymbol{V} = (V_x, V_\perp)$, $V_\perp = (V_y, V_z)$
- deviation of temperature from conduction profile θ, pressure p
- Rayleigh number $\mathcal{R}$, Prandtl number $\mathcal{P}$

Boussinesq approximation :

$$\rho = \rho_0 \left(1 - \alpha(T - T_0)\right) \qquad \mathcal{P} = \frac{\nu}{\kappa},\ \mathcal{R} = \frac{\alpha g d^3 (T_0 - T_1)}{\nu \kappa}$$

ρ density, α volume expansion coefficient, ν viscosity, κ thermal diffusivity, d distance between the planes, T temperature, $T_0 > T_1$

Governing equations

three-dimensional convection

$$\begin{aligned}\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} + \nabla p &= \mathcal{P}(\theta e_z + \Delta\boldsymbol{V})\\ \nabla\cdot\boldsymbol{V} &= 0\\ \frac{\partial\theta}{\partial t} + \boldsymbol{V}\cdot\nabla\theta &= \Delta\theta + \mathcal{R}(\boldsymbol{V}\cdot e_z)\end{aligned}$$

- velocity field $\boldsymbol{V} = (V_x, V_\perp)$, $V_\perp = (V_y, V_z)$
- deviation of temperature from conduction profile θ, pressure p
- Rayleigh number $\mathcal{R}$, Prandtl number $\mathcal{P}$

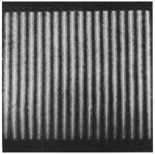
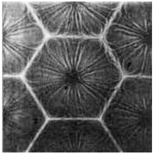
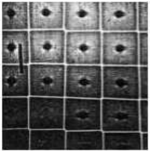
boundary conditions : rigid-rigid, rigid-free, or free-free

- rigid : $\boldsymbol{V} = \theta = 0, \quad z = 0$ and/or 1
- free : $\partial_z V_x = \partial_z V_y = V_z = \theta = 0, \quad z = 0$ and/or 1

symmetries : *horizontal translations and rotations*

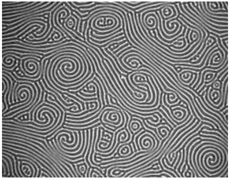
Regular patterns

- rolls, hexagons, squares, . . .

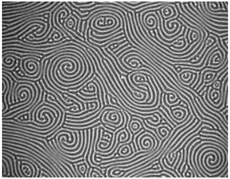
[Yudovich ’65 – ’67, Rabinowitz ’68, Sattinger ’70 – ’71, Iooss ’71, Kielhöfer & Kirchgässner ’73, Golubitsky, Stewart & Schaeffer ’88, . . .]

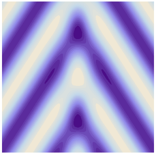
- **grain boundaries/domain walls, dislocations, . . .**

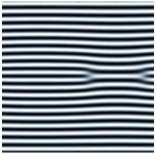


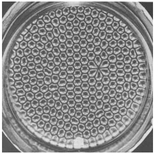
DEFECTS

grain boundaries/domain walls, dislocations, . . .





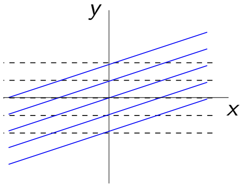


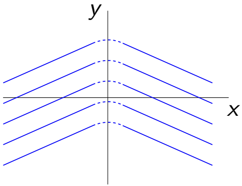


- *many results for simpler models . . .*
- **Swift-Hohenberg equation : [H. & Scheel, 2012]**
  see also [Lloyd, Scheel, 2017; Ercolani, Kamburov, Lega, 2018]

Rolls and domain walls

- rolls ⇝ rotated rolls ⇝ domain walls



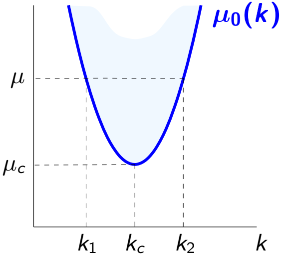


Existence of rolls

- **classical 2d convection problem**
- **approach :** *e.g., use a Lyapunov-Schmidt reduction*

[Yudovich, 1966-67]

