Nečas Center for Mathematical Modeling

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MINISTRY OF EDUCATION, YOUTH AND SPORTS

On nonlinear problems of parabolic type with implicit constitutive equations involving flux

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Section 1

Introduction and Problem formulation

For any given

$$\Omega \subset \mathbb{R}^d, \quad T > 0, \quad f : Q \to \mathbb{R},$$
$$u_0 : \Omega \to \mathbb{R}, \quad u_D : \Sigma_D \to \mathbb{R}, \quad g : \Sigma_N \to \mathbb{R}$$

find a function $u : Q \to \mathbb{R}$ satisfying

$$\begin{aligned} \partial_t u - \Delta u &= f && \text{in } Q \\ u(0, \cdot) &= u_0 && \text{in } \Omega \\ u &= u_D && \text{on } \Sigma_D \\ \nabla u \cdot \mathbf{n} &= g && \text{on } \Sigma_N \end{aligned}$$

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$$\begin{aligned} \partial_t u - \Delta u = f & \quad \text{in } Q \\ u(0, \cdot) = u_0 & \quad \text{in } \Omega \\ u = 0 & \quad \text{on } \Sigma_D \\ \nabla u \cdot \mathbf{n} = 0 & \quad \text{on } \Sigma_N \end{aligned}$$

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$$\Omega \subset \mathbb{R}^d, \quad T > 0, \quad f : Q \to \mathbb{R},$$
$$u_0 : \Omega \to \mathbb{R}$$

find a function $u : Q \to \mathbb{R}$ and $\mathbf{j} : Q \to \mathbb{R}^d$ satisfying

$$\begin{aligned} \partial_t u - \operatorname{div} \mathbf{j} &= f && \text{in } Q \\ \mathbf{j} &= \nabla u && \text{in } Q \\ u(0, \cdot) &= u_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma_D \\ \nabla u \cdot \mathbf{n} &= 0 && \text{on } \Sigma_N \end{aligned}$$

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$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \int_\Omega \mathbf{j} \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}t = \int_0^t \int_\Omega f u \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{2}\|u_0\|_2^2$$

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$$\|u(t)\|_2^2 + \int_0^t \|\mathbf{j}\|_2^2 \, \mathrm{d}t + \int_0^t \|\nabla u\|_2^2 \, \mathrm{d}t = 2 \int_0^t \int_\Omega f u \, \mathrm{d}x \, \mathrm{d}t + \|u_0\|_2^2$$

For any given

$$\Omega \subset \mathbb{R}^d, \quad T > 0, \quad f : Q \to \mathbb{R},$$
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find a function $u : Q \to \mathbb{R}$ and $\mathbf{j} : Q \to \mathbb{R}^d$ satisfying

$$\begin{aligned} \partial_t u - \operatorname{div} \mathbf{j} &= f && \text{in } Q \\ \mathbf{j} &= \nabla u && \text{in } Q \\ u(0,\cdot) &= u_0 && \text{in } \Omega \\ u &= 0 && \text{on } \Sigma_D \\ \nabla u \cdot \mathbf{n} &= 0 && \text{on } \Sigma_N \end{aligned}$$

$$\|u(t)\|_2^2 + \int_0^t \|\mathbf{j}\|_2^2 \, dt + \int_0^t \|\nabla u\|_2^2 \, dt = 2 \int_0^t \int_\Omega f u \, dx \, dt + \|u_0\|_2^2$$

There is unique $(u, \mathbf{j})$ solving weakly the problem.

For any given

$$\Omega \subset \mathbb{R}^d, \quad T > 0, \quad f : Q \to \mathbb{R},$$
$$u_0 : \Omega \to \mathbb{R}, \quad \mathbf{g} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$$

find a function $u : Q \to \mathbb{R}$ and $\mathbf{j} : Q \to \mathbb{R}^d$ satisfying

$$\begin{aligned} \partial_t u - \operatorname{div} \mathbf{j} &= f \quad &&\text{in } Q \\ \mathbf{g}(\mathbf{j}, \nabla u) &= \mathbf{0} \quad &&\text{in } Q \\ u(0, \cdot) &= u_0 \quad &&\text{in } \Omega \\ u &= 0 \quad &&\text{on } \Sigma_D \\ \nabla u \cdot \mathbf{n} &= 0 \quad &&\text{on } \Sigma_N \end{aligned}$$

