Spontaneous periodic orbits in the Navier-Stokes flow

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Joint work with

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The Navier-Stokes equations for a fluid of constant density ρ can be expressed as

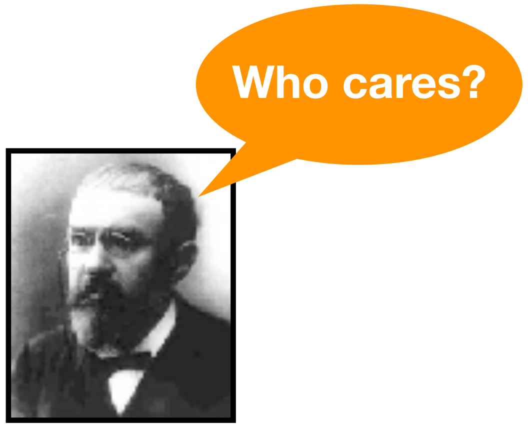
$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f \\ \nabla \cdot u = 0, \end{cases}$$

where $u = u(x,t)$ is the velocity, $p(x,t) = P(x,t)/\rho$ is the pressure scaled by the density, ν is the kinematic viscosity and $f = f(x,t)$ is an external forcing term.

Navier (1822) *Stokes (1845)*

Millennium Prize problem

In three space dimensions and time, given an initial velocity field and identically zero forcing term, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier-Stokes equations.



From a dynamical systems perspective, this is not the most important question.

Henri Poincaré

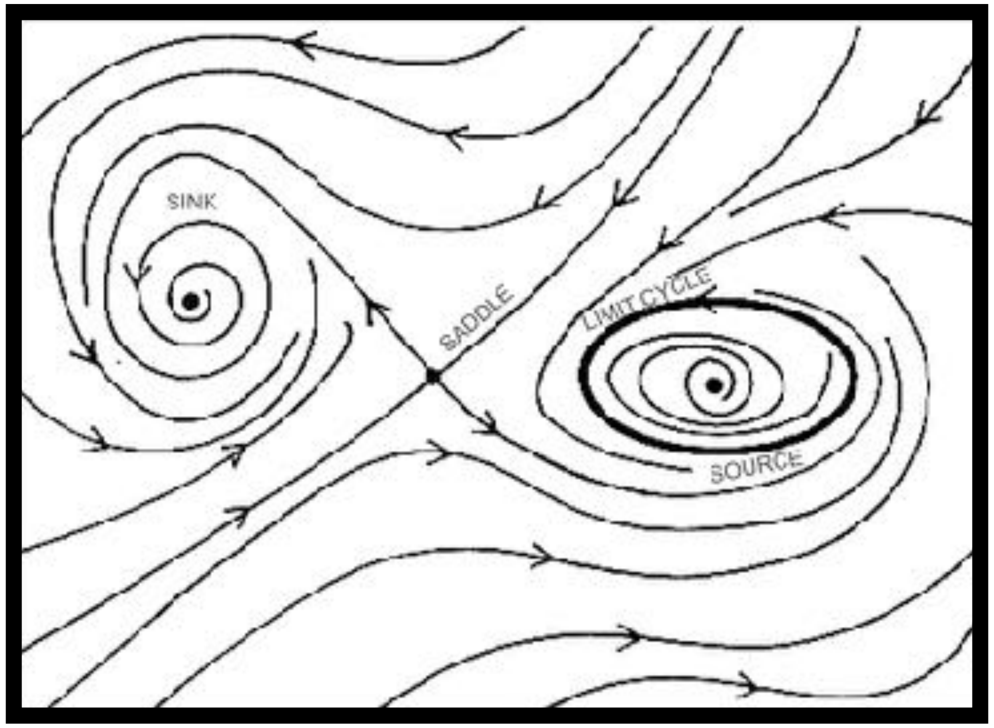
What shall we care about then ?

In any dynamical system, it is the bounded solutions which are most important and which should be investigated first.

Henri Poincaré

Compact invariant sets

Exploit smoothness, boundedness and low dimensionality.



- Equilibrium solutions.
- Time periodic solutions.
- Connecting orbits.
- Global attractors.

$$\mathcal{F}(x) = 0$$

In 1959, James Serrin published two papers on the existence and stability of certain solutions to the Navier-Stokes equations in the limit of large viscosity.

- Existence of globally stable equilibrium solutions;
- Existence of periodic solutions on a three-dimensional bounded domain subject to time-periodic boundary data and body forces.

James Serrin

Many authors followed Serrin in studying the periodically forced (non-autonomous) Navier-Stokes system dominated by viscosity.

- [**Kaniel & Shinbrot, 1967**] Existence of periodic strong solutions for small time-periodic forcing f (for 3D bounded domains with fixed boundaries);
- [**Takeshita, 1969**] Existence of periodic strong solutions for any time-periodic forcing f (for 2D bounded domains with fixed boundaries);
- many more proofs of existence of periodic orbits for non-autonomous NS [**Teramoto, Maremonti, Kozono & Nakao, Kato, Farwig & Okabe, Hsia**]

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- The same cannot be said about spontaneous periodic motions, that is periodic flows driven by a time-independent forcing.

- The regular vortex shedding in the wake of a cylinder, for instance, arises in the absence of a body force and as a consequence of the nonlinearity in NS, not by virtue of the advection being dominated by viscous damping.

<u>Goal</u>: Develop a general (computer-assisted) approach to prove existence of spontaneous periodic orbits in the Navier-Stokes flow for some time-independent f.

Computer-assisted proofs (CAPs) in dynamics

The main idea is to construct algorithms that provide an approximate solution to a problem together with precise and possibly efficient bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense.

This field draws inspiration from the ideas in

- Scientific computing
- Functional analysis
- Approximation theory
- Nonlinear analysis
- Numerical analysis
- Topological methods

Early pioneer works

Cesari [1964] ***Functional analysis and Galerkin's method.***
Lanford [1982] ***A computer-assisted proof of the Feigenbaum conjectures.***
Mischaikow & Mrozek [1995] ***Chaos in the Lorenz equations.***
Tucker [1999] ***The Lorenz attractor exists.***

A functional analytic approach to CAPs in dynamics

A general nonlinear problem

$$\boxed{\mathcal{F}(x) = 0}$$

The unknown x could be a

- **solution to an initial value problem of an ODE**
- **periodic orbit of an ODE**
- **local (un)stable manifold of a fixed point of an ODE**
- **normal bundle of a periodic orbit of an ODE**
- **local (un)stable manifold of a periodic orbit of an ODE**
- **connecting orbit of an ODE**
- **periodic orbit of a functional delay equation**
- **critical point of an action functional**
- **solution to a boundary value problem**
- **steady state of a PDE**
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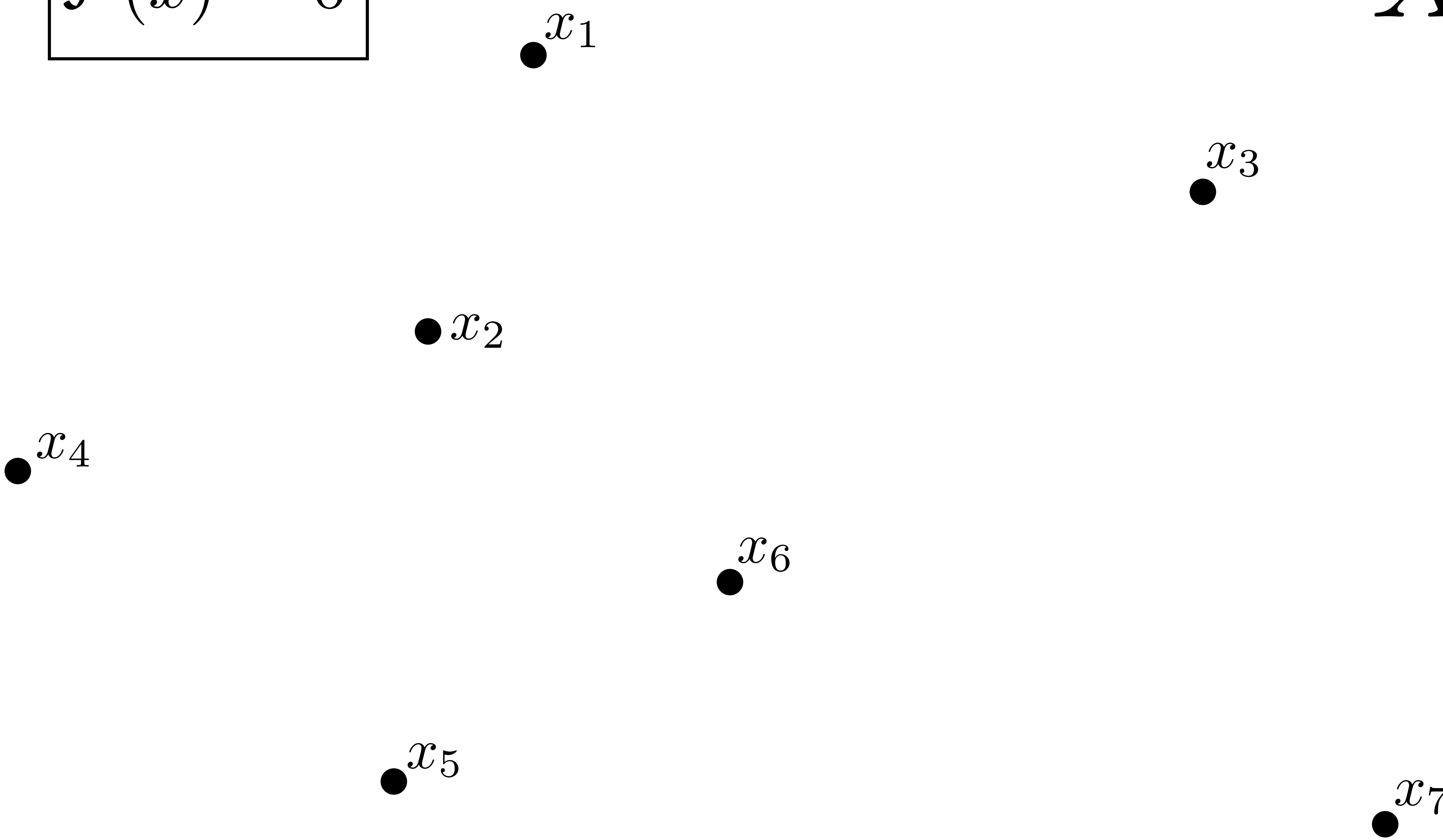