- bifurcation parameter : $\boxed{\mu = \mathcal{R}^{1/2}}$
- wavenumber $\boxed{k}$ in the horizontal variable y
- rolls exist in the shaded region above the curve $\mu_0(k)$:

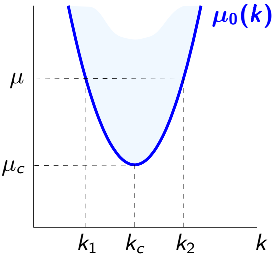


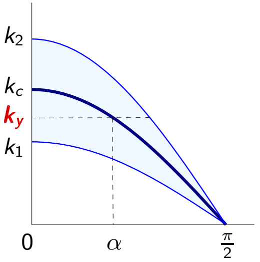
[Anne Pellew & Southwell, 1940]

Rotated rolls

invariance under horizontal rotations

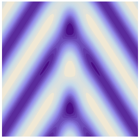
- rotation with angle $\boxed{\alpha}$
- wavenumber $\boxed{k_y = k\cos\alpha}$ in the horizontal variable y





Approach : spatial dynamics

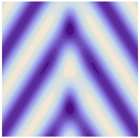
domain walls



- *steady solutions*
- *connect rolls with different orientations*

Approach : spatial dynamics

domain walls



- *steady solutions*
- *connect rolls with different orientations*

---

steady problem : *dynamical system*

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\boldsymbol{x}} = \mathcal{F}(\mathbf{U}; \mu, \mathcal{P})$$

- *rolls* $\longleftrightarrow$ *equilibria (relative)*
- *domain walls* $\longleftrightarrow$ *heteroclinic orbits*

Main steps

- **dynamical system** *(infinite-dimensional)*

$$\boxed{\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\boldsymbol{x}} = \mathcal{F}(\mathbf{U};\, \mu, \mathcal{P})}$$

Main steps

- **dynamical system** *(infinite-dimensional)*

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\boldsymbol{x}} = \mathcal{F}(\mathbf{U}; \mu, \mathcal{P})$$

- **parameters :** *find the bifurcation points*
  - particular solution $\boldsymbol{U}_* = \mathbf{0}$
  - linear operator $\boldsymbol{D_U}\mathcal{F}(\boldsymbol{U}_*; \mu, \mathcal{P})$
  - *ex. purely imaginary eigenvalues*

Main steps

- **dynamical system** *(infinite-dimensional)*

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\boldsymbol{x}} = \mathcal{F}(\mathbf{U}; \mu, \mathcal{P})$$

- **parameters :** *find the bifurcation points*
- **reduction :** *use center manifolds*
  - locally invariant manifolds
  - *contain all small bounded solutions*

Main steps

- **dynamical system** *(infinite-dimensional)*

$$\boxed{\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\boldsymbol{x}} = \mathcal{F}(\mathbf{U}; \mu, \mathcal{P})}$$

- **parameters :** *find the bifurcation points*
- **reduction :** *use center manifolds*
- **reduced dynamics :** *finite-dimensional dynamical system*
  - use normal forms
  - study the leading order dynamics *(ex. heteroclinic orbits)*
  - show persistence of heteroclinic orbits

Dynamical system

- **steady system**

$$\begin{aligned} \boldsymbol{V}\cdot\nabla\boldsymbol{V}+\nabla p &= \mathcal{P}(\theta e_z+\Delta\boldsymbol{V}) \\ \nabla\cdot\boldsymbol{V} &= 0 \\ \boldsymbol{V}\cdot\nabla\theta &= \Delta\theta+\mathcal{R}(\boldsymbol{V}\cdot e_z) \end{aligned}$$

- **new variables**

$$\boxed{\mathbf{U}=(\boldsymbol{V},\boldsymbol{W},\theta,\varphi)}, \quad \boxed{\boldsymbol{W}=\partial_x\boldsymbol{V}-\mathcal{P}^{-1}pe_x, \quad \varphi=\partial_x\theta}$$

- **dynamical system**

$$\boxed{\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}\boldsymbol{x}}=\mathcal{F}(\mathbf{U};\mu,\mathcal{P})}$$

Dynamical system

symmetries

- **reversibility :** *the vector field anti-commutes with*

$$\boxed{\mathcal{S}_1 \mathbf{U} = (-V_x, V_\perp, W_x, -W_\perp, \theta, -\varphi)}$$

