Weak solutions for a version of compressible Oldroyd-B model without stress diffusion

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joint work with Y. Lu (Nanjing) and based on a joint paper with A. Novotný (Toulon)

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Lecture

- Introduction
  - Compressible Oldroyd-B model
  - Reformulation of the problem
  - Weak solution
- Multifluid system
  - Existence result, main assumptions
  - Construction of a solution
- Back to the reformulated compressible Oldroyd-B model
  - Proof of the existence of a solution

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Compressible Oldroyd-B model I

The following model was derived in

J. W. Barrett, Y. Lu, E. Süli: Existence of large-data finite-energy global weak solutions to a compressible Oldroyd–B model. Comm. Math. Sci. 15 (2017) 1265–1323.

$$
\begin{aligned}
\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) &= 0,\\
\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) - \mathrm{div}_x\, \mathbb{S}(\nabla_x \mathbf{u}) &= \mathrm{div}_x \left(\mathbb{T} - (kL\eta + \zeta\, \eta^2)\, \mathbb{I} \right) + \varrho\, \mathbf{f},\\
\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u}) &= \varepsilon \Delta_x \eta,\\
\partial_t \mathbb{T} + \mathrm{Div}_x(\mathbf{u}\, \mathbb{T}) - \left(\nabla_x \mathbf{u}\, \mathbb{T} + \mathbb{T}\, \nabla_x^{\mathrm{T}} \mathbf{u} \right) &= \varepsilon \Delta_x \mathbb{T} + \frac{k}{2\lambda} \eta\, \mathbb{I} - \frac{1}{2\lambda} \mathbb{T}.
\end{aligned}
\tag{1}
$$

Above,

$$
\mathbb{S}(\nabla_x \mathbf{u}) = \mu^S \left(\frac{\nabla_x \mathbf{u} + \nabla_x^{\mathrm{T}} \mathbf{u}}{2} - \frac{1}{3} (\mathrm{div}_x\, \mathbf{u}) \mathbb{I} \right) + \mu^B (\mathrm{div}_x\, \mathbf{u}) \mathbb{I},
$$

is the Newtonian stress tensor with $\mu^S > 0$ and $\mu^B \geq 0$, $\mathbb{T}$ is the extra stress tensor, $p(\varrho) = a\varrho^\gamma$ is the fluid pressure, $kL\eta + \zeta\eta^2$ can be interpreted as polymer pressure, ε, k, λ, ζ and L are positive numbers.

Compressible Oldroyd-B model II

The authors showed existence of global in time solutions for this model, under a fundamental assumption that $\varepsilon > 0$. A similar result was derived for a slightly different model in

M. Bulíček, E. Feireisl, J. Málek: On a class of compressible viscoelastic rate-type fluids with stress-diffusion. Nonlinearity 32 (2019) 4665–4681.

They also considered so-called stress diffusion which corresponds to $\varepsilon > 0$.

In reality, the stress diffusion is by several orders lower than other physical constants and therefore is often in modeling neglected. We shall try to address this problem, at least in some specific situations.
We shall introduce several simplification/variants of the original model in order to be able to study the problem from the point of view of existence of global in time solution for arbitrarily large data.

Compressible Oldroyd-B model III

The derivative

$$\partial_t \mathbb{T} + \mathrm{Div}_x(\mathbf{u}\,\mathbb{T}) - \left(\nabla_x \mathbf{u}\,\mathbb{T} + \mathbb{T}\,\nabla_x^{\mathrm{T}} \mathbf{u}\right)$$

is called the upper convected derivative and it is an example of a frame invariant derivative of the tensor field. Inspired by the famous result for the incompressible Oldroyd-B model

P. L. Lions and N. Masmoudi: Global solutions for some Oldroyd models of non-Newtonian flows. Chin. Ann. Math., Ser. B 21(2) (2000), 131–146.

we replace it by the corotational derivative

$$\partial_t \mathbb{T} + \mathrm{Div}_x(\mathbf{u}\,\mathbb{T}) - (\omega(\nabla \mathbf{u})\,\mathbb{T} - \mathbb{T}\,\omega(\nabla \mathbf{u}))\,,$$

where $\omega(\nabla \mathbf{u}) = \left(\nabla_x \mathbf{u} - \nabla_x^{\mathrm{T}} \mathbf{u}\right)/2$ is the vorticity tensor. Next, similarly as in the paper by Bulíček, Feireisl, Málek we assume that the extra stress tensor has a simpler form

$$\mathbb{T} = \tau \mathbb{I}$$

with τ a scalar function. Assuming $\varepsilon = 0$ we get instead of $(1)_4$

$$\partial_t \tau + \mathrm{div}_x(\tau \mathbf{u}) = \frac{k}{2\lambda}\eta - \frac{1}{2\lambda}\tau.$$

By introducing $\tilde{\tau} = \tau - k\eta$, we deduce from $(1)_3$ and the equation above

$$\partial_t \tilde{\tau} + \mathrm{div}_x(\tilde{\tau} \mathbf{u}) = -\frac{1}{2\lambda}\tilde{\tau}.$$

Compressible Oldroyd-B model IV

By neglecting the tilde we end up with the following system of equations

$$
\begin{aligned}
\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) &= 0,\\
\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x\big(p(\varrho) + q(\eta) - \tau\big) - \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) &= \varrho\, \mathbf{f},\\
\partial_t \eta + \operatorname{div}_x(\eta \mathbf{u}) &= 0,\\
\partial_t \tau + \operatorname{div}_x(\tau \mathbf{u}) &= -\frac{1}{2\lambda}\tau.
\end{aligned}
\tag{2}
$$

Above, $p(\varrho) = a\varrho^\gamma$ (as above) while $q(\eta) = k(L-1)\eta + \zeta\eta^2$. We consider system (2) in $Q_T = (0,T) \times \Omega$ with $\Omega \subset \mathbb{R}^3$ and consider the boundary condition

$$
\mathbf{u} = \mathbf{0} \qquad \text{on } (0,T) \times \partial\Omega. \tag{3}
$$

Finally we prescribe the initial conditions for ϱ, $\varrho\mathbf{u}$, η and τ in Ω.