Impossible to compute exactly !

A general nonlinear problem **to solve in a Banach space** X

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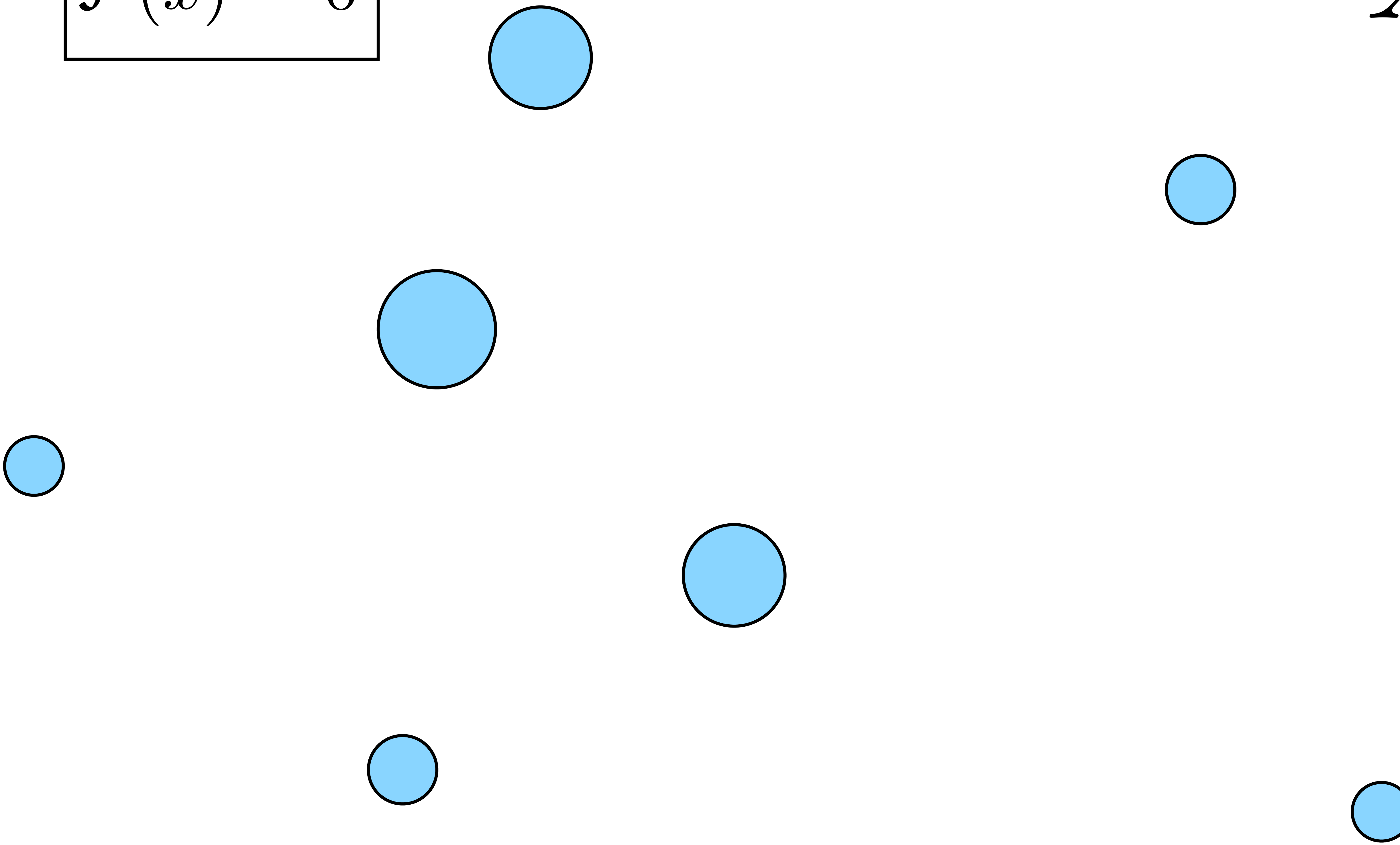

Alternative: find small balls in which it is demonstrated (in a mathematically rigorous sense) that a unique solution exists.

How to find these small isolating balls ?

1. Let $\bar{x}$ a numerical approximation of $\mathcal{F}(x) = 0$ in X computed using a finite dimensional reduction.

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6. Find $r > 0$ such that $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction mapping (tool : **radii polynomials**).

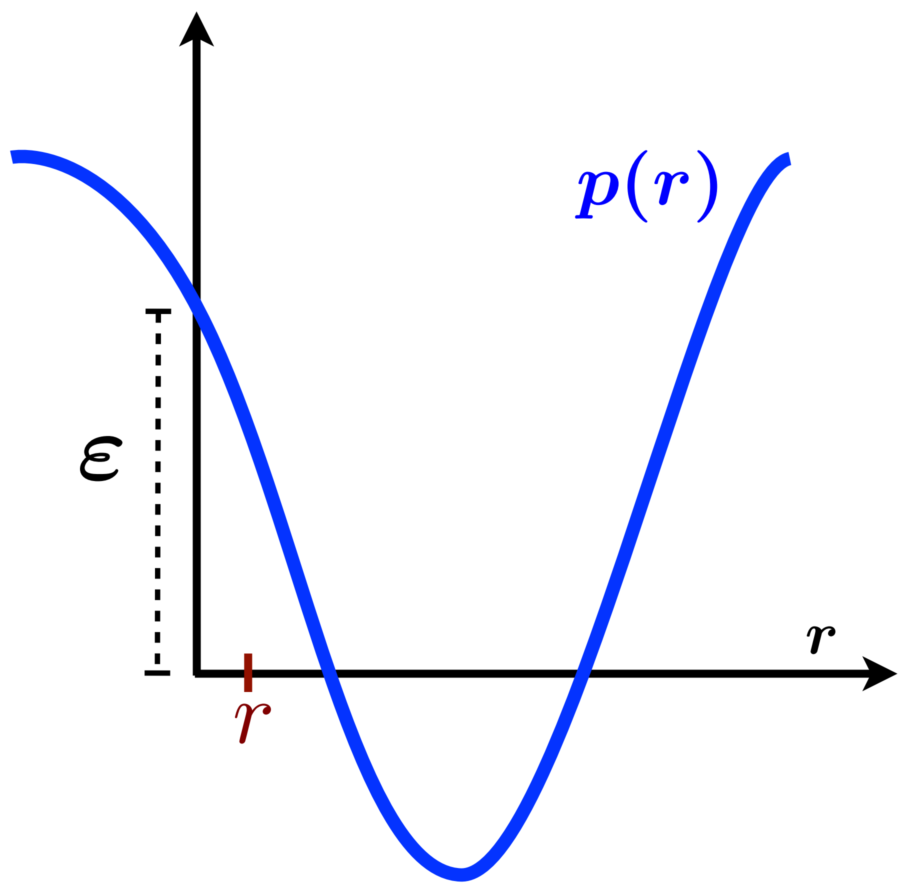
Theorem : Let $T : X \to X$ defined by $T(x) = x - A\mathcal{F}(x)$ with $T \in C^1(X)$. Let $r > 0$ and consider bounds ε and $\kappa = \kappa(r)$ satisfying