- **reflection :** *the vector field commutes with*

$$\boxed{\mathcal{S}_2 \mathbf{U}(y, z) = (V_x, -V_y, V_z, W_x, -W_y, W_z, \theta, \varphi)(-y, z)}$$

- **translations in** y

Parameters

- **particular solution $U_* = 0$; linear operator**

$$\boxed{\mathcal{L}_\mu = D_U \mathcal{F}(0;\mu,\mathcal{P})}$$

- **bifurcation points**
  - *ex. purely imaginary eigenvalues (here : discrete spectrum)*
  - *co-existence of rolls with different orientations*

PARAMETERS

- **particular solution $U_* = 0$; linear operator**

$$\boxed{\mathcal{L}_\mu = D_U \mathcal{F}(0; \mu, \mathcal{P})}$$

- **bifurcation points**
  - *ex. purely imaginary eigenvalues (here : discrete spectrum)*
  - *co-existence of rolls with different orientations*

- **connection with the classical evolution problem :**
  three-dimensional convection with periodic boundary conditions in the horizontal directions (x, y)

$$\boxed{\frac{dU}{dt} = \mathbf{L}_\mu U + \mathbf{R}(U)}$$

Parameters

spatial dynamics

- *eigenvalue problem :* $ik_x U = \mathcal{L}_\mu U$
- *with Fourier series in y :* $U(y,z) = e^{ink_y y} U_n(z)$

$\rightsquigarrow$ *solutions of the linear system :*

$$\boxed{\mathbf{U}(x,y,z) = e^{ik_x x + ink_y y} U_n(z)}$$

Parameters

spatial dynamics

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---

temporal dynamics

- *solutions in the kernel of* $\mathbf{L}_\mu$ *are of the form*

$$U(x,y,z) = e^{ik_1 x + ik_2 y} U_{(k_1,k_2)}(z)$$

temporal dynamics

- *solutions in the kernel of* $\mathbf{L}_\mu$ *are of the form*

$$U(x,y,z) = e^{ik_1x+ik_2y}U_{(k_1,k_2)}(z)$$

- *with* $k^2 = k_1^2 + k_2^2$, *for any* $k > 0$, *there exists a sequence*

$$\mu_0(k) < \mu_1(k) < \ldots \to \infty,\ k \to 0,\infty$$

for which such solutions exist

- $\mu_0(k)$ *has a unique global minimum* μ_c *for* $k = k_c$ *(numerical result)*

[Yudovich, 1966]

Parameters

spatial dynamics

- **bifurcation parameter** $\boxed{\varepsilon = \mu - \mu_c}$

  $\rightsquigarrow$ *for* $\varepsilon = 0$*, purely imaginary eigenvalues satisfy :*

$$\boxed{k_x^2 + n^2 k_y^2 = k_c^2, \quad n \in \mathbb{Z}}$$

Parameters

spatial dynamics

- **bifurcation parameter** $\boxed{\varepsilon = \mu - \mu_c}$

  $\rightsquigarrow$ *for $\varepsilon = 0$, purely imaginary eigenvalues satisfy :*

  $$\boxed{k_x^2 + n^2 k_y^2 = k_c^2, \quad n \in \mathbb{Z}}$$

- **choose** $\boxed{k_y = k_c \cos\alpha, \quad 0 < \alpha < \pi/3}$

  (co-existence of rolls with different orientations)

  $\rightsquigarrow$ *six (counting geometric multiplicities) purely imaginary eigenvalues for $n = 0, \pm 1$:*

  $$\boxed{\pm ik_c, \quad \pm ik_x, \quad k_x = k_c \sin\alpha}$$

- *for $\pi/3 < \alpha < \pi/2$ there are additional purely imaginary eigenvalues for some $n = 2, 3, \ldots, N$* **[Scheel & Wu, 2013]**

Reduction

- **center manifold theorem :** *close to the bifurcation point (small ε), small bounded solutions lie on a finite-dimensional center manifold*
  - *dimension : sum of the algebraic multiplicities of the purely imaginary eigenvalues* $\pm ik_c,\ \pm ik_x \rightsquigarrow$ **12-dimensional**