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \int_\Omega \mathbf{j} \cdot \nabla u \, \mathrm{d}x \, \mathrm{d}t = \int_0^t \int_\Omega f u \, \mathrm{d}x \, \mathrm{d}t + \frac{1}{2}\|u_0\|_2^2$$

$$\boxed{\mathbf{g}(\mathbf{j}, \nabla u) = \mathbf{0}}$$

Aim: to develop PDE theory

- self-contained, robust, simple, elegant
- accessible to broad scientific audience
- accessible for numerical analysis and computing

Particularly

- to avoid the assumption on the existence of *measurable* selection or to avoid the dependence on specific form
- to avoid using convolution
- to avoid introducing generalized concept of solution (using subdifferential, variational inequalities)

Beyond Fourier, Darcy, Fick, Hooke's, Navier-Stokes linear relations

$\mathbf{j} = \mathbf{k}(\nabla u)$	$\nabla u = \mathbf{k}(\mathbf{j})$
$\mathbf{j} = \lvert\nabla u\rvert^{p-2}\nabla u$	$\nabla u = \lvert\mathbf{j}\rvert^{p'-2}\mathbf{j}$
$\mathbf{j} = (1 + \lvert\nabla u\rvert)^{p-2}\nabla u$	$\nabla u = (1 + \lvert\mathbf{j}\rvert)^{p'-2}\mathbf{j}$
$\mathbf{j} = (1 + \lvert\nabla u\rvert^2)^{\frac{p-2}{2}}\nabla u$	$\nabla u = (1 + \lvert\mathbf{j}\rvert^2)^{\frac{p'-2}{2}}\mathbf{j}$
$\mathbf{j} = (\lvert\nabla u\rvert - \delta_*)^+ \frac{\nabla u}{\lvert\nabla u\rvert}$	$\nabla u = (\lvert\mathbf{j}\rvert - \sigma_*)^+ \frac{\mathbf{j}}{\lvert\mathbf{j}\rvert}$

Table: $p \in (1, +\infty)$, $p' = p/(p-1)$, and $\delta_*, \sigma_* > 0$. The structures motivated by the classification of incompressible fluid models presented in Blechta et al. SIAM J Math Analysis (2020).

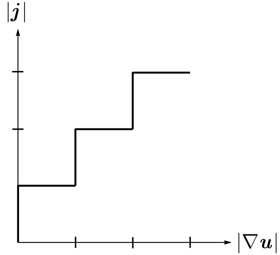

Figure: Left: the step function both from $\mathbf{j}$ and ∇u viewpoint. Obtained by considering the $\sqrt{2}$-periodic zig-zag function with the magnitude $\sqrt{2}/2$ rotated by 45 degrees in $(\mathbf{j}, \nabla u)$-plane. Right: one simple step followed by the linear relation $\mathbf{j} = \nabla \mathbf{u}$. Both curves are continuous, none of then can be written in the form $\mathbf{j} = \mathbf{k}(\nabla u)$ or $\mathbf{k} = \mathbf{k}(\nabla u)$.

Assumptions on the admissible form of $\mathbf{g}$

(g1) $\mathbf{g} \in \mathcal{C}^{0,1}(\mathbb{R}^d \times \mathbb{R}^d)^d$ and $\mathbf{g}(\mathbf{0},\mathbf{0}) = \mathbf{0}$

(g2)

$$\mathbf{g_j}(\mathbf{j},\mathbf{d}) \geq 0, \quad \mathbf{g_d}(\mathbf{j},\mathbf{d}) \leq 0, \quad \mathbf{g_j}(\mathbf{j},\mathbf{d}) - \mathbf{g_d}(\mathbf{j},\mathbf{d}) > 0$$

$$\mathbf{g_d}(\mathbf{j},\mathbf{d})(\mathbf{g_j}(\mathbf{j},\mathbf{d}))^T \leq 0$$

(g3)

$$\text{either} \quad \forall \mathbf{d} \in \mathbb{R}^d \quad \liminf_{|\mathbf{j}| \to +\infty} \mathbf{g}(\mathbf{j},\mathbf{d}) \cdot \mathbf{j} > 0$$

$$\text{or} \quad \forall \mathbf{j} \in \mathbb{R}^d \quad \limsup_{|\mathbf{d}| \to +\infty} \mathbf{g}(\mathbf{j},\mathbf{d}) \cdot \mathbf{d} < 0$$

(g4)

$$\mathbf{j} \cdot \mathbf{d} \geq c_1(|\mathbf{j}|^{p'} + |\mathbf{d}|^p) - c_2$$

For any $\Omega \subset \mathbb{R}^d$, $T > 0, \mathbf{u}_0, f, p \in (1,\infty), \mathbf{g}$ satisfying (g1)–(g4), there is $(u, \mathbf{j})$ solving the studied problem.

