Uniqueness and regularity of flows of non-Newtonian fluids with critical power-law growth

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joint work with M. Bulíček, F. Ettwein, D. Pražák

Model (generalised Newtonian flow):

$$\partial_t u + \operatorname{div}(u \otimes u) = f - \nabla\pi + \operatorname{div}(\mathcal{T}), \quad \operatorname{div} u = 0 \tag{1}$$

in $Q = \Omega \times I$, $I = (0, T)$, $\Omega \subset \mathbb{R}^3$,

$u : Q \to \mathbb{R}^3 \ldots$ velocity, $\pi : Q \to \mathbb{R} \ldots$ pressure

$f : Q \to \mathbb{R}^3 \ldots$ external forces,

$\mathcal{T} : Q \to \mathbb{R}^{3\times 3} \ldots$ extra stress tensor

Extra stress tensor

Typical examples with $p > 1$

$$\mathcal{T}_1(D) = \nu(1 + |D|^2)^{\frac{p-2}{2}} D$$
$$\mathcal{T}_2(D) = \nu(1 + |D|^{p-2}) D$$

$\mathcal{T}_1$, $\mathcal{T}_2$, $p = 2$... Navier-Stokes system
$\mathcal{T}_1$... Ladyzhenskaya, O. A.: The mathematical theory of viscous incompressible flow. Gordon and Breach, Science Publishers, New York-London-Paris 1969. Supplement.
$\mathcal{T}_2$, $p = 3$... Smagorinsky, J.: General Circulation Experiments with the Primitive Equation I the Basic Experiment. Monthly Weather Review, 91, 99-164.

Boundary and initial conditions

Boundary conditions

homogeneous Dirichlet boundary conditions
$u = 0$ on $\partial\Omega \times I$

Initial conditions

$u(0) = u_0 \in L^2(\Omega)$ or better

Basic approach to existence

Equation

$$\partial_t u - \operatorname{div}(\mathcal{T}) + \operatorname{div}(u \otimes u) + \nabla \pi = f, \quad \operatorname{div} u = 0$$

Natural regularity of weak solution

$$u \in L^p(I, W^{1,p}_{0,\mathrm{div}}(\Omega)) \cap L^\infty(I, L^2(\Omega))$$

Diening, L., Růžička, M., Wolf, J.: Existence of weak solutions for unsteady motions of generalized Newtonian fluids. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 9 (2010), no. 1, 1–46.

$p > 6/5$ ... Existence of weak solution... $W^{1,p} \hookrightarrow L^2$

Additional regularity if $p \geq 11/5$

$$u \in N^{1/2,2}(I, L^2(\Omega)) \quad \partial_t u \in L^{p'}(I, W^{1,p}_{0,\mathrm{div}}(\Omega)^*) \tag{2}$$

Basic approach to uniqueness

Lemma

$\boldsymbol{u}_1$, $\boldsymbol{u}_2$ weak solutions corresponding to $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$, $p \geq 11/5$,

$$p_{\mathrm{uniq}} := \frac{2p}{2p-3}. \tag{3}$$

Then

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_2^2 &+ c\big(\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,2}}^2 + \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_{W^{1,p}}^p\big) \\ &\leq C\|\boldsymbol{u}_2\|_{W^{1,p}}^{p_{\mathrm{uniq}}}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_2^2 + C\|\boldsymbol{f}_1 - \boldsymbol{f}_2\|_{(W^{1,p})^*}^{p'}.\end{aligned} \tag{4}$$

existence of one weak solution in $L^{p_{\mathrm{uniq}}}(0,T;W^{1,p}) \implies$ uniqueness

$p \geq 5/2 \quad \implies p \geq p_{\mathrm{uniq}}$

Regularity in time and testing with time derivatives

$$\partial_t u - \operatorname{div}(\mathcal{T}) + \operatorname{div}(u \otimes u) + \nabla \pi = f$$

Test with $\partial_t u$

- $\|u(t)\|_2^2 + \partial_t \|Du(t)\|_p^p \leq c \int_\Omega |u \nabla u \partial_t u|(t)$
- need of potentiality of $\mathcal{T}$, $p \geq 12/5$

Test with $\partial_t^2 u$

- $\partial_t \|\partial_t u(t)\|_2^2 + \int_\Omega (1 + |Du|)^{p-2} |D\partial_t u|^2 \leq \int_\Omega |\nabla u||\partial_t u|^2(t)$
- $p \geq 5/2$

Maybe, test with $\partial_t u, \partial_t^2 u$ is too much. We need fractional derivatives.

