Concentration (or not) in the Vlasov-Navier-Stokes system

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Outline

Fluid/kinetic systems

Describe a dispersed phase moving within a continous one

- $f(t, \mathbf{x}, \mathbf{v})$: density function (kinetic equation)
- $\mathbf{u}(t, \mathbf{x})$ and $p(t, \mathbf{x})$: velocity and pressure (fluid mechanics equations)

Zoology

First models by O'Rourke and Williams ('80) and then :

Fluid	Aerosol	Interaction
Incompressible	Fragmentation	Drag force
Viscous	Coalescence	Brinkman
Homogeneous	Polydispersed	...

→ # Models $\gg 1$

→ « revelancy » *vs.* « mathematical challenge »

The Vlasov-Navier-Stokes system

- Aerosol : monodispersed and without internal interactions
- Fluid : viscous, homogeneous, incompressible
- Interaction : **drag** and **Brinkman** forces

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = -\int_{\mathbf{v}} (\mathbf{u} - \mathbf{v})\ f \ \mathrm{d}\mathbf{v}, \\ \mathrm{div}\ \mathbf{u} = 0, \\ \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0. \end{cases}$$

Outline

The energy-dissipation identity

Defining

$$\mathsf{E}(t) = \frac{1}{2}\int_{\mathbf{x}} |\mathbf{u}(t)|^2 + \frac{1}{2}\int_{\mathbf{x},\mathbf{v}} f(t)|\mathbf{v}|^2,$$

$$\mathsf{D}(t) = \int_{\mathbf{x}} |\nabla \mathbf{u}(t)|^2 + \int_{\mathbf{x},\mathbf{v}} f(t)|\mathbf{u}(t) - \mathbf{v}|^2,$$

we have formally

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathsf{E}(t) + \mathsf{D}(t) = 0.$$

Setting

- $t \geq 0$, $\mathbf{x} \in \mathbb{T}^d$, $\mathbf{v} \in \mathbb{R}^d$, $d \in \{2, 3\}$;
- Admissible initial data :
  - $0 \leq f_0 \in \mathrm{L}^\infty \cap \mathrm{L}^1(\mathbb{T}^d \times \mathbb{R}^d)$;
  - $\displaystyle\int_{\mathbb{T}^d \times \mathbb{R}^d} |\mathbf{v}|^2 f_0 < +\infty$;
  - $\mathbf{u}_0 \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{T}^d)$.

Global weak solutions

Théorème (Boudin, Desvillettes, Grandmont, M. – 2009)

For $d = 2, 3$ and admissible initial data, there exists a global weak solution $(\mathbf{u}, f)$ to the VNS system that satisfies for almost all $0 \leq s \leq t$ the energy-dissipation inequality

$$\mathsf{E}(t) + \int_s^t \mathsf{D}(\tau)\,\mathrm{d}\tau \leq \mathsf{E}(s). \qquad (\star)$$

- Leray solution for $\mathbf{u}$, renormalized for f ;
- More realistic framework [Boudin, Grandmont, M. – 2017] ;
- Here : large time behavior for solutions satisfying $(\star)$.

Outline

What to expect ?

- Consider the **uncoupled** Vlasov equation on $\mathbb{T}^d \times \mathbb{R}^d$

$$\partial_t g + \mathbf{v} \cdot \nabla_\mathbf{x} g - \nabla_\mathbf{v} \cdot ((\mathbf{U} - \mathbf{v}) g) = 0.$$

- If $\mathbf{U} = 0$ (linearizing around $(\mathbf{u} = 0, f = 0)$), **concentration** in velocity :

$$g(t, \mathbf{x}, \mathbf{v}) \underset{t \to +\infty}{\rightharpoonup} \left(\int_{\mathbb{R}^d} g_0(\mathbf{x} - \mathbf{v}, \mathbf{v}) \, \mathrm{d}\mathbf{v} \right) \otimes \delta_0.$$

- Similar for $t \mapsto \mathbf{U}(t) \in \mathbb{R}^d$, concentration around $\mathbf{v} = \mathbf{U}(t)$

$\rightarrow$ $\nexists\, g \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{T}^d \times \mathbb{R}^d) \setminus \{0\}$ stationary solution

What to expect ?

- Navier-Stokes on $\mathbb{T}^d$: $\|\mathbf{u}(t) - \langle \mathbf{u}(t) \rangle\|_2 \to 0$

$\to$ **Concentration** is expected for the full VNS system

- Attempts
  - [Jabin – 2000] (Vlasov-Stokes)
  - [Choi, Kwon – 2015] (if-theorem)

How to quantify the concentration ...