Section 2

Main result

For any given

$$\Omega \subset \mathbb{R}^d, T > 0, \mathbf{u}_0 : \Omega \to \mathbb{R}^N, \mathbf{f} : Q \to \mathbb{R}^N \text{ and}$$
$$\mathbf{G} : \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N} \text{ satisfying} \quad \text{(G1)–(G4)}$$

find a couple $\mathbf{u} : Q \to \mathbb{R}^N$ and $\mathbf{J} : Q \to \mathbb{R}^{d\times N}$ solving the problem

$$\begin{aligned}
\partial_t \mathbf{u} - \operatorname{div} \mathbf{J} &= \mathbf{f} && \text{in } Q \\
\mathbf{G}(\mathbf{J}, \nabla \mathbf{u}) &= \mathbf{0} && \text{in } Q \\
\mathbf{u}(0, \cdot) &= \mathbf{u}_0 && \text{in } \Omega \\
\mathbf{u} &= \mathbf{0} && \text{on } \Sigma_D \\
\mathbf{J}\mathbf{n} &= \mathbf{0} && \text{on } \Sigma_N
\end{aligned}$$

(G1) $\mathbf{G} \in \mathcal{C}^{0,1}(\mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N})^{d\times N}$ and $\mathbf{G}(\mathbf{0},\mathbf{0}) = \mathbf{0}$,

(G2) for almost all $(\mathbf{J},\mathbf{D}) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}$:

$$\mathbf{G}_{\mathbf{J}}(\mathbf{J},\mathbf{D}) \geq 0, \quad \mathbf{G}_{\mathbf{D}}(\mathbf{J},\mathbf{D}) \leq 0, \quad \mathbf{G}_{\mathbf{J}}(\mathbf{J},\mathbf{D}) - \mathbf{G}_{\mathbf{D}}(\mathbf{J},\mathbf{D}) > 0,$$
$$\text{and} \quad \mathbf{G}_{\mathbf{D}}(\mathbf{J},\mathbf{D})(\mathbf{G}_{\mathbf{J}}(\mathbf{J},\mathbf{D}))^T \leq 0$$

(G3) one of the following holds:

$$\text{either} \quad \forall \mathbf{D} \in \mathbb{R}^{d\times N} \quad \liminf_{|\mathbf{J}|\to+\infty} \mathbf{G}(\mathbf{J},\mathbf{D}) : \mathbf{J} > 0$$
$$\text{or} \quad \forall \mathbf{J} \in \mathbb{R}^{d\times N} \quad \limsup_{|\mathbf{D}|\to+\infty} \mathbf{G}(\mathbf{J},\mathbf{D}) : \mathbf{D} < 0,$$

(G4) there exist $c_1, c_2 > 0$ such that for all $(\mathbf{J},\mathbf{D}) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}$ fulfilling $\mathbf{G}(\mathbf{J},\mathbf{D}) = \mathbf{0}$ we have

$$\mathbf{J} : \mathbf{D} \geq c_1(|\mathbf{J}|^{p'} + |\mathbf{D}|^p) - c_2.$$

The Maxwell–Stefan system

For $u_\nu : (0, T) \times \Omega \to \mathbb{R}$, $0 \le u_\nu \le 1$, $\nu = 1, \dots, N$, $N \ge 2$

$$\partial_t u_\nu - \operatorname{div} \mathbf{j}_\nu = r_\nu(\mathbf{u})$$

$$\nabla u_\nu = \sum_{\mu=1,\, \mu\neq\nu}^{N} \alpha_{\mu\nu} \left(u_\mu \mathbf{j}_\nu - u_\nu \mathbf{j}_\mu \right)$$

The constants $\alpha_{\nu\mu} > 0$ for $\nu \neq \mu$ and $\mathbf{u} := (u_1, \dots, u_N)$.
Denoting $\mathbf{d}_\nu := \nabla u_\nu$, $\mathbf{D} := (\mathbf{d}_1, \dots, \mathbf{d}_N)^T$ and $\mathbf{J} := (\mathbf{j}_1, \dots, \mathbf{j}_N)^T$:

$$\mathbf{D} = \mathbb{B}(\mathbf{u})\mathbf{J},$$

where $\mathbb{B}$ is $N \times N$-matrix. Then

$$\mathbf{G}(\mathbf{J}, \mathbf{D}) = \mathbb{B}(\mathbf{u})\mathbf{J} - \mathbf{D}$$

fulfill the conditions (G1)-(G3).
Note that theory is not directly applicable as **G** and also the right-hand side depends on **u**.