The resulted problem is very similar to a multifluid system studied recently by

A. Novotný, M. Pokorný: Weak solutions for some compressible multicomponent fluid models. Arch. Rational Mech. Anal. 235 (2020) 355–403.

Compressible Oldroyd-B model V

The main differences to the problems studied before are

- The "pressure"contains a negative contribution from τ
- The transport equation for τ has nontrivial right-hand side

These problems can be overcome, as will be shown later.
There are further results dealing with the same problem (in the case of two fluids)

A. Vasseur, H. Wen, C. Yu: Global weak solution to the viscous two-fluid model with finite energy, J. Math. Pures Appl. 125 (2019) 247–282.

D. Bresch, P.B. Mucha, E. Zatorska: Finite-Energy Solutions for Compressible Two-Fluid Stokes System, Arch. Ration. Mech. Anal. 232 (2019) 987–1029.

H. Wen: Global existence of weak solution to compressible two-fluid model without any domination condition in three dimensions, arXiv: 1902.05190.

Weak solution I

We assume

$$\begin{aligned} &\varrho_0 \geq 0 \text{ a.e. in } \Omega, \quad \varrho_0 \in L^\gamma(\Omega), \\ &\varrho_0 \mathbf{u}_0 \in L^1(\Omega; R^3), \quad \varrho_0 |\mathbf{u}_0|^2 \in L^1(\Omega), \\ &\eta_0 \geq 0 \text{ a.e. in } \Omega, \quad \eta_0 \in L^2(\Omega), \\ &\tau_0 \geq 0 \text{ a.e. in } \Omega, \quad \tau_0 \log \tau_0 \in L^1(\Omega). \end{aligned} \tag{4}$$

Definition

Let $T > 0$ and $\Omega \subset \mathbb{R}^3$ be a bounded $C^{2,\beta}$ domain with $0 < \beta \leq 1$. We say that $(\varrho, \mathbf{u}, \eta, \tau)$ is a finite-energy weak solution in Q_T to the system of equations (2)–(3), supplemented by the initial data (4), if:

- $\varrho \geq 0$ a.e. in $(0,T) \times \Omega$, $\varrho \in C_w([0,T]; L^\gamma(\Omega))$,
  $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega; R^3))$,
  $$\begin{aligned} &\varrho \mathbf{u} \in L^\infty([0,T]; L^1(\Omega; R^3)), \quad \varrho |\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)), \\ &\eta \geq 0 \text{ a.e. in } (0,T) \times \Omega, \quad \eta \in C_w([0,T]; L^2(\Omega)), \\ &\tau \geq 0 \text{ a.e. in } (0,T) \times \Omega, \quad \tau \log \tau \in C_w([0,T]; L^1(\Omega)). \end{aligned}$$

Weak solution II

- For any $t \in (0,T)$ and any test function $\phi \in C^\infty([0,T] \times \overline{\Omega})$, one has

$$\int_0^t \int_\Omega \left[\varrho \partial_t \phi + \varrho \mathbf{u} \cdot \nabla \phi\right] \mathrm{d}x \, \mathrm{d}t' = \int_\Omega \varrho(t,\cdot)\phi(t,\cdot) \, \mathrm{d}x - \int_\Omega \varrho_0 \phi(0,\cdot) \, \mathrm{d}x, \quad (5)$$

$$\int_0^t \int_\Omega \left[\eta \partial_t \phi + \eta \mathbf{u} \cdot \nabla \phi\right] \mathrm{d}x \, \mathrm{d}t' = \int_\Omega \eta(t,\cdot)\phi(t,\cdot) \, \mathrm{d}x - \int_\Omega \eta_0 \phi(0,\cdot) \, \mathrm{d}x, \quad (6)$$

$$\int_0^t \int_\Omega \left[\tau \partial_t \phi + \tau \mathbf{u} \cdot \nabla \phi - \frac{1}{2\lambda} \tau \phi\right] \mathrm{d}x \, \mathrm{d}t' = \int_\Omega \eta(t,\cdot)\phi(t,\cdot) \, \mathrm{d}x - \int_\Omega \eta_0 \phi(0,\cdot) \, \mathrm{d}x. \quad (7)$$

Weak solution III

- For any $t \in (0,T)$ and any test function $\varphi \in C^\infty([0,T]; C_c^\infty(\Omega; R^3))$, one has

$$\int_0^t \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \varphi + (\varrho \mathbf{u} \otimes \mathbf{u}) : \nabla \varphi + \big(p(\varrho) + q(\eta) - \tau\big) \operatorname{div}_x \varphi - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla \varphi\right] \mathrm{d}x\, \mathrm{d}t'$$
$$= - \int_0^t \int_\Omega \varrho\, \mathbf{f} \cdot \varphi \, \mathrm{d}x\, \mathrm{d}t' + \int_\Omega \varrho \mathbf{u}(t,\cdot) \cdot \varphi(t,\cdot)\, \mathrm{d}x - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \varphi(0,\cdot)\, \mathrm{d}x. \tag{8}$$