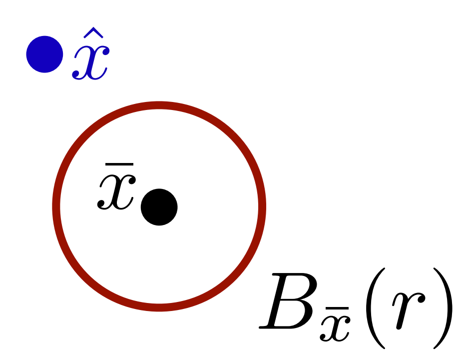
$$\begin{aligned} \|T(\bar{x}) - \bar{x}\|_X &= \|A\mathcal{F}(\bar{x})\|_X \leq \varepsilon \\ \sup_{w \in B_{\bar{x}}(r)} \|DT(w)\|_X &= \sup_{w \in B_{\bar{x}}(r)} \|I - A \cdot D\mathcal{F}(w)\|_X \leq \kappa(r). \end{aligned}$$

If

$$p(r) \stackrel{\text{def}}{=} \varepsilon + r\kappa(r) - r < 0 \quad \textbf{(radii polynomial)}$$

then $T : B_{\bar{x}}(r) \to B_{\bar{x}}(r)$ is a contraction with Lipschitz constant $\kappa(r) < 1$. Moreover A is injective and therefore $\mathcal{F} = 0$ has a unique solution in $B_{\bar{x}}(r)$.





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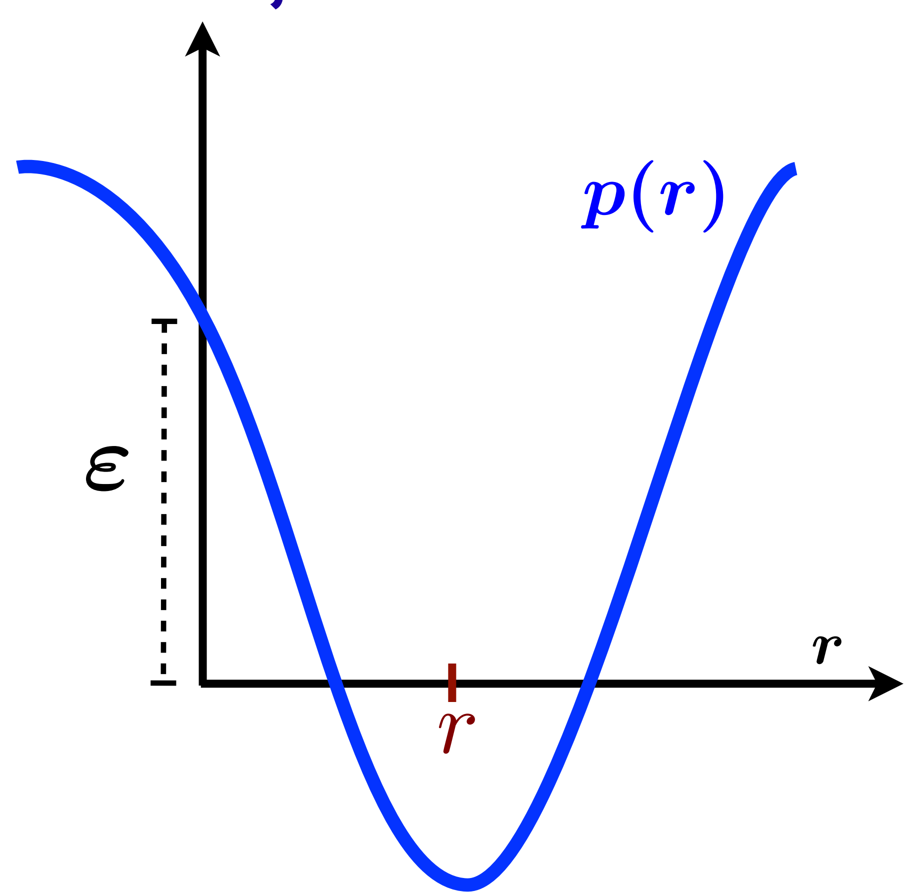
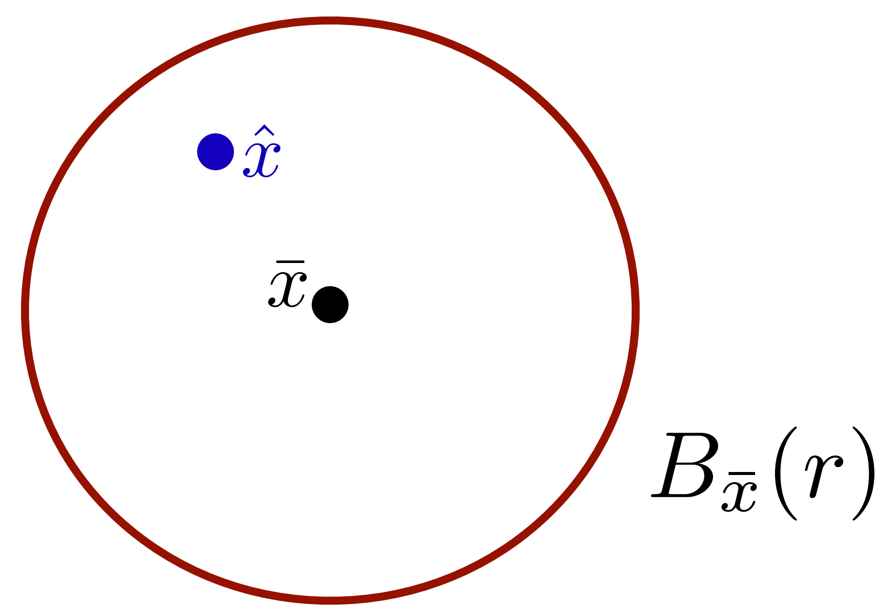
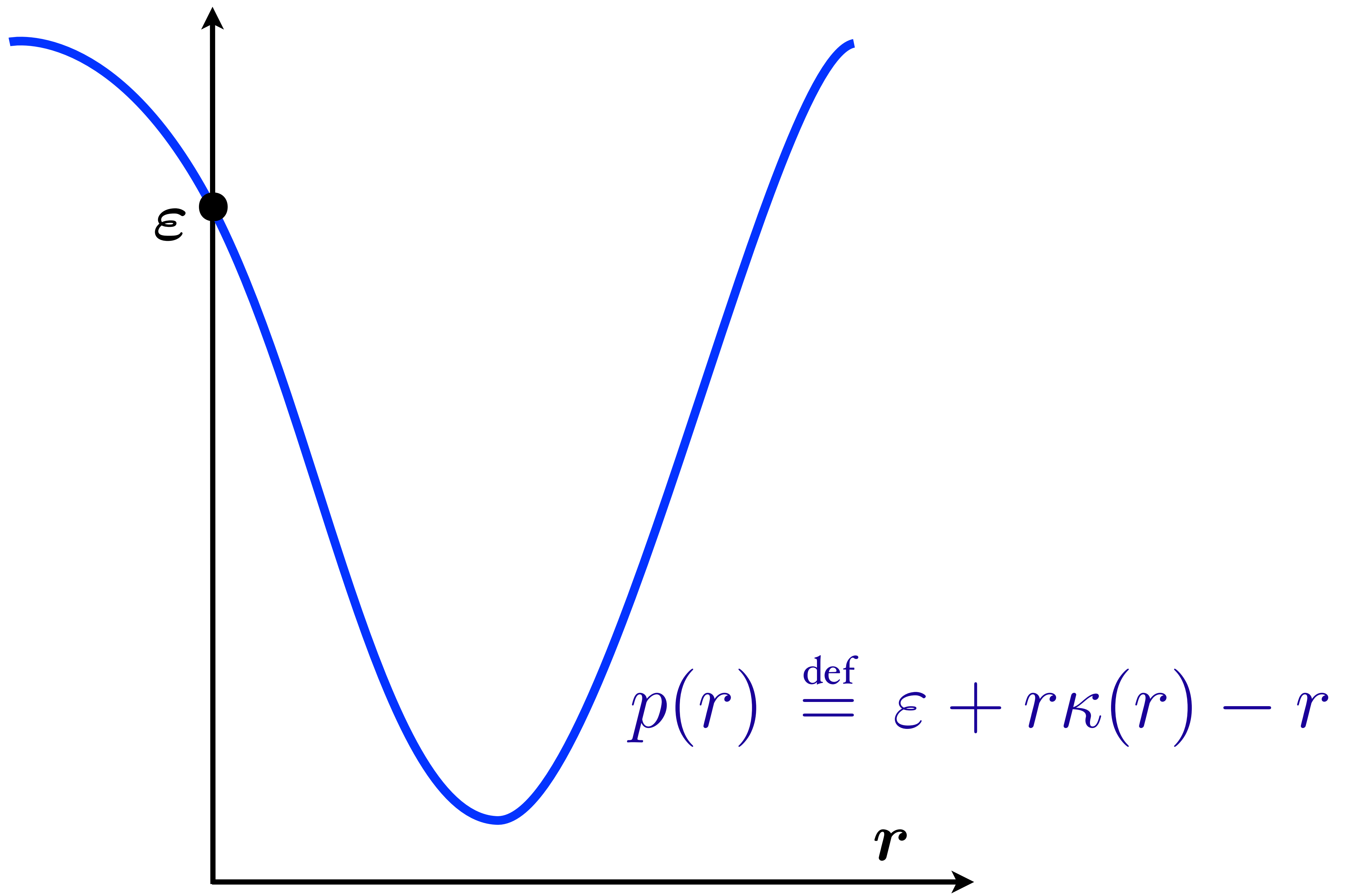