... without knowing the limiting profile ?

Conservative equation :

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (\mathbf{A} f) = 0.$$

Define

$$\mathbf{j}_f(t,x) := \int_{\mathbb{R}^d} f(t,\mathbf{x},\mathbf{v})\mathbf{v}\,\mathrm{d}\mathbf{v}, \qquad \rho_f(t,\mathbf{x}) := \int_{\mathbb{R}^d} f(t,\mathbf{x},\mathbf{v})\,\mathrm{d}\mathbf{v},$$

If $f \approx \rho_f \otimes \delta_{\mathbf{U}}$, then $\mathbf{j}_f(t,x) \approx \rho_f(t,\mathbf{x})\mathbf{U}$

$\longrightarrow$ Up to normalization, $\langle\, \mathbf{j}_f(t)\,\rangle \approx \mathbf{U}$.

The modulated energy of Choi and Kwon

Recall :

$$\mathsf{E}(t) = \frac{1}{2}\int_{\mathbb{T}^d} |\mathbf{u}(t)|^2 + \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t)|\mathbf{v}|^2.$$

Choi and Kwon introduced

$$\mathscr{E}(t) = \frac{1}{2}\int_{\mathbb{T}^d} |\mathbf{u}(t) - \langle \mathbf{u}(t)\rangle|^2 + \frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f(t)|\mathbf{v} - \langle\, \mathbf{j}_f(t)\,\rangle|^2 + |\langle\, \mathbf{j}_f(t)\,\rangle - \langle \mathbf{u}(t)\rangle|^2.$$

The modulated energy of Choi and Kwon

Transport distance :

$$\mathsf{W}_1(\mu,\nu) = \inf_{\gamma\in\mu\bullet\nu} \int_{\Omega\times\Omega} |\omega_1-\omega_2| \,\mathrm{d}\gamma(\omega_1,\omega_2).$$

W_1 metrizes vague convergence of measures.

The modulated energy of Choi and Kwon

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$\mathscr{E}$ captures concentration because of

$$\mathsf{W}_1\Big(f(t), \rho_f(t)\otimes\delta_{\mathbf{v}=\langle\, \mathbf{j}_f(t)\,\rangle}\Big) \lesssim \mathscr{E}(t)^{1/2}.$$

$\longrightarrow$ $\mathscr{E}(t) \underset{t\to+\infty}{\longrightarrow} 0 \Longrightarrow$ concentration.

If-theorem ?

The modulated energy satisfies

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathscr{E}(t) + \mathrm{D}(t) = 0.$$

Lemme (Choi, Kwon (2015))

If $(\mathbf{u}, f)$ *is a solution of VNS such that*

$$\sup_{t\geq 0} \|\rho_f(t)\|_{3/2} < \infty,$$

then $\mathscr{E}(t) \lesssim \mathrm{D}(t)$ *and thus (exponentially)*

$$\mathscr{E}(t) \underset{t\to+\infty}{\longrightarrow} 0.$$

The uniform bound

The estimation $\rho_f \in \mathsf{L}^\infty(\mathbb{R}_+; \mathsf{L}^{3/2}(\mathbb{T}^d))$ is **structurally** hard to achieve.

Energy estimate + interpolation gives

$$\rho_f \in \mathsf{L}^\infty_{\mathrm{loc}}(\mathbb{R}_+; \mathsf{L}^p(\mathbb{T}^d)),$$

for some $p < \infty$.

$\longrightarrow$ *A priori* estimates are not sufficient here !

Main result

Theorem (Han-Kwan, M., Moyano – 2020)

Consider an admissible initial data $(\mathbf{u}_0, f_0)$ *such that*

- « $f_0 \searrow 0$ *in* $\mathbf{v}$ *uniformly in* $\mathbf{x}$ » ;
- $\mathscr{E}(0) \ll 1$;
- *(d=3)* $\|\mathbf{u}_0\|_{\dot{\mathrm{H}}^{1/2}(\mathbb{T}^3)} \ll 1$.