- For a.e. $t \in (0,T)$, the following *energy inequality* holds

$$\int_\Omega \left[\frac{1}{2}\varrho |\mathbf{u}|^2 + H(\varrho,\eta,\tau)\right](t,\cdot)\, \mathrm{d}x + \int_0^t \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\, \mathrm{d}x\, \mathrm{d}t'$$
$$\leq \int_\Omega \left[\frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + H(\varrho_0,\eta_0,\tau_0)\right] \mathrm{d}x + \int_0^t \int_\Omega \varrho\, \mathbf{f} \cdot \mathbf{u}\, \mathrm{d}x\, \mathrm{d}t' \tag{9}$$
$$+ \frac{1}{2\lambda} \int_0^t \int_\Omega (\tau \log \tau + \tau)\, \mathrm{d}x\, \mathrm{d}t',$$

where the Helmholtz free energy is defined as

$$H(\varrho,\eta,\tau) = P(\varrho) + Q(\eta) - \tau \log \tau \tag{10}$$

with

$$Q(\eta) = \delta\, \eta^2 + k(L-1)\eta \log \eta, \quad P(\varrho) = \begin{cases} \dfrac{a}{\gamma - 1}\varrho^\gamma, & \text{if } \gamma \neq 1, \\ a\varrho \log \varrho, & \text{if } \gamma = 1. \end{cases} \tag{11}$$

Main result

Theorem

Let $\Omega \subset \mathbb{R}^3$ be a bounded $C^{2,\beta}$ domain with $\beta \in (0,1]$. Let $0 < \gamma \leq 2$, the constant parameters λ, δ be positive, and k, L be non-negative. We further assume that the initial data satisfies the domination relation:

$$\varrho_0 \leq \overline{C}\eta_0, \quad \tau_0 \leq \overline{C}\eta_0 \quad \text{a.e. in } \Omega \text{ for some } \overline{C} > 0. \tag{12}$$

Then, for any $T > 0$, there exists a finite-energy weak solution $(\varrho, \mathbf{u}, \eta, \tau)$ in the sense of Definition above with initial data (4) by replacing the integrability on ϱ and τ by

$$\varrho \in C_w([0, T]; L^2(\Omega)), \quad \tau \in C_w([0, T]; L^2(\Omega)).$$

Moreover, the domination condition preserves for all times:

$$\varrho(t,x) \leq \overline{C}\eta(t,x), \quad \tau(t,x) \leq \overline{C}\eta(t,x) \quad \text{for a.a. } (t,x) \in Q_T. \tag{13}$$

Multifluid system I

Recall that in the paper by Novotný and Pokorný (ARMA) we studied the following problem

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= 0, \\ \partial_t Z_i + \operatorname{div}(Z_i \mathbf{u}) &= 0, \quad i = 1, 2, \dots, K, \\ \partial_t \Big((\varrho + \sum_{i=1}^K Z_i) \mathbf{u} \Big) + \operatorname{div} \Big((\varrho + \sum_{i=1}^K Z_i) \mathbf{u} \otimes \mathbf{u} \Big) + \nabla P(\varrho, Z_0, Z_1, \dots, Z_K) & \\ = \mu \Delta \mathbf{u} + (\mu + \lambda) \nabla \operatorname{div} \mathbf{u}, & \end{aligned} \tag{14}$$

together with the boundary condition $\mathbf{u} = \mathbf{0}$ on $(0, T) \times \partial\Omega$, and the initial conditions in Ω

$$\begin{aligned} \varrho(0, x) &= \varrho_0(x), \\ Z_i(0, x) &= Z_{i0}(x), \quad i = 1, 2, \dots, K, \\ \Big(\varrho + \sum_{i=1}^K Z_i \Big) \mathbf{u}(0, x) &= \mathbf{m}_0(x). \end{aligned} \tag{15}$$

Multifluid system II

We assumed the following

Hypothesis (H1).

$$(\varrho_0, Z_{10}, Z_{20}, \dots, Z_{K0}) \in \mathcal{O}_{\vec{\underline{a}}} := \big\{(\varrho, Z_1, Z_2, \dots, Z_K) \in \mathbb{R}^{K+1} | \varrho \in [0, \infty), \\ \underline{a}_i \varrho < Z_i < \overline{a}_i \varrho \big\}, \tag{16}$$

where $0 \leq \underline{a}_i < \overline{a}_i < \infty$, $i = 1, 2, \dots, K$.

Hypothesis (H2).

$$\varrho_0 \in L^{\gamma}(\Omega), \; Z_{i0} \in L^{\beta_i}(\Omega) \;\; \text{if } \beta_i > \gamma, \\ \mathbf{m}_0 \in L^1(\Omega; \mathbb{R}^3), \; (\varrho_0 + \sum_{i=1}^{K} Z_{i0})|\mathbf{u}_0|^2 \in L^1(\Omega), \; i = 1, 2, \dots, K. \tag{17}$$

Multifluid system III

Hypothesis (H3). Function $P \in C(\overline{\mathcal{O}_{\vec{a}}}) \cap C^1(\mathcal{O}_{\vec{a}})$ and

$$\forall \varrho \in (0,1), \quad \sup_{s \in \Pi_{i=1}^K [\underline{a}_i, \overline{a}_i]} |P(\varrho, \varrho s_1, \varrho s_2, \ldots, \varrho s_K)| \leq C \varrho^\alpha \text{ with some } C > 0 \text{ and } \alpha > 0, \tag{18}$$

and

$$\underline{C}(\varrho^\gamma + \sum_{i=1}^K Z_i^{\beta_i} - 1) \leq P(\varrho, Z_1, \ldots, Z_K) \leq \overline{C}(\varrho^\gamma + \sum_{i=1}^K Z_i^{\beta_i} + 1) \text{ in } \mathcal{O}_{\vec{a}} \tag{19}$$

with $\gamma \geq \frac{9}{5}$, $\beta_i > 0$, $i = 1, 2, \ldots, K$. We moreover assume for $i = 1, 2, \ldots, K$

$$|\partial_{Z_i} P(\varrho, Z_1, Z_2, \ldots, Z_K)| \leq C(\varrho^{-\underline{\Gamma}} + \varrho^{\overline{\Gamma}-1}) \text{ in } \mathcal{O}_{\vec{a}} \tag{20}$$

with some $0 \leq \underline{\Gamma} < 1$, and with some $0 < \overline{\Gamma} < \gamma + \gamma_{BOG}$ if $\underline{a}_i = 0$,
$0 < \overline{\Gamma} < \max\{\gamma + \gamma_{BOG}, \beta_i + (\beta_i)_{BOG}\}$ if $\underline{a}_i > 0$.