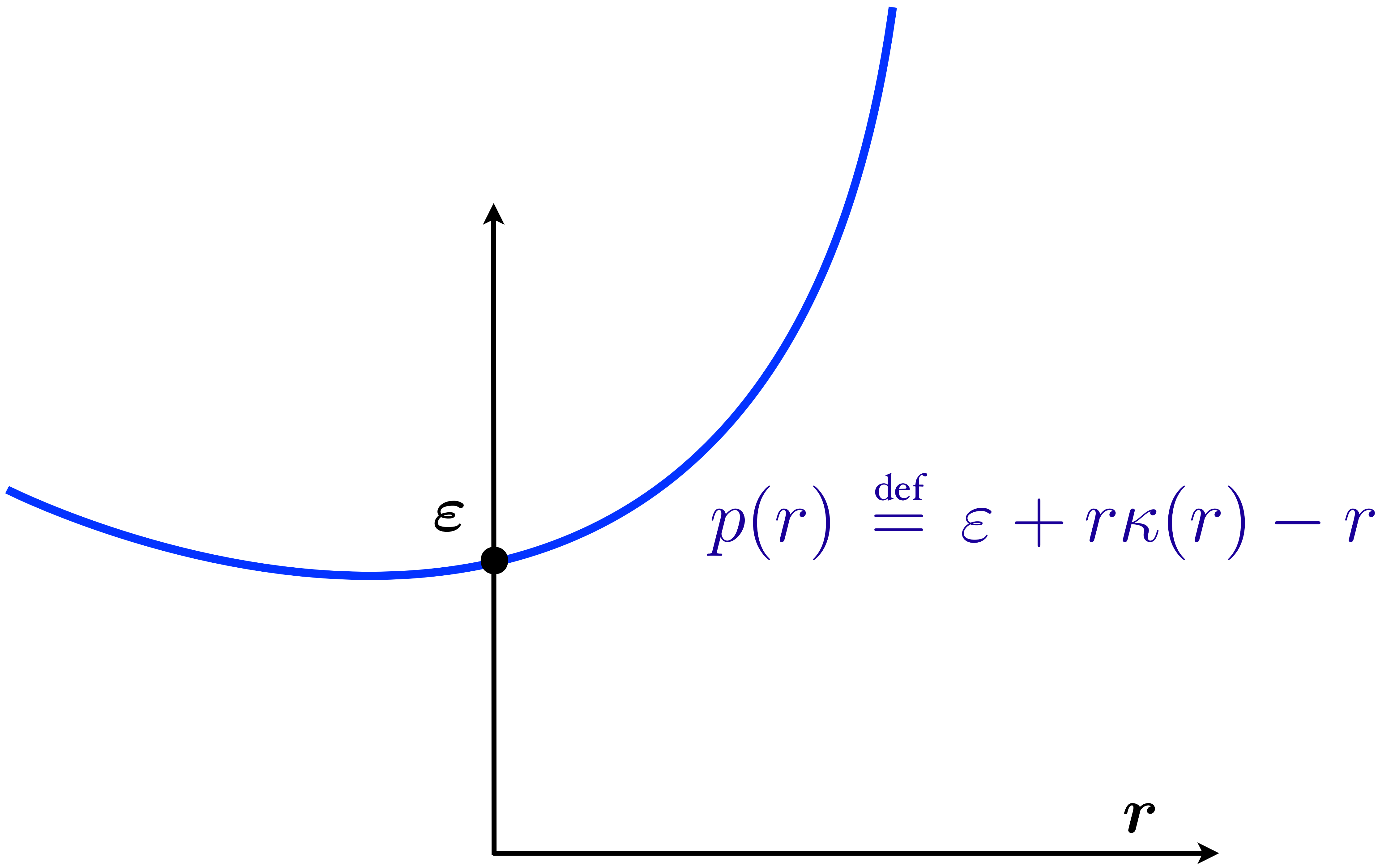
The method fails if the approximate solution is not good enough

$$\|A\mathcal{F}(\bar{x})\|_X \leq \varepsilon$$



The method fails if the approximate inverse is not good enough

$$\sup_{w \in B_{\bar{x}}(r)} \|I - A \cdot D\mathcal{F}(w)\|_X \leq \kappa(r)$$



A functional analytic approach to CAPs in dynamics

This requires an a priori setup that allows analysis and numerics to go hand in hand:

- the choice of function spaces,
- the choice of the basis functions and Galerkin projections,
- the analytic estimates,
- and the computational parameters

must all work together to bound the errors due to approximation, rounding and truncation sufficiently tightly for the verification proof to go through.

A zero-finding problem for periodic orbits in NS

Applying the curl operator to Navier-Stokes yields the vorticity equation

$$\partial_t \omega - \nu \Delta \omega + \text{nonlinear terms} = f^\omega \quad \text{on } \mathbb{T}^3 \times \mathbb{R},$$

where $\omega \stackrel{\text{def}}{=} \nabla \times u$ and $f^\omega \stackrel{\text{def}}{=} \nabla \times f$.

Plugging the space-time Fourier expansion of the vorticity

$$\omega(x,t) = \sum_{n \in \mathbb{Z}^4} \omega_n e^{i(\tilde{n}\cdot x + n_4 \Omega t)}, \quad \tilde{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3,$$

in the vorticity equation yields having to solve the zero-finding problem

$$\boxed{F_n(W) \stackrel{\text{def}}{=} i\Omega n_4 \omega_n + \nu \tilde{n}^2 \omega_n - f^\omega_n + \text{nonlinear terms} = 0,}$$