Theorem

Let $\Omega \subset \mathbb{R}^d$ Lipschitz domain, $T > 0$, $p \in (1, \infty)$,
$\mathbf{f} \in L^{p'}(0, T; V_p^*)$, $\mathbf{u}_0 \in H$ and $\mathbf{G}$ satisfy (G1)–(G4).

Then there exists a couple $(\mathbf{u}, \mathbf{J})$ fulfilling

$$\begin{aligned} \mathbf{u} &\in L^p(0, T; V_p) \cap \mathcal{C}([0, T]; H) \\ \partial_t \mathbf{u} &\in L^{p'}(0, T; V_p^*) \\ \mathbf{J} &\in L^{p'}(Q; \mathbb{R}^{d \times N}) \end{aligned}$$

$$\langle \partial_t \mathbf{u}, \varphi \rangle_{V_p} + \int_\Omega \mathbf{J} : \nabla \varphi \, \mathrm{d}x = \langle \mathbf{f}, \varphi \rangle_{V_p} \quad \text{a.a. } t \text{ and } \forall \varphi \in V_p$$

$$\mathbf{G}(\mathbf{J}, \nabla \mathbf{u}) = \mathbf{0} \ \text{ a.e. in } Q$$

$$\lim_{t \to 0_+} \|\mathbf{u}(t) - \mathbf{u}_0\|_H = 0.$$

In addition, $\mathbf{u}$ is uniquely determined.

Observation

Let $\mathbf{G}$ satisfy assumptions (G1)–(G4). Let

$$\mathcal{A} := \{(\mathbf{J}, \mathbf{D}) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N} : \ \mathbf{G}(\mathbf{J}, \mathbf{D}) = \mathbf{0}\}.$$

Then $\mathcal{A}$ is a maximal monotone p-coercive graph:

(A1) $(\mathbf{0}, \mathbf{0}) \in \mathcal{A}$

(A2) For any $(\mathbf{J}_1, \mathbf{D}_1), (\mathbf{J}_2, \mathbf{D}_2) \in \mathcal{A}$

$$(\mathbf{J}_1 - \mathbf{J}_2) : (\mathbf{D}_1 - \mathbf{D}_2) \geq 0.$$

(A3) If for some $(\mathbf{J}, \mathbf{D}) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}$ and for all $(\overline{\mathbf{J}}, \overline{\mathbf{D}}) \in \mathcal{A}$

$$(\mathbf{J} - \overline{\mathbf{J}}) : (\mathbf{D} - \overline{\mathbf{D}}) \geq 0,$$

then $(\mathbf{J}, \mathbf{D}) \in \mathcal{A}$.

(A4) There exist $C_1, C_2 > 0$ such that for all $(\mathbf{J}, \mathbf{D}) \in \mathcal{A}$

$$\mathbf{J} : \mathbf{D} \geq C_1(|\mathbf{J}|^{p'} + |\mathbf{D}|^p) - C_2.$$

M. Bulíček, J. Málek, **E. Maringová**: *On nonlinear problems of parabolic type with implicit constitutive equations involving flux.* ArXiv:2009.06917

J. Blechta, J. Málek, **K. R. Rajagopal**: *On the classification of incompressible fluids and a mathematical analysis of the equations that govern their motion.* SIAM J. Math. Anal. **52** (2020) 1232-1289

M. Bulíček, P. Gwiazda, J. Málek, **A. Swierczewska-Gwiazda**: *On unsteady flows of implicitly constituted incompressible fluids* SIAM J. Math. Anal. **44** (2012) 2756–2801

- Theory is based on ideas of Minty; Brezis, Crandall, Pazy; Alberti, Ambrosio; Francfort, Murat, Tartar
- Numerical analysis and computing: Süli, Kreuzer, Diening, Tscherpel, Farrell, Orozco
- Implicit constitutive theory: Rajagopal (since 2003)