- **12-dimensional reduced system (ODEs)**

- **center manifold theorem :** *close to the bifurcation point (small ε), small bounded solutions lie on a finite-dimensional center manifold*
  - *dimension : sum of the algebraic multiplicities of the purely imaginary eigenvalues* $\pm ik_c, \; \pm ik_x$ $\rightsquigarrow$ **12-dimensional**

- **12-dimensional reduced system (ODEs)**
  - **normal form :** *apply a general method ...*

$$
\begin{aligned}
A_0' &= ik_c A_0 + B_0 + iA_0 P_{0i} + V_0 \\
B_0' &= ik_c B_0 + iB_0 P_{0i} + A_0 Q_0 + V_0' + W_0 \\
A_+' &= ik_x A_+ + B_+ + iA_+ P_{+i} + A_- R_+ + \beta_{11} v_{+0} \\
B_+' &= ik_x B_+ + iB_+ P_{+i} + B_- R_+ + A_+ Q_+ + ib_{11} v_{+0} + \beta_{11} w_{+0} \\
A_-' &= -ik_x A_- + B_- + iA_- P_{-i} - A_+ \overline{R_+} + \beta_{11} v_{-0} \\
B_-' &= -ik_x B_- + iB_- P_{-i} - B_+ \overline{R_+} + A_- Q_- + ib_{11} v_{-0} + \beta_{11} w_{-0}
\end{aligned}
$$

in which ...

$$
\begin{aligned}
P_0 &= i(\alpha_0\delta\mu + \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3(u_3+u_5) + \alpha_4(u_4+u_6)),\\
Q_0 &= a_0\delta\mu + a_1 u_1 + a_2 u_2 + a_3(u_3+u_5) + a_4(u_4+u_6),\\
P_+ &= i(\beta_0\delta\mu + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3 + \beta_4 u_4 + \beta_5 u_5 + \beta_6 u_6),\\
P_- &= i(\beta_0\delta\mu + \beta_1 u_1 + \beta_2 u_2 + \beta_5 u_3 + \beta_6 u_4 + \beta_3 u_5 + \beta_4 u_6),\\
Q_+ &= b_0\delta\mu + b_1 u_1 + b_2 u_2 + b_3 u_3 + b_4 u_4 + b_5 u_5 + b_6 u_6,\\
Q_- &= b_0\delta\mu + b_1 u_1 + b_2 u_2 + b_5 u_3 + b_6 u_4 + b_3 u_5 + b_4 u_6,\\
R_+ &= i\gamma_7 u_7 + \gamma_9 u_{9,} \quad R_- = -\overline{R_+}, \quad S_+ = c_7 u_7 + ic_9 u_9, \quad S_- = \overline{S_+},
\end{aligned}
$$

$$
\begin{aligned}
V_0 &= \alpha_{11}(v_{0+} + v_{0-}), \quad W_0 = \alpha_{11}(w_{0+} + w_{0-}), \quad V_0' = ia_{11}(v_{0+} + v_{0-}),\\
V_+ &= \beta_{11} v_{+0} + \beta_{14} v_{-+}, \; W_+ = \beta_{11} w_{+0} + \beta_{14} w_{-+}, \; V_+' = i(b_{11} v_{+0} + b_{14} v_{-+}),\\
V_- &= \beta_{11} v_{-0} + \beta_{14} v_{+-}, \; W_- = \beta_{11} w_{-0} + \beta_{14} w_{+-}, \; V_-' = i(b_{11} v_{-0} + b_{14} v_{+-}).
\end{aligned}
$$