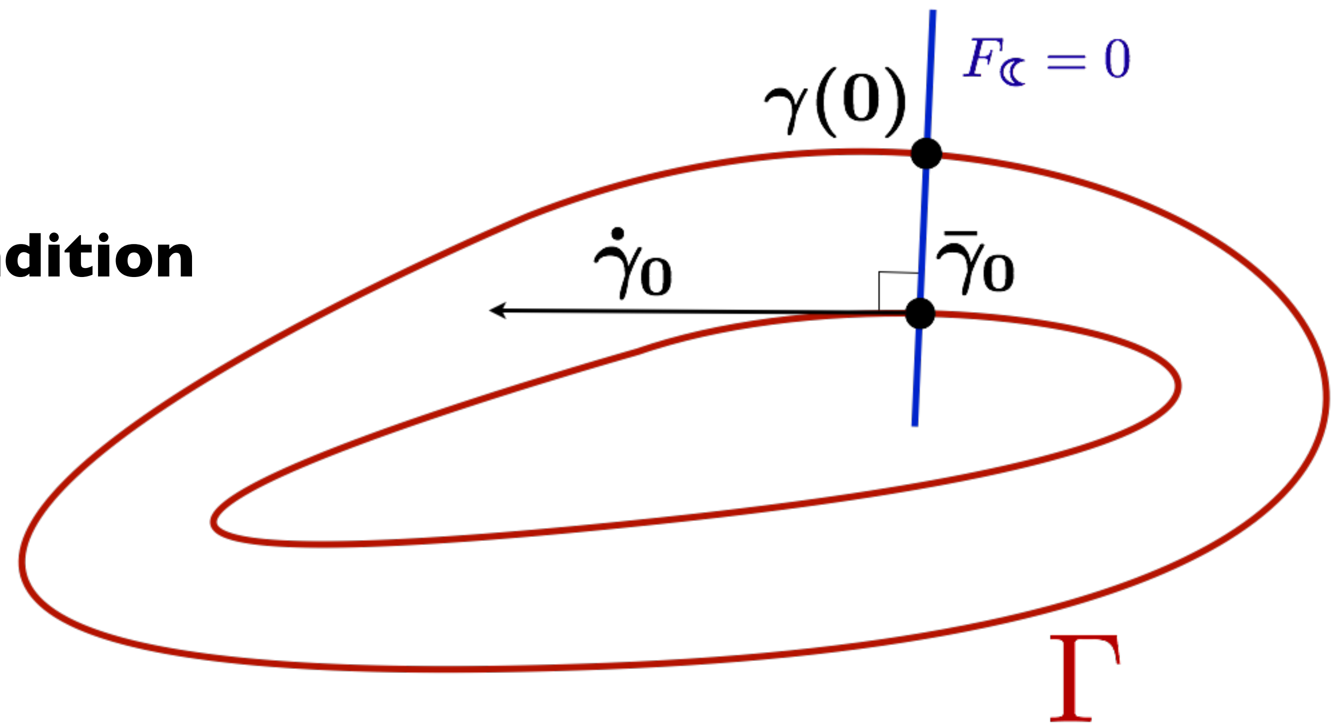
where Ω is the a-priori unknown time-frequency of the periodic orbit and

$$W = \begin{pmatrix} \Omega \\ (\omega_n)_{n \in \mathbb{Z}^4 \setminus \{0\}} \end{pmatrix}.$$

A zero-finding problem for periodic orbits in NS

Lemma: Let W be such that the vorticity ω is analytic. Assume that $F(W)=0$ and $\nabla\cdot\omega=0$. Assume also that f does not depend on time and has space average zero. Define $u=M\omega$ (that is u solves $\omega=\nabla\times u$). Then there exists a pressure function $p:\mathbb{T}^3\times\mathbb{R}\to\mathbb{R}$ such that (u,p) is a $\frac{2\pi}{\Omega}$-periodic solution of NS.

Phase condition



$$\mathcal{F}(W)=\begin{pmatrix} F_{☾}(W) \\ (F_n(W))_{n\in\mathbb{Z}^4_*}\end{pmatrix}=0$$

$$\boxed{F_n(W) \stackrel{\text{def}}{=} i\Omega n_4\omega_n+\nu\tilde{n}^2\omega_n-f_n^{\omega}+\text{nonlinear terms}}$$

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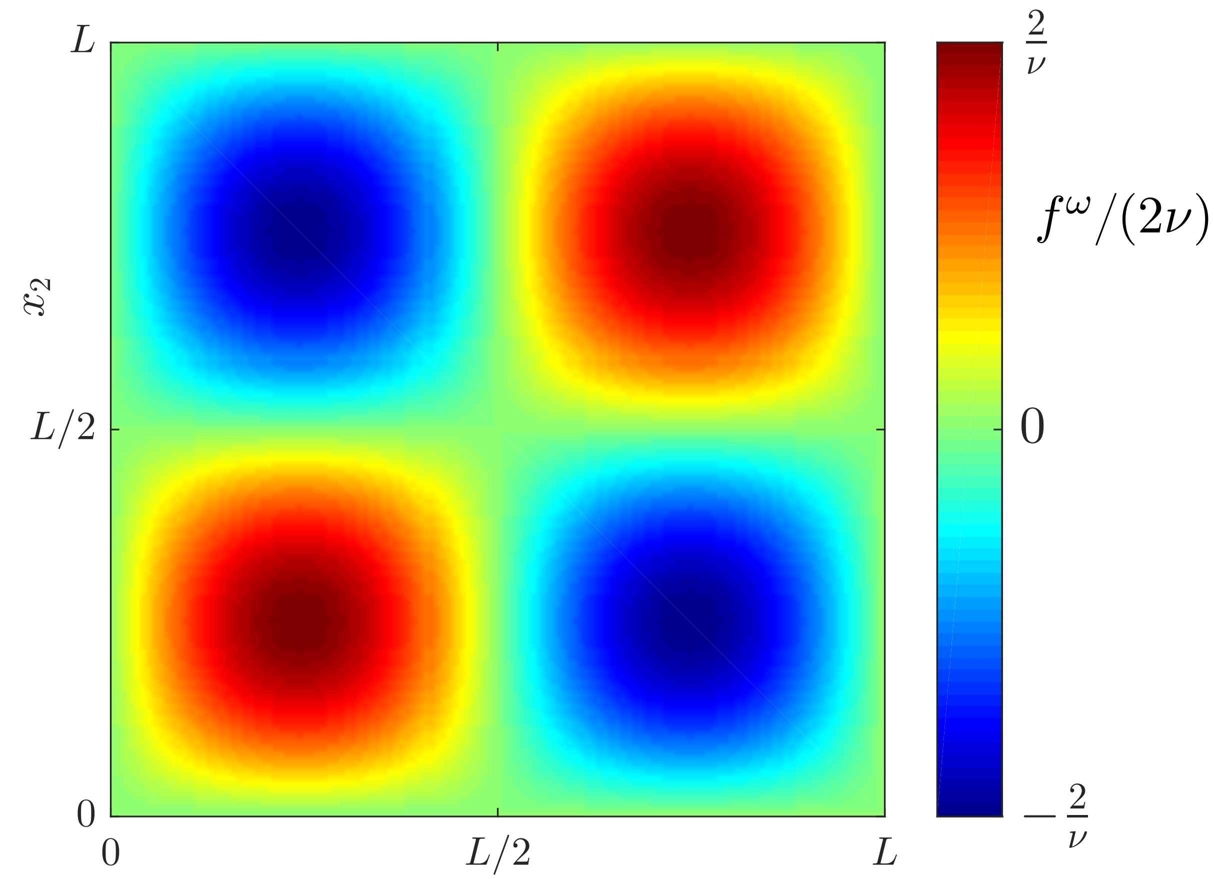
Spontaneous periodic orbits in the Navier-Stokes flow

$$\begin{cases} \partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f, & \text{on } \mathbb{T}^3 \text{ of size } L = 2\pi \\ \nabla \cdot u = 0. \end{cases}$$

Taylor-Green (time-independent) forcing term

$$f = \begin{pmatrix} 2 \sin x_1 \cos x_2 \\ -2 \cos x_1 \sin x_2 \\ 0 \end{pmatrix}$$

$$f^\omega \stackrel{\text{def}}{=} \nabla \times f = \begin{pmatrix} 0 \\ 0 \\ 4 \sin x_1 \sin x_2 \end{pmatrix}$$



Density plot of the vertical vorticity of $f^\omega/(2\nu)$. Red corresponds to vortices with a counter clockwise rotation and blue clockwise.

The autonomous Navier-Stokes equations under this time-independent forcing term admit a viscous equilibrium solution for which we have the analytic expression

$$u^* = \frac{1}{2\nu} f, \qquad p^* = \frac{1}{4\nu^2} \left(\cos 2x_1 + \cos 2x_2 \right).$$

Spontaneous periodic orbits in the Navier-Stokes flow

$$\mathcal{F}(W) = \begin{pmatrix} F_{\mathbb{C}}(W) \\ (F_n(W))_{n \in \mathbb{Z}^4_*} \end{pmatrix} = 0$$

$$\boxed{F_n(W) \stackrel{\text{def}}{=} i\Omega n_4 \omega_n + \nu \tilde{n}^2 \omega_n - f^\omega_n + \text{nonlinear terms}}$$

Banach space: $X = \mathbb{C} \times \left(\ell^1_\eta(\mathbb{C})\right)^3$

Norm: $\|W\| = |\Omega| + \sum_{1 \le l \le 3} \|\omega^{(l)}\|_{\ell^1_\eta}.$

What is A?