Any weak solution of the VNS system initiated with $(\mathbf{u}_0, f_0)$ *satisfies* $\mathscr{E}(t) \to 0$, *with exponential rate.*

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- maintain regularity of solutions

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- ~~maintain regularity of solutions~~ consider a smooth solution ;

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- ~~maintain regularity of solutions~~ consider a smooth solution ;
- produce a strong uniform (in time) bound :

$$\sup_{t \geq 0} \|\rho_f(t)\|_\infty < \infty.$$

Characteristics

For $(s, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R}^d$, define $t \mapsto (\mathbf{X}_s^t, \mathbf{V}_s^t)(\mathbf{x}, \mathbf{v})$, value at time t of

$$\begin{cases} \dot{\mathbf{X}} = \mathbf{V}, \\ \dot{\mathbf{V}} = \mathbf{u}(t, \mathbf{X}) - \mathbf{V}, \end{cases}$$

with $(\mathbf{X}_s^s, \mathbf{V}_s^s)(\mathbf{x}, \mathbf{v}) = (\mathbf{x}, \mathbf{v})$.

The solution of

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0,$$

is then given by

$$f(t, \mathbf{x}, \mathbf{v}) = e^{dt} f_0(\mathbf{X}_t^0(\mathbf{x}, \mathbf{v}), \mathbf{V}_t^0(\mathbf{x}, \mathbf{v})).$$

Tracking the uniform bound

In particular

$$\rho_f(t,\mathbf{x}) = \int_{\mathbb{R}^d} e^{dt} f_0\big(\mathbf{X}_t^0(\mathbf{x},\mathbf{v}),\mathbf{V}_t^0(\mathbf{x},\mathbf{v})\big)\,\mathrm{d}\mathbf{v}.$$

[Bardos, Degond – 1985] (for Vlasov-Poisson) :

- if $\Gamma_{t,\mathbf{x}} : \mathbf{v} \mapsto \mathbf{V}_t^0(\mathbf{x},\mathbf{v})$ is $\mathscr{C}^1$-diffeomorphism
- with $|\det \mathrm{D}_{\mathbf{v}}\Gamma_{t,\mathbf{x}}| \gtrsim e^{dt}$, then $\mathbf{w} = \Gamma_{t,\mathbf{x}}(\mathbf{v})$

$$|\rho_f(t,\mathbf{x})| \lesssim \int_{\mathbb{R}^d} \|f_0(\cdot,\mathbf{w})\|_\infty \,\mathrm{d}\mathbf{w} < +\infty.$$

The straightening change of variable

$\Gamma_{t,\mathbf{x}}$ satisfies the implicit equation

$$\Gamma_{t,\mathbf{x}}(\mathbf{v}) = \mathbf{V}_t^0(\mathbf{x},\mathbf{v}) = e^t\mathbf{v} - \int_0^t e^s \mathbf{u}(s, \mathbf{X}_t^s(\mathbf{x},\mathbf{v}))\,\mathrm{d}s,$$

which relates $\mathbf{v} \mapsto e^{-t}\Gamma_{t,\mathbf{x}}(\mathbf{v})$ to the identity

Lemme

If $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is such that $\|\nabla\Phi\|_\infty < 1$, then

- $\Theta := \mathrm{Id}_{\mathbb{R}^d} + \Phi$ *is a $\mathscr{C}^1$-diffeomorphism of $\mathbb{R}^d$;*
- $\inf_{\mathbb{R}^d} |\det \nabla\Theta| > 0$.

Another estimate to track

Corollaire

There exists $\mathsf{c}_d > 0$ *such that, if*

$$\int_0^t \|\nabla \mathbf{u}(s)\|_\infty \, \mathrm{d}s < \mathsf{c}_d,$$

then $\Gamma_{t,\mathbf{x}}$ *is a* $\mathscr{C}^1$*-diffeomorphism with* $|\det \mathrm{D}_{\mathbf{v}}\Gamma_{t,\mathbf{x}}| \gtrsim e^{dt}$.

Again $\nabla \mathbf{u} \in \mathrm{L}^1(\mathbb{R}_+; \mathrm{L}^\infty(\mathbb{T}^d))$ not encoded in the energy estimate.

$$\text{Defining } t^\star = \sup\left\{t > 0, \int_0^t \|\nabla \mathbf{u}(s)\|_\infty \, \mathrm{d}s < \mathsf{c}_d\right\},$$

the goal is now to prove $t^\star := +\infty$.