Section 3

Proof in five steps

Step 1

$$\boxed{\mathcal{A} := \{(\mathbf{J},\mathbf{D}) : \ \mathbf{G}(\mathbf{J},\mathbf{D}) = \mathbf{0}\}}$$

Two different approximations for $\varepsilon > 0$:

- Let $\mathcal{A}$ be a maximal monotone p=coercive graph

$$\mathcal{A}_\varepsilon := \{(\tilde{\mathbf{J}},\tilde{\mathbf{D}}) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}; \exists(\overline{\mathbf{J}},\overline{\mathbf{D}}) \in \mathcal{A}, \tilde{\mathbf{J}} = \overline{\mathbf{J}}, \tilde{\mathbf{D}} = \overline{\mathbf{D}} + \varepsilon\overline{\mathbf{J}}\}$$
$$\mathcal{A}_\varepsilon^\varepsilon := \{(\mathbf{J},\mathbf{D}) \in \mathbb{R}^{d\times N} \times \mathbb{R}^{d\times N}; \exists(\tilde{\mathbf{J}},\tilde{\mathbf{D}}) \in \mathcal{A}_\varepsilon, \mathbf{J} = \tilde{\mathbf{J}} + \varepsilon\tilde{\mathbf{D}}, \mathbf{D} = \tilde{\mathbf{D}}\}$$

- Let $\mathbf{G}$ satisfy (G1)–(G4). Set

$$\mathbf{G}_\varepsilon(\mathbf{J},\mathbf{D}) := \mathbf{G}(\mathbf{J} - \varepsilon\mathbf{D}, \mathbf{D} - \varepsilon\mathbf{J})$$

$\mathcal{A}_\varepsilon^\varepsilon$ is a maximal monotone 2-coercive graph and $\exists!$ single-valued mapping $\mathbf{J}_\varepsilon^* : \mathbb{R}^{d\times N} \to \mathbb{R}^{d\times N}$ satisfying

$$(\mathbf{J}, \mathbf{D}) \in \mathcal{A}_\varepsilon^\varepsilon \iff \mathbf{J} = \mathbf{J}_\varepsilon^*(\mathbf{D}).$$

Moreover, $\mathbf{J}_\varepsilon^*$ is Lipschitz continuous and uniformly monotone, i.e. $\exists C_1,\ C_2 > 0$ such that $\forall \mathbf{D}_1,\ \mathbf{D}_2 \in \mathbb{R}^{d\times N}$

$$\begin{aligned} |\mathbf{J}_\varepsilon^*(\mathbf{D}_1) - \mathbf{J}_\varepsilon^*(\mathbf{D}_2)| &\le C_2 |\mathbf{D}_1 - \mathbf{D}_2|, \\ (\mathbf{J}_\varepsilon^*(\mathbf{D}_1) - \mathbf{J}_\varepsilon^*(\mathbf{D}_2)) : (\mathbf{D}_1 - \mathbf{D}_2) &\ge C_1 |\mathbf{D}_1 - \mathbf{D}_2|^2. \end{aligned}$$

Step 2

For every $\varepsilon \in (0,1)$, there is a unique couple $(\mathbf{u}^\varepsilon, \mathbf{J}^\varepsilon)$

$$(\mathbf{u}^\varepsilon, \mathbf{J}^\varepsilon) \in \big(L^2(0,T;V) \cap \mathcal{C}([0,T];H)\big) \times L^2(Q;\mathbb{R}^{d\times N})$$

$$\begin{aligned} \langle \partial_t \mathbf{u}^\varepsilon, \varphi \rangle_V + \int_\Omega \mathbf{J}^\varepsilon : \nabla\varphi \,\mathrm{d}x = \langle \mathbf{f}, \varphi \rangle_V \quad & \text{a.a. } t \text{ and } \forall \varphi \in V \\ \mathbf{J}^\varepsilon = \mathbf{J}^*_\varepsilon(\nabla \mathbf{u}^\varepsilon) \quad & \text{almost everywhere in } Q \\ \lim_{t\to 0_+} \|\mathbf{u}^\varepsilon(t) - \mathbf{u}_0\|_H = 0 & \end{aligned}$$