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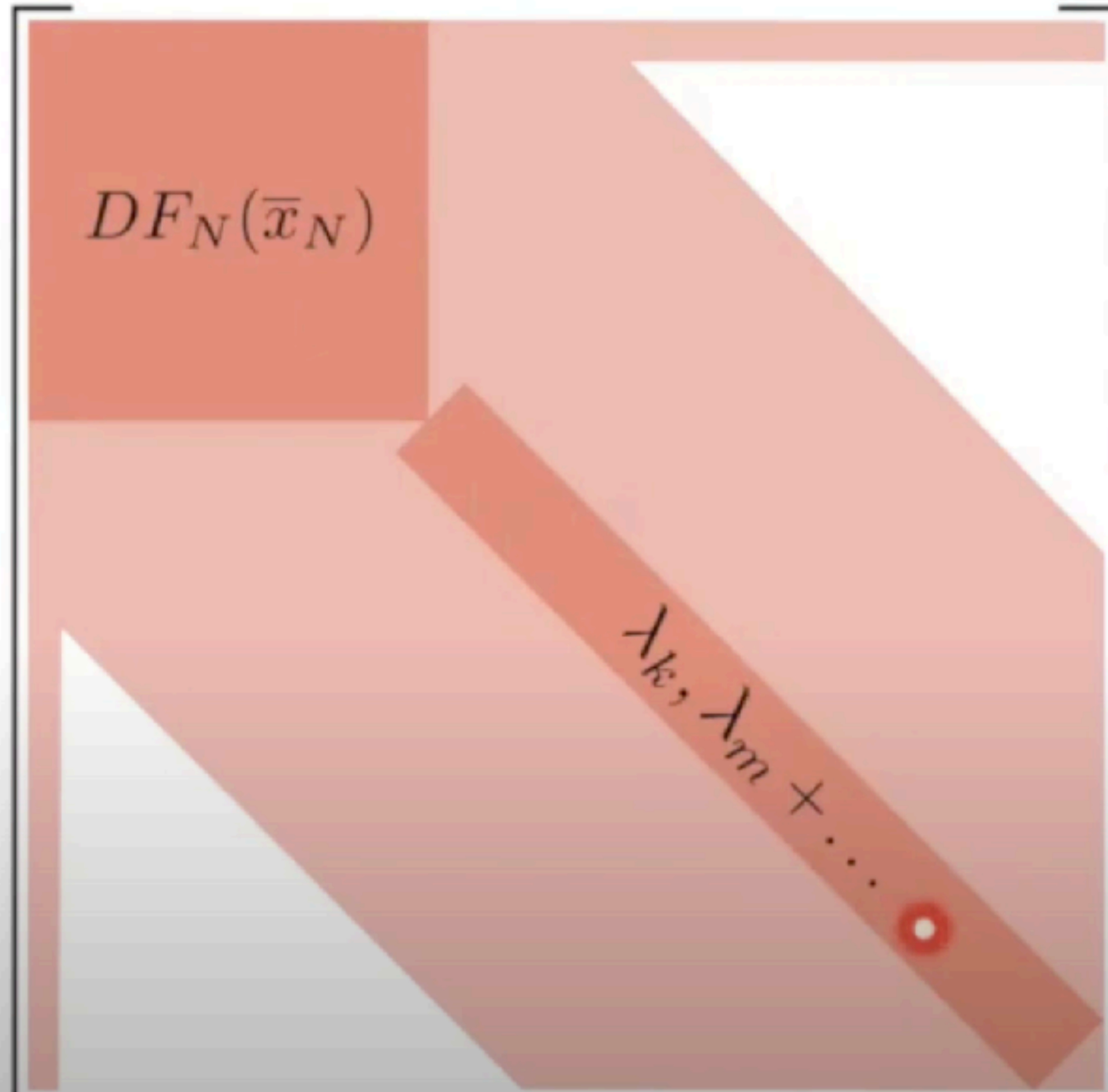

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$$T(x) = x - AF(x)$$

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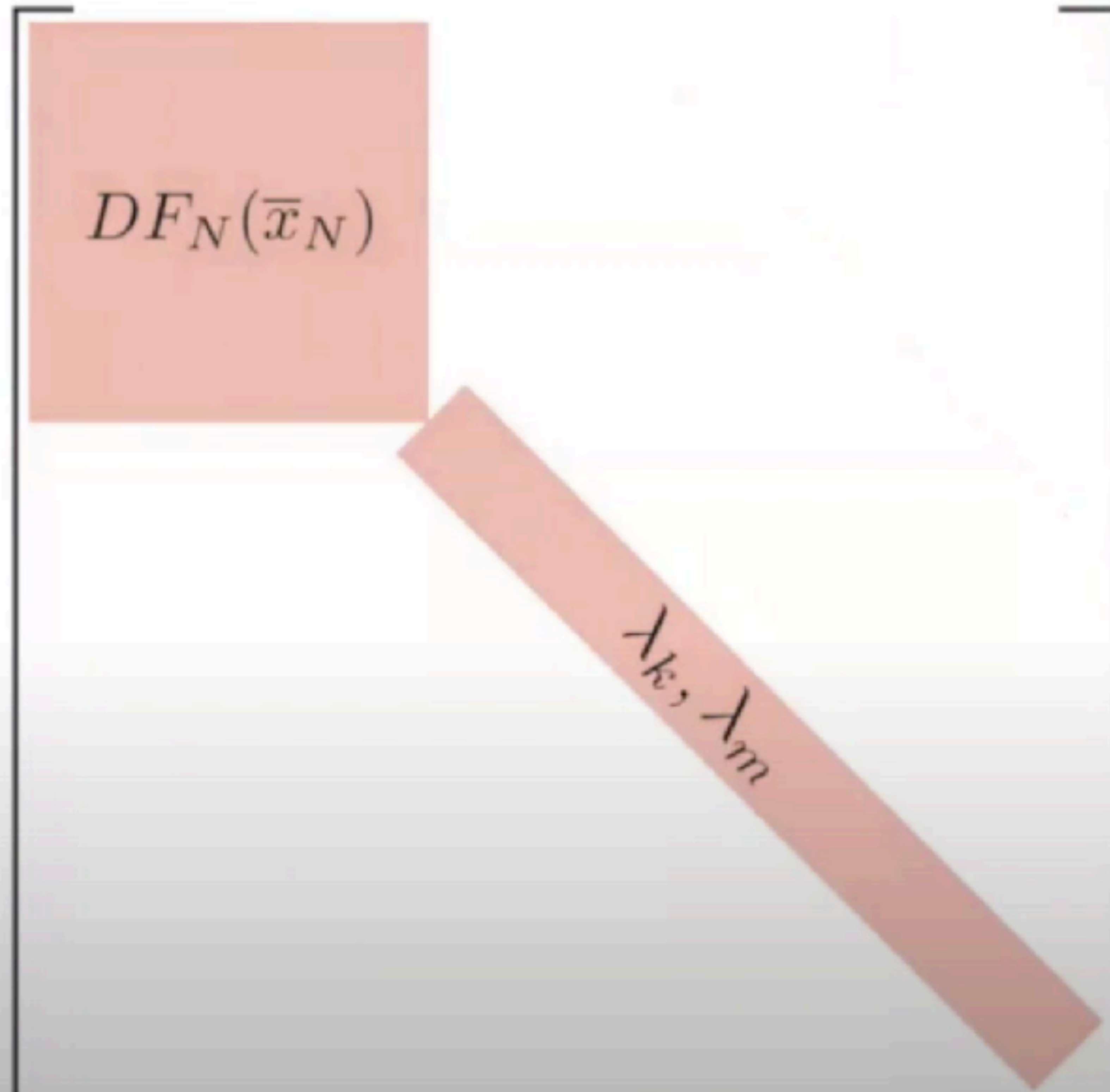