Maximal regularity

Maximal regularity for the heat flow :

$$\mathbf{w}(0,\cdot) = 0 \Rightarrow \|\Delta \mathbf{w}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathbb{T}^d))} \lesssim \|\partial_t \mathbf{w} - \Delta \mathbf{w}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathbb{T}^d))}.$$

Here $\mathbf{u}$ solution of

$$\partial_t \mathbf{u} - \Delta \mathbf{u} = \mathbb{P}\big[\mathbf{j}_f - \rho_f \mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u}\big] := \mathbf{F},$$

decomposed as $\mathbf{u} = e^{t\Delta}\mathbf{u}_0 + \mathbf{w}$, where $\mathbf{w}(0,\cdot) = 0$ and

$$\partial_t \mathbf{w} - \Delta \mathbf{w} = \mathbf{F},$$

$$\Longrightarrow \quad \|\mathrm{D}^2\mathbf{u}\|_{\mathrm{L}^p(\varepsilon,T;\mathrm{L}^q(\mathbb{T}^d))} \lesssim \|\mathbf{F}\|_{\mathrm{L}^p(0,T;\mathrm{L}^q(\mathbb{T}^d))} + \|\mathbf{u}_0\|_2.$$

The bootstrap argument

Combine the previous with

- estimates that are valid on $[0, t^\star)$ + Choi-Kwon Lemma ;
- Gagliardo-Nirenberg-Sobolev estimate

$$\|\nabla \mathbf{u}(s)\|_\infty \lesssim \|D^2\mathbf{u}(s)\|_q^\alpha \|\mathbf{u}(s) - \langle \mathbf{u}(s)\rangle\|_2^{1-\alpha};$$

- $\|\mathbf{u}(s) - \langle \mathbf{u}(s)\rangle\|_2 \lesssim \mathscr{E}(s)^{1/2}$;

$$\begin{aligned}\int_0^{t^\star} \|\nabla \mathbf{u}(s)\|_\infty \,\mathrm{d}s &\lesssim \mathscr{E}(0)^\gamma \Big(1 + \|\mathbf{F}\|_{L^p(0,t^\star;L^q(\mathbb{T}^d))}\Big) \\ &\lesssim \mathscr{E}(0)^\gamma \\ &\Rightarrow t^\star = +\infty.\end{aligned}$$

The regularity issue

- Computations on Vlasov : renormalized solutions
- Computations on NS and instantaneous regularization
  - $d = 2$: true for all positive times (! of Leray solution)
  - $d = 3$: true for $t \geq 0$ as long as

$$\|\mathbf{u}_0\|^2_{\mathrm{H}^{1/2}(\mathbb{T}^3)} + \mathrm{C}\int_0^t \|(\mathbf{j}_f - \rho_f \mathbf{u})(s)\|^2_{\mathrm{H}^{-1/2}(\mathbb{T}^3)}\,\mathrm{d}s \leq \frac{1}{\mathrm{C}^2}.$$

- Computations on the system (energy identities)
  - $d = 2$: true for all positive times ([HKM^3 – 2017])
  - $d = 3$: part of the assumption

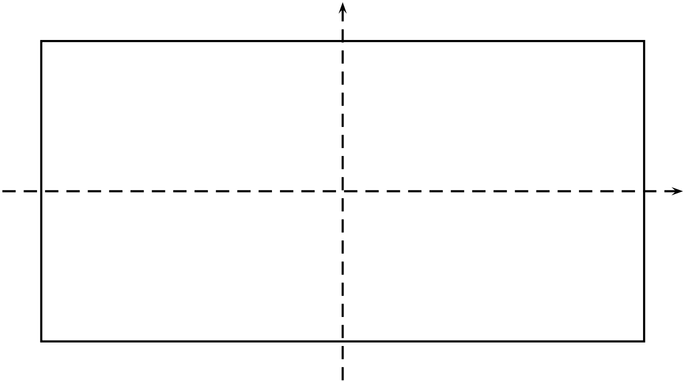
Outline

Avoiding the concentration scenario ?

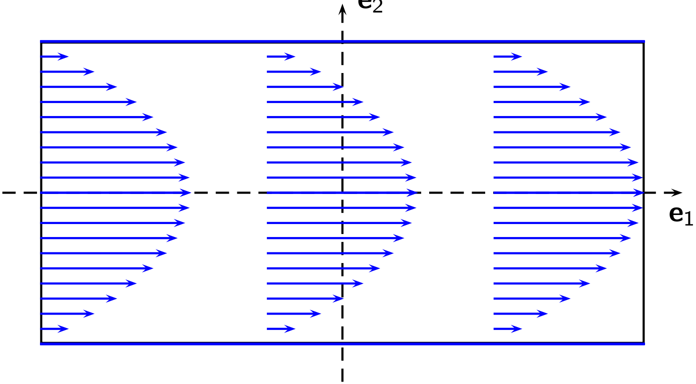
- $\mathbb{R}^d$? Damping to 0 for $\mathbf{u}$ ([Han-Kwan '21])
- Common feature : absence of boundary
  - Eternity of influence of $\mathbf{u}$ on f
  - Escape of particles ?
- $\neq$ existence issues, geometry is important

$\longrightarrow$ An elementary example can reveal this phenomenon.