Results on uniqueness for generalized NS flows

- Ladyzhenskaya, O.A. ... (1969) - $p \geq 5/2$, (1970) - $p \geq 12/5$
- Du, Q. and Gunzburger, M.D. ... (1991) - $p \geq 11/5$
- BEKP ... (2010) ... $p > 11/5$, local regularity ... uniqueness in trajectory sense
- BKP ... (2019) ... $p \geq 11/5$, global regularity ... uniqueness

(J. Málek, J. Nečas, M. Rokyta, M. Růžička 1996, periodic bc, results based on spatial regularity)

Improved regularity of the weak solution - (2010)

Bulíček, Ettwein, Kaplický, Pražák 2010

- $p > 11/5$
- $f \in W^{1,2}_{\mathrm{loc}}(I, L^2(\Omega))$
- u is weak solution of (1), boundary conditions, initial conditions

Then

$$u \in W^{1,\infty}_{\mathrm{loc}}(I, L^2(\Omega)) \cap N^{2/p,p}_{\mathrm{loc}}(I, W^{1,p}(\Omega)) \cap W^{1,2}_{\mathrm{loc}}(I, W^{1,2}(\Omega))$$

Uniqueness, since

$$N^{2/p,p}(I, W^{1,p}(\Omega)) \hookrightarrow L^{p_{\mathrm{uniq}}}(I, W^{1,p}(\Omega))$$

but only in the sense of trajectories.

Improved regularity of the weak solution - (2019)

Bulíček, Kaplický, Pražák 2019

- $p \geq 11/5$
- $f \in N^{\delta,p'}(0,T;(W^{1,p}_{div})^*)$ with $\delta \geq 0$ sufficiently large
- u is a weak solution

$$\begin{aligned} u_0 \in L^2_{div} \quad &\Longrightarrow \quad \forall t_0 \in (0,T) : \boldsymbol{u} \in L^{p_{\mathrm{uniq}}}(t_0,T;W^{1,p}), \\ u_0 \in W^{1,p}_{div} \quad &\Longrightarrow \quad \boldsymbol{u} \in L^{p_{\mathrm{uniq}}}(0,T;W^{1,p}). \end{aligned}$$

Uniqueness for $p \geq 11/5$.

Proof of the theorem-local version, $p > 11/5$

Proof

starting information: $\boldsymbol{u} \in L^p_{loc}(0, T; W^{1,p})$. Assume $\boldsymbol{u} \in N^{a,p}_{loc}(0, T; W^{1,p})$.

- Step 1: show that convective term belongs to some $N^{b,p'}(t_0, T; (W^{1,p})^*)$
- Step 2: test eqn with localized second time differences, use apriori regularity $\implies$ improvement of differentiability

  $$\boldsymbol{u} \in N^{\sigma,p}(t_0, T; W^{1,p})$$

  with $\sigma > a$ if $p > 11/5$.
- iterate Step 1 and Step 2

$\square$

New features of proof -global version, $p \geq 11/5$

- To start iteration if $p = 11/5$: initial improvement of regularity by Gehring argument $\boldsymbol{u} \in L^q(t_0, T; W^{1,p})$ with $t_0 \in [0, T)$ and $q > p$.
- To avoid localization it time and to get results up to time $t = 0$ we need

$$\exists c > 0, \forall h \in (0, T - t_0) : \|\boldsymbol{u}(t_0 + h) - \boldsymbol{u}(t_0)\|_2^2 \leq c h^{2\tau}.$$

Components of the proof

- *M. Bulíček, F. Ettwein, P. Kaplický, D. Pražák: On uniqueness and time regularity of flows of power-law like non-Newtonian fluids. Math. Methods Appl. Sci. 33 (2010).*
- *M. Bulíček, P. Kaplický, D. Pražák: Uniqueness and regularity of flows of non-Newtonian fluids with critical power-law growth. Math. Models Methods Appl. Sci. 29 (2019), no. 6, 1207–1225.*

Thank you for your attention!!!