Multifluid system IV

Hypothesis (H4). We assume

$$P(\varrho, \varrho s_1, \varrho s_2 \dots, \varrho s_K) = \mathcal{P}(\varrho, s_1, s_2, \dots, s_K) - \mathcal{R}(\varrho, s_1, s_2, \dots, s_K), \tag{21}$$

where $[0,\infty) \ni \varrho \mapsto \mathcal{P}(\varrho, s_1, s_2, \dots, s_K)$ is non decreasing for any $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2, \dots, K$, and $\varrho \mapsto \mathcal{R}(\varrho, s_1, s_2, \dots, s_K)$ is for any $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2, \dots, K$ a non-negative C^2-function in $[0, \infty)$ with uniformly bounded C^2-norm with respect to $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2, \dots, K$ and with compact support uniform with respect to $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2, \dots, K$. Here, $\underline{a}_i, \overline{a}_i$ are the constants from relation (16).

Hypothesis (H5). Functions $\varrho \mapsto P(\varrho, Z_1, Z_2, \dots, Z_K)$, $Z_i > 0$, $i = 1, 2, \dots, K$ resp. $(Z_1, Z_2, \dots, Z_K) \mapsto \partial_{Z_j} P(\varrho, Z_1, Z_2, \dots, Z_K)$, $\varrho > 0$, are Lipschitz on $\cap_{i=1}^K (Z_i/\overline{a}_i, Z_i/\underline{a}_i) \cap (\underline{r}, \infty)^K$ resp. $\Pi_{i=1}^K (\underline{a}_i \varrho, \overline{a}_i \varrho) \cap (\underline{r}, \infty)^K$ for all $\underline{r} > 0$ with Lipschitz constants

$$\widetilde{L}_P \le C(\underline{r})(1 + |Z|^A) \ \text{ resp. } \widetilde{L}_P \le C(\underline{r})(1 + \varrho^A) \tag{22}$$

with some non negative number A. Number $C(\underline{r})$ may diverge to $+\infty$ as $\underline{r} \to 0^+$.

Multifluid system V

In the paper with A. Novotný we proved:

Theorem
Let $\gamma > \frac{9}{5}$. Then under Hypotheses (H1–H5)*, there exists at least one weak solution to problem* (14)–(15)*. Moreover, the densities*
$\varrho \in C_{weak}([0,T); L^{\gamma}(\Omega))$, $Z_i \in C_{weak}([0,T); L^{\max\{\gamma,\beta_i\}}(\Omega))$, $i = 1, 2, \dots, K$,
$(\varrho + \sum_{i=1}^{K} Z_i)\mathbf{u} \in C_{weak}([0,T); L^q(\Omega;\mathbb{R}^3))$ *for some* $q > 1$, *and*
$P(\varrho, Z_1, Z_2, \dots, Z_K) \in L^q(\Omega)$ *for some* $q > 1$.

Note that also the case $\gamma = \frac{9}{5}$ was covered, but we do not need the case here. Recall also that the main purpose of the paper was to study a more complex problem which is the "real"multifluid model, the problem presented above was only an auxiliary problem to which we were able to transform it. Finally note that the proof for multifluid or bi-fluid problem is in principle the same and we will therefore consider now only the bi-fluid system.

Approximation I

We first take $\delta > 0$ and a sufficiently large $B \gg 1$. Further, we take $\eta_\delta(x) = \eta(x/\delta)$ a smooth cut-off function

$$\eta(z) = \left\{ \begin{array}{cc} 1 & \text{for } 0 \le z \le 1/2 \\ 0 & \text{for } 1 < z \\ \in (0,1) & \text{for } 1/2 < z < 1 \end{array} \right\}, \tag{23}$$

$$0 \le -\eta'(z) \le 2 \ \text{ for all } z$$

and define

$$\Pi_\delta(\varrho, Z) = P_\delta(\varrho, Z) + \delta\Big(\varrho^B + Z^B + \frac{1}{2}\varrho^2 Z^{B-2} + \frac{1}{2} Z^2 \varrho^{B-2}\Big), \tag{24}$$

where

$$P_\delta(\varrho, Z) = \big(1 - \eta_\delta(\sqrt{\varrho^2 + Z^2})\big) P(\varrho, Z).$$

Without loss of generality the initial conditions are regular enough with densities (ϱ_0, Z_0) out of vacuum (if not we regularize it), namely

$$0 < (\underline{a} + \delta)\varrho_0 \le Z_0 \le \overline{a}\varrho_0,\ (\varrho_0, Z_0) \in C^3(\overline{\Omega}),\ (\partial_{\mathbf{n}}\varrho_0, \partial_{\mathbf{n}} Z_0)|_{\partial\Omega} = (0,0) \tag{25}$$

$$\mathbf{u}_0 \in C^3(\overline{\Omega}; R^3) \cap W_0^{1,2}(\Omega; R^3).$$

We take $\varepsilon > 0$ to regularize the continuity equations. Finally we take $\{\mathbf{\Phi}^j\}_{j=1}^\infty \subset C^2(\overline{\Omega}; R^3)) \cap W_0^{1,2}(\Omega; R^3)$ an orthonormal basis in $L^2(\Omega; R^3))$ and consider for a fixed $N \in \mathbb{N}$ an orthogonal projection of the momentum equation onto the linear hull $\mathrm{LIN}\{\mathbf{\Phi}^j\}_{j=1}^N$.