The domain : a 2D box

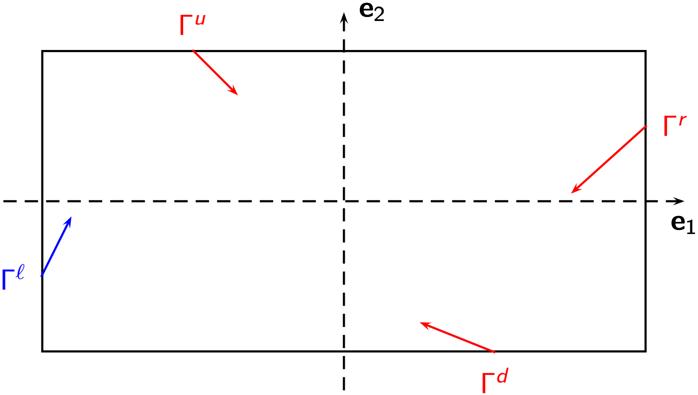


The Poiseuille flow



$\mathbf{u}_{\text{pois}}(t, \mathbf{x}) := (1 - \langle \mathbf{x}, \mathbf{e}_2 \rangle^2)\, \mathbf{e}_1$ is a stationnary solution of NS

Entering phase space boundary



$$\Gamma^\ell \cup \Gamma^u \cup \Gamma^r \cup \Gamma^d = \Big\{(\mathbf{x},\mathbf{v}) \in \partial\Omega \times \mathbb{R}^2 \,:\, \mathbf{v}\cdot\mathbf{n}(\mathbf{x}) < 0\Big\}$$

The whole system

For $(t, \mathbf{x}, \mathbf{v}) \in \mathbb{R}_+ \times \Omega \times \mathbb{R}^2$ we consider

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla p = -\displaystyle\int_{\mathbf{v}} (\mathbf{u} - \mathbf{v}) \ f \ \mathrm{d}\mathbf{v}, \\ \mathrm{div} \ \mathbf{u} = 0, \\ \partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot (f(\mathbf{u} - \mathbf{v})) = 0, \end{cases}$$

with initial conditions $(\mathbf{u}_0, f_0)$ and boundary conditions :

$$\mathbf{u} = \mathbf{u}_{\mathrm{pois}} \text{ on } \mathbb{R}_+ \times \partial\Omega,$$
$$f = \psi \text{ on } \Gamma^{\ell},$$
$$f = 0 \text{ on } \Gamma^u \cup \Gamma^d \cup \Gamma^r.$$

The exit geometric condition

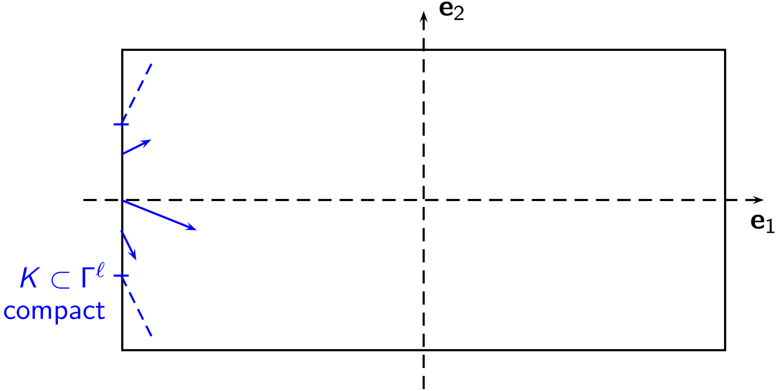
- $T > 0$;
- $\mathbf{u}$ a vector field ;
- K a compact set of the phase space $(\mathbf{x}, \mathbf{v})$;

Definition

$(\mathbf{u}, K)$ satisfies the EGC in time T if any trajectory starting from K has a lifetime $< T$ inside Ω (with transversal exit).