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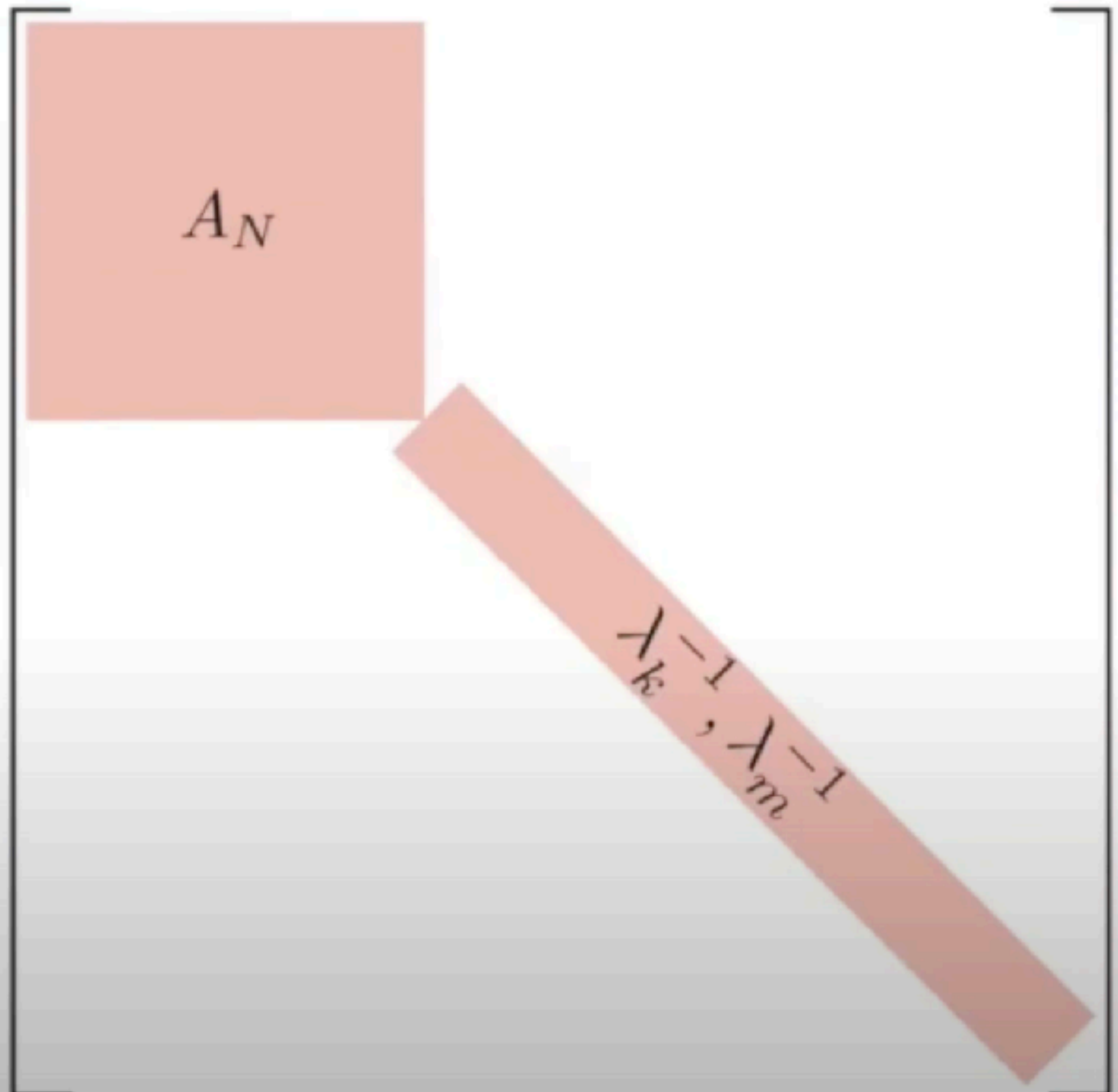

$A_N \approx DF_N(\overline{x}_N)^{-1}$

Banach contraction Theorem

- T maps $B_r(\overline{x}) \subset X$ into itself
- $\|T(x) - T(\tilde{x})\|_X \leq \kappa \|x - \tilde{x}\|_X \quad \kappa < 1$

Analytic estimates

$$\|T(\overline{x}) - \overline{x}\|_X \leq Y$$

$$\|DT(\overline{x})\|_{B(X)} \leq Z$$

$$\|D^2 T(x)\|_{.....} \leq W(r) \quad \forall x \in B_r(\overline{x})$$

Inequality $Y + Z\hat{r} + \frac{1}{2} W(\hat{r})\hat{r}^2 < \hat{r}$

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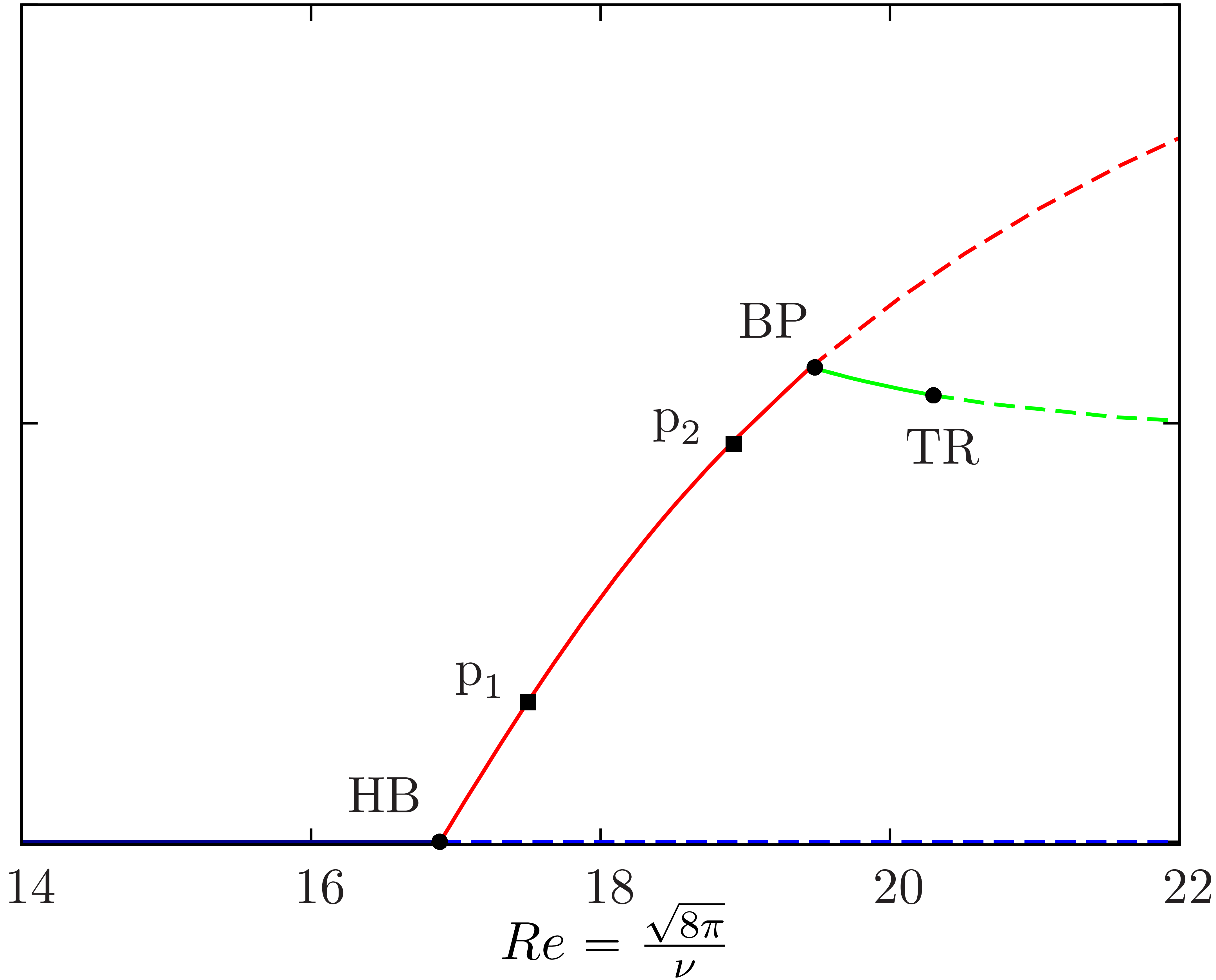
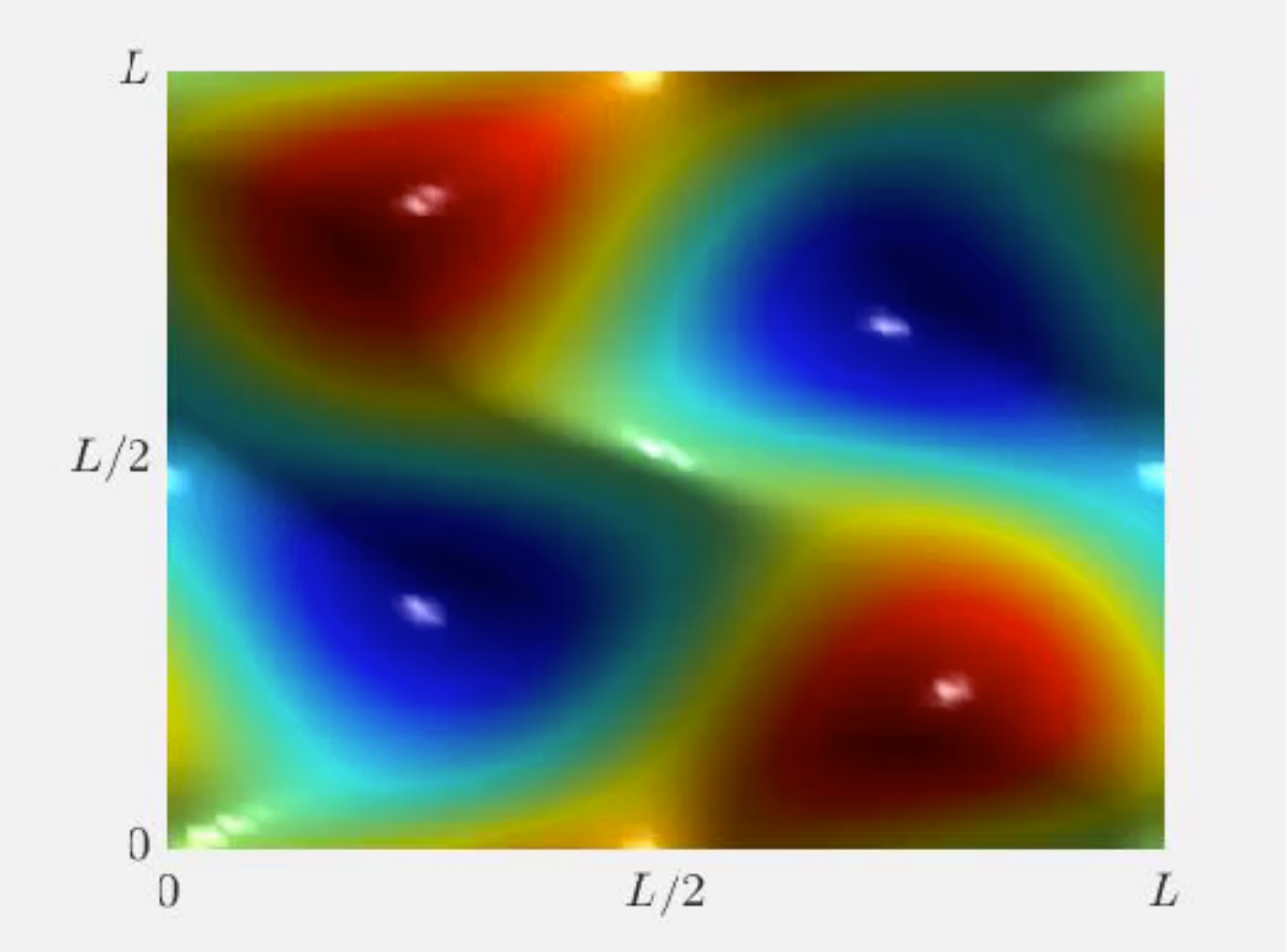