Step 2

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$$\langle \partial_t \mathbf{u}^\varepsilon, \varphi\rangle_V + \int_\Omega \mathbf{J}^\varepsilon : \nabla\varphi \,\mathrm{d}x = \langle \mathbf{f}, \varphi\rangle_V \qquad \text{a.a. } t \text{ and } \forall \varphi \in V$$

$$\mathbf{J}^\varepsilon = \mathbf{J}^*_\varepsilon(\nabla \mathbf{u}^\varepsilon) \quad \text{almost everywhere in } Q$$

$$\lim_{t\to 0_+} \|\mathbf{u}^\varepsilon(t) - \mathbf{u}_0\|_H = 0$$

Uniform estimates

$$\frac{1}{2}\|\mathbf{u}^\varepsilon(t)\|_H^2 + \int_{Q_t} \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \langle \mathbf{f}, \mathbf{u}^\varepsilon\rangle_{V_\mu}\,\mathrm{d}\tau + \frac{1}{2}\|\mathbf{u}_0\|_H^2$$

Step 2

For every $\varepsilon \in (0,1)$, there is a unique couple $(\mathbf{u}^\varepsilon, \mathbf{J}^\varepsilon)$

$$(\mathbf{u}^\varepsilon, \mathbf{J}^\varepsilon) \in \big(L^2(0,T;V) \cap \mathcal{C}([0,T];H)\big) \times L^2(Q;\mathbb{R}^{d\times N})$$

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Uniform estimates

$$\frac{1}{2}\|\mathbf{u}^\varepsilon(t)\|_H^2 + \int_{Q_t} \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \langle \mathbf{f}, \mathbf{u}^\varepsilon\rangle_{V_\mu} \,\mathrm{d}\tau + \frac{1}{2}\|\mathbf{u}_0\|_H^2$$

$$\mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \geq \tilde{C}_1(|\mathbf{J}^\varepsilon|^{\min\{p',2\}}) + |\nabla \mathbf{u}^\varepsilon|^{\min\{p,2\}}) - \tilde{C}_2.$$

$$\mu := \min\{p, 2\}, \quad \mu' := \max\{p', 2\},$$
$$\nu := \min\{p', 2\}, \quad \nu' := \max\{p, 2\}.$$

$$\boxed{\mathbf{f} \in L^{\mu'}(0, T; V_\mu^*)}$$

Step 3

As $\varepsilon \to 0_+$,

$$\begin{aligned} \mathbf{u}^\varepsilon &\rightharpoonup \mathbf{u} && \text{weakly in } L^\mu(0,T;V_\mu) \\ \mathbf{J}^\varepsilon &\rightharpoonup \mathbf{J} && \text{weakly in } L^\nu(Q;\mathbb{R}^{d\times N}) \\ \mathbf{u}^\varepsilon &\rightharpoonup^* \mathbf{u} && \text{weakly}^* \text{ in } L^\infty(0,T;H) \\ \partial_t \mathbf{u}^\varepsilon &\rightharpoonup \partial_t \mathbf{u} && \text{weakly in } L^\nu(0,T;V^*_{\nu'}) \end{aligned}$$

The evolutionary governing equation is fulfilled.

Step 4

As $(\mathbf{J}^\varepsilon, \nabla \mathbf{u}^\varepsilon) \in \mathcal{A}^\varepsilon_\varepsilon$ a.e. in Q satisfy

$$\int_U \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leq C \quad \text{uniformly with respect to } \varepsilon.$$

Then $\exists \mathbf{J} \in L^{p'}(U; \mathbb{R}^{d\times N})$, $\nabla \mathbf{u} \in L^p(U; \mathbb{R}^{d\times N})$:

$$\begin{aligned} \mathbf{J}^\varepsilon &\rightharpoonup \mathbf{J} && \text{weakly in } L^{\min\{2,p'\}}(U; \mathbb{R}^{d\times N}), \\ \nabla \mathbf{u}^\varepsilon &\rightharpoonup \nabla \mathbf{u} && \text{weakly in } L^{\min\{2,p\}}(U; \mathbb{R}^{d\times N}). \end{aligned}$$

Step 4

As $(\mathbf{J}^\varepsilon, \nabla \mathbf{u}^\varepsilon) \in \mathcal{A}^\varepsilon_\varepsilon$ a.e. in Q satisfy

$$\int_U \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leq C \quad \text{uniformly with respect to } \varepsilon.$$