Approximation II

Our approximation looks as follows:

Definition

The triple $(\varrho^{N,\varepsilon,\delta}, Z^{N,\varepsilon,\delta}, \mathbf{u}^{N,\varepsilon,\delta}) = (\varrho, Z, \mathbf{u})$ is a solution to our approximate problem, provided $\partial_t \varrho$, $\partial_t Z$, $\nabla^2 \varrho$ and $\nabla^2 Z \in L^r(I \times \Omega)$ for some $r \in (1, \infty)$, $\mathbf{u}(t,x) = \sum_{j=1}^N c_j^N(t) \mathbf{\Phi}_j(x)$ with $c_j^N \in C^1(0,T) \cap C([0,T])$ for $j = 1, 2, \cdots, N$, the regularized continuity equation problems

$$\begin{aligned} \partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) &= \varepsilon \Delta \varrho \\ \frac{\partial \varrho}{\partial \mathbf{n}}\Big|_{\partial\Omega} &= 0 \\ \varrho(0,x) &= \varrho_0, \end{aligned} \tag{26}$$

and

$$\begin{aligned} \partial_t Z + \operatorname{div}(Z \mathbf{u}) &= \varepsilon \Delta Z \\ \frac{\partial Z}{\partial \mathbf{n}}\Big|_{\partial\Omega} &= 0 \\ Z(0,x) &= Z_0 \end{aligned} \tag{27}$$

hold in the a.a. sense, and the Galerkin approximation for the momentum equation

Approximation III

$$\int_0^T \int_\Omega \Big(\partial_t ((\varrho + Z)\mathbf{u})\boldsymbol{\varphi} - (\varrho + Z)(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} - \Pi_\delta(\varrho, Z)\, \mathrm{div}\, \boldsymbol{\varphi} \Big) \mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_\Omega \Big(\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\mu + \lambda)\, \mathrm{div}\, \mathbf{u}\ \mathrm{div}\, \boldsymbol{\varphi} - \varepsilon \big(\nabla(\varrho + Z) \cdot \nabla \mathbf{u} \big) \cdot \boldsymbol{\varphi} \Big) \mathrm{d}x\mathrm{d}t \tag{28}$$

holds for any $\boldsymbol{\varphi} \in \mathrm{LIN}\{\boldsymbol{\Phi}\}_{j=1}^N$, and

$$\mathbf{u}(0, x) = \mathcal{P}_N(\mathbf{u}_0) \tag{29}$$

with $\mathcal{P}_N$ the orthogonal projection onto $\mathrm{LIN}\{\boldsymbol{\Phi}\}_{j=1}^N$ in $L^2(\Omega; R^3)$.

Existence for the approximate problem I

The existence proof is standard and is based on estimates for suitable linear problems, energy equality and a fixed point argument. We additionally use the following comparison principle.

Proposition

Suppose that $\varrho_0 \in W^{1,2}(\Omega)$, $\mathbf{u} \in L^\infty(0,T;W^{1,\infty}(\Omega;R^3))$, $\mathbf{u}|_{(0,T)\times\partial\Omega} = \mathbf{0}$. *Then we have:*

1. *The parabolic problem* (26) *admits a unique solution in the class*
$$\varrho \in L^2(I;W^{2,2}(\Omega)) \cap W^{1,2}(I;L^2(\Omega)). \qquad (30)$$
2. *If moreover* $0 < \underline{\varrho} \le \varrho_0 \le \overline{\varrho} < \infty$ *a.a. in* Ω, *then there is* $0 < \underline{c} < \overline{c} < \infty$ *dependent on* $\tau, \underline{\varrho}, \overline{\varrho}$ *and* $\|\mathrm{div}\,\mathbf{u}\|_{L^1(I;L^\infty(\Omega))}$ *such that*
$$\text{for all } \tau \in \overline{I},\ \underline{c} \le \varrho(\tau,x) \le \overline{c} \text{ for a.a. } x \in \Omega.$$

It ensures that
$$(\underline{a}+\delta)\varrho \le Z \le \overline{a}\varrho$$
in Q_T.

Existence for the approximate problem II

Note further that the estimate of the density in the energy equality are provided via the term $\mathcal{H}_\delta(\varrho, Z) = H_{P_\delta}(\varrho, Z) + h_\delta(\varrho, Z)$, where $h_\delta(\varrho, Z) = \frac{\delta}{B-1}(\varrho^B + Z^B + \frac{1}{2}\varrho^2 Z^{B-2} + \frac{1}{2} Z^2 \varrho^{B-2})$ and

$$H_{P_\delta}(\varrho, Z) = \varrho \int_1^\varrho \frac{P_\delta(s, s\frac{Z}{\varrho})}{s^2} \,\mathrm{d}s \text{ if } \varrho > 0,\ \ H_P(0,0) = 0.$$

We end up with the energy inequality

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\big(\|(\varrho^N + Z^N)|\mathbf{u}^N|^2\|_{L^1(\Omega)} + \|\varrho^N, Z^N\|^2_{L^2(\Omega)}\big)\\
&+\int_\Omega \mathcal{H}_\delta(\varrho^N, Z^N)\mathrm{d}x\Big)\\
&+\varepsilon \int_\Omega 1_{\{\varrho^N + Z^N \geq K\}}(\varrho^N, Z^N)\Big(\frac{\partial^2 H_{P_\delta}}{\partial \varrho^2}(\varrho^N, Z^N)|\nabla \varrho^N|^2\\
&+2\frac{\partial^2 H_{P_\delta}}{\partial \varrho \partial Z}(\varrho^N, Z^N)\nabla \varrho^N \cdot \nabla Z^N + \frac{\partial^2 H_{P_\delta}}{\partial Z^2}(\varrho^N, Z^N)|\nabla Z^N|^2\Big)\mathrm{d}x\\
&+\varepsilon \int_\Omega 1_{\{\varrho^N + Z^N < K\}}(\varrho^N, Z^N)\Big(\frac{\partial^2 H_{P_\delta}}{\partial \varrho^2}(\varrho^N, Z^N)|\nabla \varrho^N|^2\\
&+2\frac{\partial^2 H_{P_\delta}}{\partial \varrho \partial Z}(\varrho^N, Z^N)\nabla \varrho^N \cdot \nabla Z^N + \frac{\partial^2 H_{P_\delta}}{\partial Z^2}(\varrho^N, Z^N)|\nabla Z^N|^2\Big)\mathrm{d}x\\
&+\varepsilon\delta B \int_\Omega \Big(\frac{\partial^2 h_\delta}{\partial \varrho^2}(\varrho^N, Z^N)|\nabla \varrho^N|^2 + 2\frac{\partial^2 h_\delta}{\partial \varrho \partial Z}(\varrho^N, Z^N)\nabla \varrho^N \cdot \nabla Z^N\\
&+\frac{\partial^2 h_\delta}{\partial Z^2}(\varrho^N, Z^N)|\nabla Z^N|^2\Big)\mathrm{d}x\\
&+\int_\Omega \big(\mu|\nabla \mathbf{u}^N|^2 + (\mu + \lambda)|\operatorname{div} \mathbf{u}^N|^2\big)\mathrm{d}x = \mathbf{0}.
\end{aligned} \tag{31}$$