$\longrightarrow$ Reminiscent of [Bardos, Lebeau, Rauch – 1992]

The exit geometric condition (EGC)



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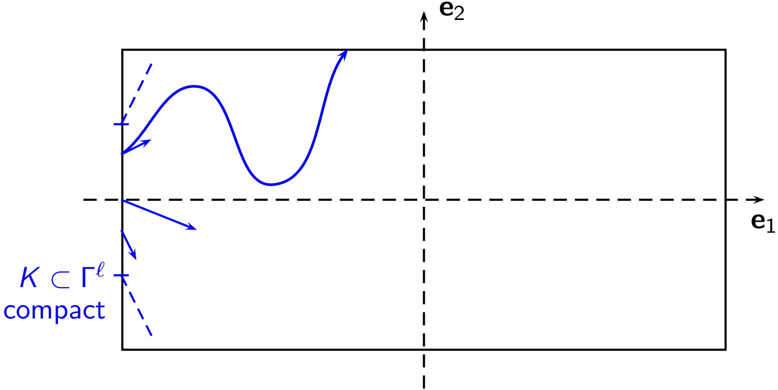

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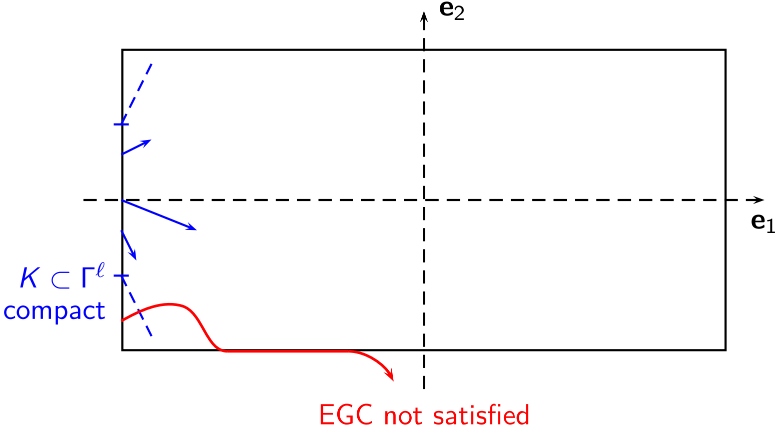

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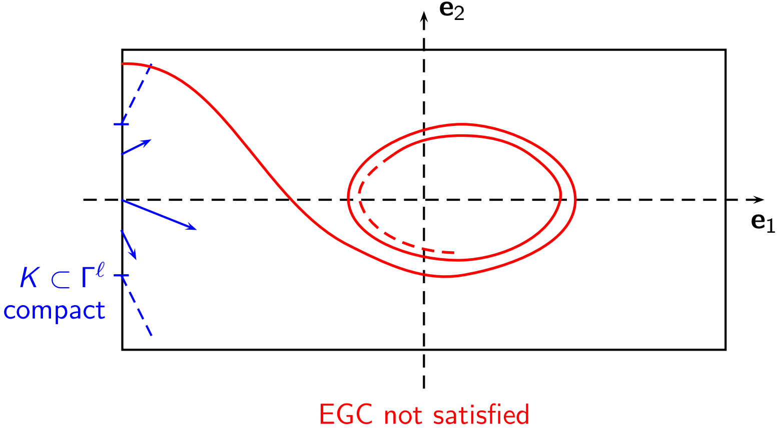

Stability of the EGC

- given $\mathbf{u}$, not obvious to exhibit a compatible K
- $\mathbf{u}_{\text{pois}}$: EGC $\Leftrightarrow |x_2 + v_2| \neq 1$ for $(\mathbf{x}, \mathbf{v}) \in K$

Lemma

If $(\mathbf{u}^\star, K)$ satisfies the EGC in time T, then for $T^\star > T$ there exists $\delta > 0$ such that

$$\sup_{t \geq 0} \int_t^{t+T^\star} \|\mathbf{u}(\tau) - \mathbf{u}^\star\|_{\mathrm{L}^\infty(\Omega)} \mathrm{d}\tau < \delta$$

$$\Rightarrow (\mathbf{u}, K) \text{ satisfies EGC in time } T^\star.$$

Main result

Theorem (Glass, Han-Kwan, M. – 2018)

There exists non trivial smooth stationnary solutions of VNS which are furthermore asymptotically stable (for adequate perturbations).

- example of non trivial non singular equilibrium ;
- stationary solutions built near $(\mathbf{u}_{\text{pois}}, 0)$;
- admissible perturbation : EGC remains satisfied.

Thank you !