$$
\begin{aligned}
v_{0+} &= A_0 A_+ \overline{B_+} - B_0 A_+ \overline{A_+}, \; w_{0+} = A_0 B_+ \overline{B_+} - B_0 \overline{A_+} B_+,\\
v_{0-} &= A_0 A_- \overline{B_-} - B_0 A_- \overline{A_-}, \; w_{0-} = A_0 B_- \overline{B_-} - B_0 \overline{A_-} B_-,\\
v_{0+-} &= A_0 \overline{B_+} A_- - B_0 \overline{A_+} A_-, \; w_{0+-} = A_0 \overline{B_+} B_- - B_0 \overline{A_+} B_-,\\
v_{0-+} &= A_0 A_+ \overline{B_-} - B_0 A_+ \overline{A_-}, \; w_{0-+} = A_0 B_+ \overline{B_-} - B_0 B_+ \overline{A_-},\\
v_{+0} &= A_0 \overline{B_0} A_+ - A_0 \overline{A_0} B_+, \; w_{+0} = B_0 \overline{B_0} A_+ - \overline{A_0} B_0 B_+,\\
v_{-0} &= A_0 \overline{B_0} A_- - A_0 \overline{A_0} B_-, \; w_{-0} = B_0 \overline{B_0} A_- - \overline{A_0} B_0 B_-,\\
v_{+-} &= \overline{A_+} B_+ A_- - A_+ \overline{A_+} B_-, \; w_{+-} = B_+ \overline{B_+} A_- - A_+ \overline{B_+} B_-,\\
v_{-+} &= A_+ \overline{A_-} B_- - B_+ A_- \overline{A_-}, w_{-+} = A_+ B_- \overline{B_-} - B_+ A_- \overline{B_-}.
\end{aligned}
$$

$$
\begin{aligned}
u_1 &= A_0\overline{A_0}, \; u_2 = i(A_0\overline{B_0} - \overline{A_0}B_0), \; u_3 = A_+\overline{A_+}, \; u_4 = i(A_+\overline{B_+} - \overline{A_+}B_+),\\
u_5 &= A_-\overline{A_-}, \; u_6 = i(A_-\overline{B_-} - \overline{A_-}B_-), \; u_7 = A_+\overline{A_-}, \; u_8 = \overline{A_+}A_-,\\
u_9 &= (A_+\overline{B_-} - \overline{A_-}B_+), \; u_{10} = (\overline{A_+}B_- - A_-\overline{B_+}).
\end{aligned}
$$

Reduced dynamics

- further 2 transformations ... **new system**

$$
\begin{aligned}
C_0'' &= C_0\left(a_0\mathrm{sign}(\varepsilon) + a_1|C_0|^2 + a_3(|C_+|^2 + |C_-|^2)\right)\\
C_+'' &= C_+\left(b_0\mathrm{sign}(\varepsilon) + b_1|C_0|^2 + b_3|C_+|^2 + b_5|C_-|^2\right)\\
C_-'' &= C_-\left(b_0\mathrm{sign}(\varepsilon) + b_1|C_0|^2 + b_5|C_+|^2 + b_3|C_-|^2\right)
\end{aligned}
\quad +O(|\varepsilon|^{1/2})
$$

- **existence of domain walls** *(heteroclinic orbit)*

Reduced dynamics

- further 2 transformations ... **new system**

$$\begin{aligned} C_0'' &= C_0\left(a_0\mathrm{sign}(\varepsilon) + a_1|C_0|^2 + a_3(|C_+|^2 + |C_-|^2)\right) \\ C_+'' &= C_+\left(b_0\mathrm{sign}(\varepsilon) + b_1|C_0|^2 + b_3|C_+|^2 + b_5|C_-|^2\right) \\ C_-'' &= C_-\left(b_0\mathrm{sign}(\varepsilon) + b_1|C_0|^2 + b_5|C_+|^2 + b_3|C_-|^2\right) \end{aligned} \quad + O(|\varepsilon|^{1/2})$$

- **existence of domain walls** *(heteroclinic orbit)*
  - $\varepsilon > 0,\ a_0 < 0,\ a_1 > 0$ *(existence of rolls)*

    [Yudovich, 1966]
  - $0 < b_3 < b_5$ *(existence of the leading order heteroclinic)*

    [van den Berg & van der Vorst, 2000]
  - $1 < \dfrac{b_5}{b_3} < (4 + \sqrt{3}),\ \dfrac{a_3}{a_1} \geq \max\left(2,\ \dfrac{b_3 + b_5}{2b_3}\right)$