Limit passages I

We will not discuss the limit passages $N \to \infty$ and $\varepsilon \to 0$ as they are either easy or the difficulties are similar to those appearing in the passage $\delta \to 0$ which we discuss in detail.

So we look at

$$\begin{aligned} &\int_0^T \int_\Omega \big(\varrho\partial_t\psi + \varrho\mathbf{u}\cdot\nabla\psi\big)\mathrm{d}x\mathrm{d}t + \int_\Omega \varrho_0\psi(0,\cdot)\mathrm{d}x = 0, \\ &\int_0^T \int_\Omega \big(Z\partial_t\psi + Z\mathbf{u}\cdot\nabla\psi\big)\mathrm{d}x\mathrm{d}t + \int_\Omega Z_0\psi(0,\cdot)\mathrm{d}x = 0 \end{aligned} \tag{32}$$

for any $\psi \in C^1_c([0,T)\times\overline{\Omega})$,

$$\begin{aligned} &\int_0^T \int_\Omega \big((\varrho+Z)\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho+Z)(\mathbf{u}\otimes\mathbf{u}) : \nabla\boldsymbol{\varphi} + \Pi_\delta(\varrho,Z)\operatorname{div}\boldsymbol{\varphi}\big)\mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \big(\mu\nabla\mathbf{u} : \nabla\boldsymbol{\varphi} + (\mu+\lambda)\operatorname{div}\mathbf{u}\,\operatorname{div}\boldsymbol{\varphi}\big)\mathrm{d}x\mathrm{d}t - \int_\Omega \mathbf{m}_0\cdot\boldsymbol{\varphi}(0,\cdot)\mathrm{d}x \end{aligned} \tag{33}$$

for any $\boldsymbol{\varphi} \in C^\infty_c([0,T)\times\Omega;R^3)$, and the energy inequality

$$\begin{aligned} &\frac{1}{2}\|(\varrho+Z)|\mathbf{u}|^2(t)\|_{L^1(\Omega)} + \int_\Omega \mathcal{H}_\delta(\varrho,Z)(t)\mathrm{d}x \\ &+ \int_0^t \int_\Omega \Big(\mu|\nabla\mathbf{u}|^2 + (\mu+\lambda)|\operatorname{div}\mathbf{u}|^2\Big)\mathrm{d}x\,\mathrm{d}\tau \\ &\le \frac{1}{2}\int_\Omega \frac{|\mathbf{m}_0|^2}{Z_0+\varrho_0}\mathrm{d}x + \int_\Omega \mathcal{H}_\delta(\varrho_0,Z_0)\mathrm{d}x. \end{aligned} \tag{34}$$

Limit passages II

This yields the estimates

$$\|(\varrho_\delta + Z_\delta)|\mathbf{u}_\delta|^2\|_{L^\infty(I;L^1(\Omega))} + \|\mathbf{u}_\delta\|_{L^2(I;W^{1,2}(\Omega))} + \|\varrho_\delta\|_{L^\infty(I;L^\gamma(\Omega))}$$
$$+ \|Z_\delta\|_{L^\infty(I;L^{q_{\gamma,\beta}}(\Omega))} + \delta^{1/B}\big(\|\varrho_\delta\|_{L^\infty(I;L^B(\Omega))} + \|Z_\delta\|_{L^\infty(I;L^B(\Omega))}\big) \leq C. \tag{35}$$

The Bogovskii estimates together with the comparison principle provide

$$\|\varrho_\delta\|_{L^{\gamma+\gamma_{\rm BOG}}(I\times\Omega)} + \|Z_\delta\|_{L^{q_{\gamma,\beta}+\gamma_{\rm BOG}}(I\times\Omega)} \leq C, \tag{36}$$

$$\delta\int_0^T \psi\int_\Omega (\varrho^{B+\gamma_{BOG}} + Z^{B+\gamma_{BOG}})\mathrm{d}x\mathrm{d}t \leq C, \tag{37}$$

$$\|P(\varrho_\delta, z_\delta)\|_{L^q(I\times\Omega)} \leq C \tag{38}$$

for some $q > 1$, and

$$\delta\int_0^T \psi\int_\Omega Z^{B+\beta_{BOG}}\mathrm{d}x\mathrm{d}t \leq C \tag{39}$$

if $\beta > \gamma$ and $\underline{a} > 0$, where $\gamma_{BOG} := \min\{\frac{2}{3}\gamma - 1, \frac{\gamma}{2}\}$.