Then $\exists \mathbf{J} \in L^{p'}(U;\mathbb{R}^{d\times N})$, $\nabla \mathbf{u} \in L^p(U;\mathbb{R}^{d\times N})$:

$$\begin{aligned} \mathbf{J}^\varepsilon &\rightharpoonup \mathbf{J} && \text{weakly in } L^{\min\{2,p'\}}(U;\mathbb{R}^{d\times N}),\\ \nabla \mathbf{u}^\varepsilon &\rightharpoonup \nabla \mathbf{u} && \text{weakly in } L^{\min\{2,p\}}(U;\mathbb{R}^{d\times N}). \end{aligned}$$

Moreover

$$\limsup_{\varepsilon\to 0_+} \int_U \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leq \int_U \mathbf{J} : \nabla \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t$$

Step 4

As $(\mathbf{J}^\varepsilon, \nabla \mathbf{u}^\varepsilon) \in \mathcal{A}^\varepsilon_\varepsilon$ a.e. in Q satisfy

$$\int_U \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leq C \quad \text{uniformly with respect to } \varepsilon.$$

Then $\exists \mathbf{J} \in L^{p'}(U; \mathbb{R}^{d\times N})$, $\nabla \mathbf{u} \in L^p(U; \mathbb{R}^{d\times N})$:

$$\begin{aligned} \mathbf{J}^\varepsilon &\rightharpoonup \mathbf{J} && \text{weakly in } L^{\min\{2,p'\}}(U; \mathbb{R}^{d\times N}), \\ \nabla \mathbf{u}^\varepsilon &\rightharpoonup \nabla \mathbf{u} && \text{weakly in } L^{\min\{2,p\}}(U; \mathbb{R}^{d\times N}). \end{aligned}$$

Moreover

$$\limsup_{\varepsilon\to 0_+} \int_U \mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leq \int_U \mathbf{J} : \nabla \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t$$

Consequently,

$$\mathbf{J}^\varepsilon : \nabla \mathbf{u}^\varepsilon \rightharpoonup \mathbf{J} : \nabla \mathbf{u} \quad \text{weakly in } L^1(U)$$

$$(\mathbf{J}, \nabla \mathbf{u}) \in \mathcal{A} \text{ a.e. in } Q$$

Step 5

$$\boxed{f \in L^{p'}(0,T;V_p^*)}$$

There exist $\{\mathbf{f}^m\}_{m\in\mathbb{N}} \subset L^{\mu'}(0,T;V_\mu^*)$:

$$\mathbf{f}^m \to \mathbf{f} \text{ in } L^{p'}(0,T;V_p^*)$$

Section 4

Final notes

Note 1

Many results: there exist a convex $\Phi : \mathbb{R}^{d\times N} \to \mathbb{R}$ and its convex conjugate $\Phi^* : \mathbb{R}^{d\times N} \to \mathbb{R}$ such that

$$\mathbf{G}(\mathbf{J}, \mathbf{D}) = \mathbf{0} \qquad \Longleftrightarrow \qquad \mathbf{J} : \mathbf{D} = \Phi(\mathbf{J}) + \Phi^*(\mathbf{D})$$

Then the condition "$\mathbf{G}(\mathbf{J}, \nabla \mathbf{u}) = \mathbf{0}$ almost everywhere in Q" can be equivalently replaced by

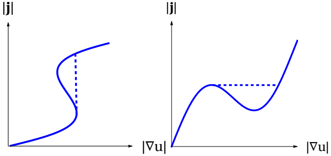
$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|_H^2 + \int_\Omega \Phi(\mathbf{J}) + \Phi^*(\nabla \mathbf{u})\,\mathrm{d}x \leq \langle \mathbf{f}, \nabla \mathbf{u}\rangle_{V_p}. \tag{3}$$

- $+$ easy to take limit
- $-$ required potential structure has its limitations
- $-$ requires the admissibility of **u** as a test function in governing equation

A. Abbatiello, E. Feireisl: *On a class of generalized solutions to equations describing incompressible viscous fluids.* Ann. Mat. Pura Appl. 199 (2020) 1183–1195.

Note 2 - open problem

non-monotone relation and hysteresis processes



$$\nabla u = \left(\alpha(1+\beta|\mathbf{j}|^2)^s + \gamma\right)\mathbf{j} \qquad s < -\frac{1}{2}$$

A. Janečka, J. Málek, **V. Pruša, G. Tierra**: *Numerical scheme for simulation of transient flows of non-Newtonian fluids characterised by a non-monotone relation between the symmetric part of the velocity gradient and the Cauchy stress tensor* Acta Mech. 230 (2019) 729–747.