    (persistence of the heteroclinic) **[H. & Scheel, 2012]**

- **solve** $\mathcal{T}(C_0, C_+, C_-; \varepsilon) = 0$

$$\mathcal{T}(C_0, C_+, C_-; \varepsilon) = \begin{pmatrix} C_0'' - C_0(a_0 + a_1|C_0|^2 + a_3(|C_+|^2 + |C_-|^2)) + O(|\varepsilon|^{1/2}) \\ C_+'' - C_+(b_0 + b_1|C_0|^2 + b_3|C_+|^2 + b_5|C_-|^2) + O(|\varepsilon|^{1/2}) \\ C_-'' - C_-(b_0 + b_1|C_0|^2 + b_5|C_+|^2 + b_3|C_-|^2) + O(|\varepsilon|^{1/2}) \end{pmatrix}$$

  - *particular solution :* $(0, C_+^\star, C_-^\star; 0)$

PERSISTENCE

- **solve** $\mathcal{T}(C_0, C_+, C_-; \varepsilon) = 0$

$$\mathcal{T}(C_0, C_+, C_-; \varepsilon) = \begin{pmatrix} C_0'' - C_0(a_0 + a_1|C_0|^2 + a_3(|C_+|^2 + |C_-|^2)) + O(|\varepsilon|^{1/2}) \\ C_+'' - C_+(b_0 + b_1|C_0|^2 + b_3|C_+|^2 + b_5|C_-|^2) + O(|\varepsilon|^{1/2}) \\ C_-'' - C_-(b_0 + b_1|C_0|^2 + b_5|C_+|^2 + b_3|C_-|^2) + O(|\varepsilon|^{1/2}) \end{pmatrix}$$

  - *particular solution :* $(0, C_+^\star, C_-^\star; 0)$

- **linear operator :** $D_{(C_0, C_+, C_-)}\mathcal{T}(0, C_+^\star, C_-^\star; 0) = \mathcal{L}_\star$
  - *space of (small) exponentially weighted functions*
  - *injective Fredholm operator with index* -1
  - *kernel of the adjoint operator spanned by* $(0, iC_+^\star, -iC_-^\star)$

Persistence

- **solve** $\mathcal{T}(C_0, C_+, C_-; \varepsilon) = 0$

$$\mathcal{T}(C_0, C_+, C_-; \varepsilon) = \begin{pmatrix} C_0'' - C_0(a_0 + a_1|C_0|^2 + a_3(|C_+|^2 + |C_-|^2)) + O(|\varepsilon|^{1/2}) \\ C_+'' - C_+(b_0 + b_1|C_0|^2 + b_3|C_+|^2 + b_5|C_-|^2) + O(|\varepsilon|^{1/2}) \\ C_-'' - C_-(b_0 + b_1|C_0|^2 + b_5|C_+|^2 + b_3|C_-|^2) + O(|\varepsilon|^{1/2}) \end{pmatrix}$$

  - *particular solution :* $(0, C_+^\star, C_-^\star; 0)$

- **linear operator :** $D_{(C_0,C_+,C_-)}\mathcal{T}(0, C_+^\star, C_-^\star; 0) = \mathcal{L}_\star$
  - *space of (small) exponentially weighted functions*
  - *injective Fredholm operator with index* -1
  - *kernel of the adjoint operator spanned by* $(0, iC_+^\star, -iC_-^\star)$

- **implicit function theorem**
  - *second parameter :* $k_x + \epsilon$
  - $\partial_\epsilon \mathcal{T}(0, C_+^\star, C_-^\star; 0, 0) \notin \mathrm{Im}(\mathcal{L}_\star)$
  - **solution** $\boxed{(C_0, C_+, C_-)(\varepsilon),\ \epsilon(\varepsilon)}$

Compute coefficients . . .

- **use the classical theory :** $\boxed{\varepsilon > 0,\ a_0 < 0,\ a_1 > 0}$

Compute coefficients ...

- **use the classical theory :** $\boxed{\varepsilon > 0,\ a_0 < 0,\ a_1 > 0}$
- **direct computation of b_5/b_3 ...**

$$\boxed{\frac{b_5}{b_3} = \frac{b_{53}(\sin^2\alpha) + b_{53}(\cos^2\alpha) + b_{53}(0)}{\frac{1}{2}b_{53}(1) + b_{53}(0)}}$$

$$b_{53}(\Theta) = A_{53}(\Theta) + B_{53}(\Theta)\mathcal{P}^{-1} + C_{53}(\Theta)\mathcal{P}^{-2}, \quad \Theta = \sin^2\alpha$$