Limit passages III

We easily get $(\varrho_\delta, Z_\delta) \to (\varrho, Z)$ in $C_{\rm weak}(\overline{I}; L^\gamma(\Omega))$, $\mathbf{u}_\delta \rightharpoonup \mathbf{u}$ in $L^2(I; W^{1,2}(\Omega))$, $(\varrho_\delta \mathbf{u}_\delta, Z_\delta \mathbf{u}_\delta) \rightharpoonup_* (\varrho\mathbf{u}, Z\mathbf{u})$ in $L^\infty(I; L^{\frac{2\gamma}{\gamma+1}}(\Omega))$, $(\varrho_\delta + Z_\delta)\mathbf{u}_\delta \to (\varrho + Z)\mathbf{u}$ in $C_{\rm weak}(\overline{I}; L^{\frac{2\gamma}{\gamma+1}}(\Omega))$.
Next we write

$$P_\delta(\varrho_\delta, Z_\delta) = -\eta_\delta(\sqrt{\varrho_\delta^2 + Z_\delta^2})P(\varrho_\delta, Z_\delta) + P(\varrho_\delta, Z_\delta),$$

where

$$\|\eta_\delta(\sqrt{\varrho_\delta^2 + Z_\delta^2})P(\varrho_\delta, Z_\delta)\|_{L^q(Q_T)} \to 0.$$

We write, using Hypothesis (H4)

$$P(\varrho_\delta, Z_\delta) = P(\varrho_\delta, \varrho_\delta s_\delta) - P(\varrho_\delta, \varrho_\delta s) + \mathcal{P}(\varrho_\delta, s) + \mathcal{R}(\varrho_\delta, s),$$

and get

$$\|P(\varrho_\delta, \varrho_\delta s_\delta) - P(\varrho_\delta, \varrho_\delta s)\|_{L^q(Q_T)} \to 0.$$

This is based on the following:

Limit passages IV

Proposition

1. Let

$$\mathbf{u}_n \in L^2(I, W_0^{1,2}(\Omega; R^3)), \ (\varrho_n, Z_n) \in \mathcal{O}_0 \cap \Big(C(\bar{I}; L^1(\Omega)) \cap L^2(Q_T)\Big)^2.$$

Suppose that

$$\sup_{n \in N} \Big(\|\varrho_n\|_{L^\infty(I;L^\gamma(\Omega))} + \|Z_n\|_{L^\infty(I;L^\gamma(\Omega))} + \|\varrho_n\|_{L^2(Q_T)} + \|\mathbf{u}_n\|_{L^2(I;W^{1,2}(\Omega))} \Big) < \infty,$$

where $\gamma > 6/5$*, and that both couples* $(\varrho_n, \mathbf{u}_n)$*,* $(Z_n, \mathbf{u}_n)$ *satisfy continuity equation. Then, up to a subsequence (not relabeled)*

$$(\varrho_n, Z_n) \to (\varrho, Z) \text{ in } (C_{\rm weak}(\bar{I}; L^\gamma(\Omega)))^2,$$
$$\mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(I; W^{1,2}(\Omega; R^3)),$$

where (ϱ, Z) *belongs to spaces*

$$\mathcal{O}_0 \cap (L^2(Q_T))^2 \cap (L^\infty(I, L^\gamma(I, \Omega)))^2 \cap (C(\bar{I}; L^1(\Omega))^2$$

and $(\varrho, \mathbf{u})$ *as well as* $(Z,\mathbf{u})$ *verify continuity equation in the renormalized sense.*
2. *We define in agreement as above for all* $t \in \overline{I}$,

$$s_n(t,x) = \frac{Z_n(t,x)}{\varrho_n(t,x)}, \quad s(t,x) = \frac{Z(t,x)}{\varrho(t,x)}. \tag{40}$$

Suppose in addition to assumptions of item 1. *that*

$$\int_\Omega \varrho_n(0,x)s_n^2(0,x)\mathrm{d}x \to \int_\Omega \varrho(0,x)s^2(0,x)\,\mathrm{d}x.$$

Then $s_n, s \in C(\overline{I}; L^q(\Omega))$, $1 \le q < \infty$ *and for all* $t \in \overline{I}$, $0 \le s_n(t,x) \le \overline{a}$, $0 \le s(t,x) \le \overline{a}$ *for a.a.* $x \in \Omega$. *Moreover, both* $(s_n, \mathbf{u}_n)$ *and* $(s, \mathbf{u})$ *satisfy transport equation.*
3. *Finally,*

$$\int_\Omega (\varrho_n|s_n - s|^p)(\tau,\cdot)\,\mathrm{d}x \to 0 \text{ with any } 1 \le p < \infty \tag{41}$$

for all $\tau \in \overline{I}$.

Limit passages VI

We have that the continuity equations are fulfilled in the renormalized sense.

Proposition

Let couples $(\varrho, \mathbf{u})$, $(Z, \mathbf{u})$

$$\varrho \in L^2(Q_T),\ (\varrho, Z) \in \mathcal{O}_0,\ \mathbf{u} \in L^2(I; W^{1,2}(\Omega; R^3)),$$

verify continuity equation. Then for any

$$b \in C^1([0,\infty)^2), (\partial_\varrho b, \partial_Z b) \in L^\infty(\mathcal{O}_0; R^2)$$

the function $b(\varrho, Z)$ *verifies the renormalized continuity equation*

$$\partial_t b(\varrho, Z) + \mathrm{div}(b(\varrho, Z)\mathbf{u}) \tag{42}$$

$$+(\varrho\partial_\varrho b(\varrho, Z) + Z\partial_Z b(\varrho, Z) - b(\varrho, Z))\mathrm{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'(Q_T).$$

Proposition

Let ϱ, Z *belong to* $C(\overline{I}; L^1(\Omega))$. *We define for all* $t \in \overline{I}$, $s(t,x) = \frac{Z(t,x)}{\varrho(t,x)}$ *as above. Then we have:*

1. If for all $t \in \overline{I}$, $0 \leq Z(t,\cdot) \leq \overline{a}\varrho(t,\cdot)$ *a.e. in* Ω, *then*

$$\text{for all } t \in \overline{I},\ 0 \leq s(t,\cdot) \leq \overline{a} \text{ a.e. in } \Omega. \tag{43}$$