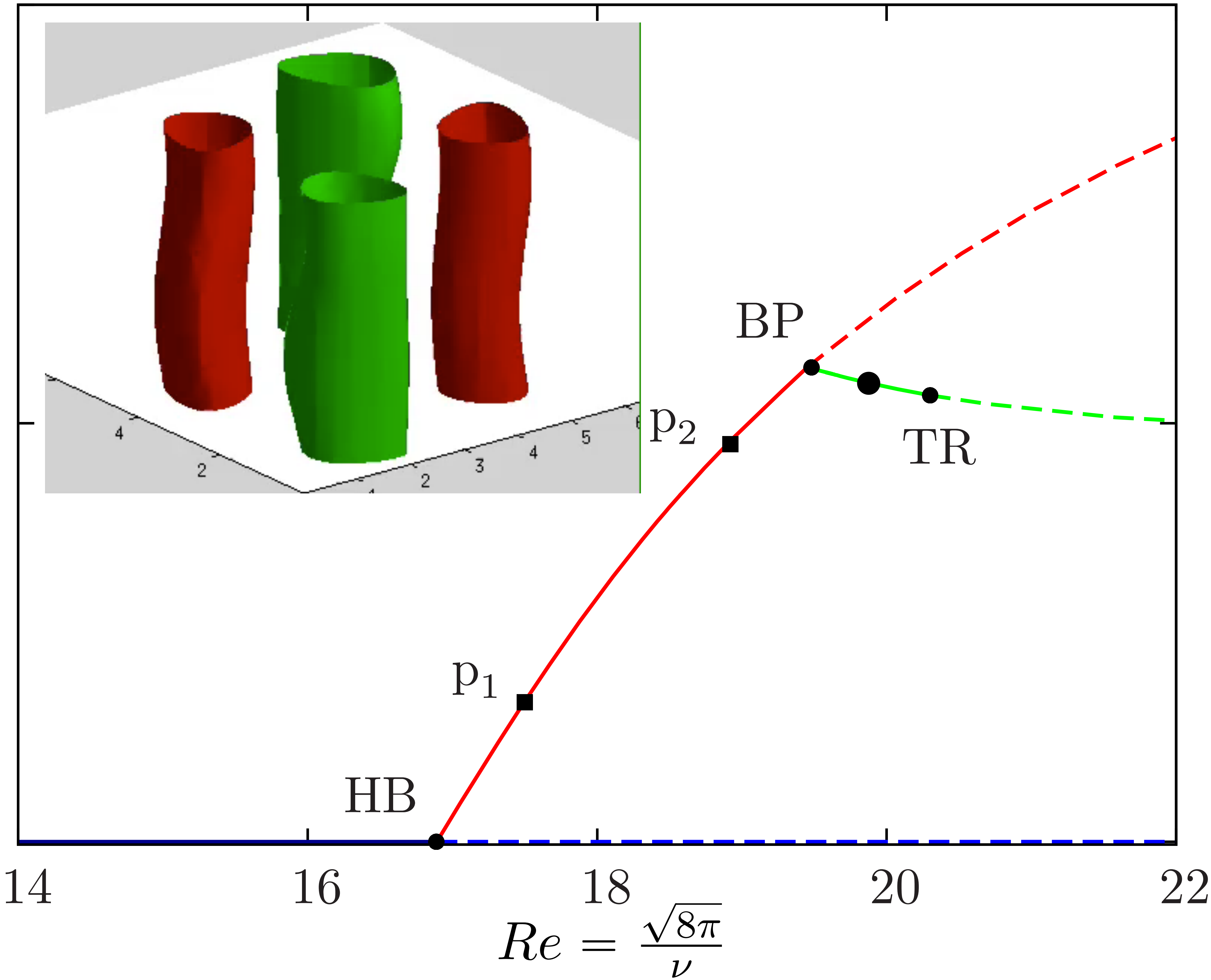
Theorem: Consider NS defined on the three-torus $\mathbb{T}^3$ (with size length $L = 2\pi$) and consider the Taylor-Green time-independent forcing term. Let $\nu = 0.265$ and $(\bar{u}, \bar{p})$ be a numerical solution computed with $N_{x_1} = N_{x_2} = 21$, $N_{x_3} = 0$ and $N_t = 16$ Fourier coefficients. Let $r = 2.2491 \cdot 10^{-6}$. There exists a $\frac{2\pi}{\Omega}$-periodic solution (u, p) of NS with $|\Omega - \bar{\Omega}| \leqslant r$ and $\|u - \bar{u}\|_{C^0} \leqslant r$.



	η	N_{x_1}	N_{x_2}	N_{x_3}	N_t	$N^\dagger$	$\widetilde{N}$	RAM (GB)	CPU days
p_1	1	17	17	0	11	130	265	10	6
p_2	1	21	21	0	16	210	425	110	95

The Galerkin projection for the solution p_2 is $\mathcal{F} : \mathbb{C}^{61018} \to \mathbb{C}^{61018}$.

Future work: a fully 3D spontaneous periodic orbit



Thank you

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