$$A_{53}(\Theta) = 2\mu_c^3\langle (D^2 - 4k_c^2\Theta)^2 V_1, R_1\rangle, \qquad B_{53}(\Theta) = 4\mu_c^3\Theta\left(\langle V_1, R_2\rangle + \langle V_2, R_1\rangle\right)$$
$$C_{53}(\Theta) = -\frac{2\mu_c\Theta}{k_c^2}\langle (D^2 - 4k_c^2\Theta)V_2, R_2\rangle$$

$$R_1 = VD\phi + (1-2\Theta)\phi DV, \quad R_2 = \left(D^2 - 4k_c^2(1-\Theta)\right)(VDV) - 4\Theta(DV)(D^2V)$$

V, ϕ solutions of the boundary value problems

$$(D^2 - k^2)^3 V + \mu^2 k^2 V = 0, \quad V = DV = (D^2 - k)^2 V = 0 \text{ in } z = 0, 1$$
$$(D^2 - k^2)\phi = V, \quad \phi = 0 \text{ in } z = 0, 1$$

V_1, V_2 (unique) solutions of the boundary value problems

$$(D^2 - 4k_c^2\Theta)^3 V_1 + 4k_c^2\mu_c^2\Theta\, V_1 = R_1, \quad V_1 = DV_1 = (D^2 - 4k_c^2\Theta)^2 V_1 = 0 \text{ in } z = 0, 1$$
$$(D^2 - 4k_c^2\Theta)^3 V_2 + 4k_c^2\mu_c^2\Theta\, V_2 = R_2, \quad V_2 = (D^2 - 4k_c^2\Theta)V_2 = (D^2 - 4k_c^2\Theta)DV_2 = 0 \text{ in } z = 0, 1$$

Compute coefficients ...

- **use the classical theory :** $\boxed{\varepsilon > 0,\ a_0 < 0,\ a_1 > 0}$
- **analytical computations :** $\boxed{2b_3 = b_5 > 0}$, **when** $\boxed{\alpha = 0}$
  - *the conditions on* b_3 *and* b_5 *hold, for small angles* α
  - *it remains to check* $\boxed{\dfrac{a_3}{a_1} \geq \max\left(2,\ \dfrac{b_3 + b_5}{2b_3}\right)\ !!}$

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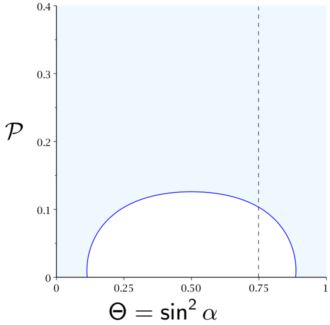
- **go back to the symmetries :** *additional reflection symmetry for the rigid-rigid and free-free boundary conditions*
  - *the subspace* $C_0 = 0$ *is invariant*
  - we can restrict to an 8-dimensional system for C_+ and C_-
  - we are left with the conditions on b_3 and b_5, which hold at least for small angles α

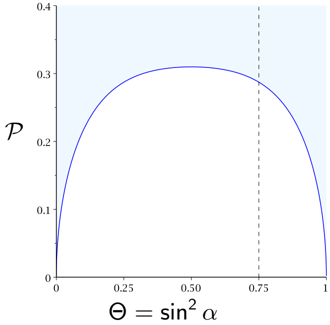
Rigid-rigid boundary conditions

- check that $$1 < \frac{b_5}{b_3} < (4 + \sqrt{3})$$
- (very long) **Maple computations :** *the inequalities hold in the shaded region*



Free-free boundary conditions

- check that $\boxed{1 < \frac{b_5}{b_3} < (4 + \sqrt{3})}$
- (easy) **computations :** *the inequalities hold in the shaded region*

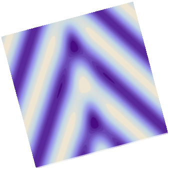


Rigid-free boundary conditions

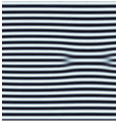
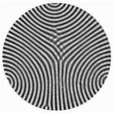
- no invariant subspace (12-dimensional system)
- check all conditions

. . . work in progress

Further issues



- large angles . . . $\pi/2$
- asymmetric domain walls
- other defects :


- stability . . .