2. Suppose moreover that

$$\varrho \in L^2(Q_T) \cap L^\infty(I; L^\gamma(\Omega)), \text{ with some } \gamma > 1$$

and that both couples $(\varrho, \mathbf{u})$ *and* $(Z, \mathbf{u})$ *satisfy continuity equation with* $\mathbf{u} \in L^2(I; W^{1,2}(\Omega; R^3))$. *Then*

$$s \in C(\overline{I}; L^q(\Omega)) \text{ for all } 1 \leq q < \infty \tag{44}$$

and the couple $(s, \mathbf{u})$ *satisfies the transport equation.*

3. If moreover $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega; R^3))$, *then transport equation holds up to the boundary in the time integrated form.*

Limit passages VIII

Proposition

1. Let

$$\varrho \in L^2(Q_T), s \in L^\infty(Q_T),\ \mathbf{u} \in L^2(I; W^{1,2}(\Omega; R^3)),$$

and let the couple $(\varrho, \mathbf{u})$ *verify the continuity equation and the couple* $(s, \mathbf{u})$ *the transport equation. Then* $s \in C(\bar{I}; L^1(\Omega))$ *and the product* $s\varrho$ *satisfies the continuity equation in the sense of distributions on* Q_T*.*
2. If moreover we have $\varrho \in L^\infty(I; L^\gamma(\Omega))$ *with some* $\gamma > 1$ *and*
$\mathbf{u} \in L^2(I, W_0^{1,2}(\Omega; R^3))$*, then*

$$\varrho \in C(\bar{I}; L^1(\Omega)),\ s\varrho \in C(\bar{I}; L^1(\Omega))$$

and the continuity equation for $s\varrho$ *holds in the time integrated form up to the boundary:*

$$\int_\Omega (s\varrho\varphi)(\tau, \cdot)\,\mathrm{d}x - \int_\Omega (s\varrho\varphi)(0, \cdot)\,\mathrm{d}x = \int_0^\tau \int_\Omega \Big(s\varrho\partial_t\varphi + s\varrho\mathbf{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \quad (45)$$

for all $\tau \in \bar{I}$ *and* $\varphi \in C^1(\overline{Q_T})$*.*

Limit passages IX

Therefore letting $\delta \to 0$

$$
\begin{aligned}
&\int_0^T \int_\Omega \left((\varrho + Z)\mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + (\varrho + Z)(\mathbf{u} \otimes \mathbf{u}) : \nabla \boldsymbol{\varphi} + \overline{\overline{P(\varrho, \varrho s)}} \operatorname{div} \boldsymbol{\varphi} \right) \mathrm{d}x \mathrm{d}t \\
= &\int_0^T \int_\Omega (\mu \nabla \mathbf{u} : \nabla \boldsymbol{\varphi} + (\mu + \lambda) \operatorname{div} \mathbf{u} \operatorname{div} \boldsymbol{\varphi}) \mathrm{d}x \mathrm{d}t - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \mathrm{d}x
\end{aligned} \tag{46}
$$

for any $\boldsymbol{\varphi} \in C_c^1([0,T) \times \Omega; R^3)$, where $\overline{\overline{P(\varrho, \varrho s)}}$ is the weak limit of $P(\varrho_\delta, s\varrho_\delta)$, and

$$
\begin{aligned}
\int_0^T \int_\Omega (\varrho \partial_t \psi + \varrho \mathbf{u} \cdot \nabla \psi) \mathrm{d}x \mathrm{d}t + \int_\Omega \varrho_0 \psi(0, \cdot) \mathrm{d}x = 0, \\
\int_0^T \int_\Omega (Z \partial_t \psi + Z \mathbf{u} \cdot \nabla \psi) \mathrm{d}x \mathrm{d}t + \int_\Omega Z_0 \psi(0, \cdot) \mathrm{d}x = 0
\end{aligned} \tag{47}
$$

for any $\psi \in C_c^1([0,T) \times \overline{\Omega})$.

It remains to prove the strong (or pointwise a.a.) convergence of ϱ_δ.

Strong convergence of densities I

We can now prove the following version of the effective viscous flux identity (T_k is the standard concave (near infinity) cut-off function)

Proposition

Identity

$$\overline{\overline{P(\varrho, \varrho s)\, T_k(\varrho)}} - (2\mu+\lambda)\,\overline{T_k(\varrho)\,\mathrm{div}\,\mathbf{u}} = \overline{\overline{P(\varrho, \varrho s)}}\;\overline{T_k(\varrho)} - (2\mu+\lambda)\,\overline{T_k(\varrho)}\,\mathrm{div}\,\mathbf{u} \quad (48)$$

holds a.a. in $I \times \Omega$.

Strong convergence of densities II

For $\gamma = 9/5$ we furthermore need the estimate of the oscillation defect measure

Proposition

The sequence ϱ_δ satisfies

$$\mathrm{osc}_{\gamma+1}[\varrho_\delta \rightharpoonup \varrho](Q_T) := \sup_{k>1} \limsup_{\delta\to 0} \int_{Q_T} |T_k(\varrho_\delta) - T_k(\varrho)|^{\gamma+1}\,\mathrm{d}x\,\mathrm{d}t < \infty. \quad (49)$$

We now follow the idea from

E. Feireisl: Compressible Navier-Stokes equations with a non-monotone pressure law, J. Differential Equations **184**, 97–108, 2002.

We take L_k, the solution to $zL_k'(z) - L_k(z) = T_k(z)$ and using that ϱ_k and ϱ both satisfy the renormalized continuity equation we get

$$\begin{aligned}\int_\Omega (L_k(\varrho_\delta) - L_k(\varrho))(\tau,\cdot)\mathrm{d}x = &\int_0^\tau \int_\Omega (T_k(\varrho)\,\mathrm{div}\,\mathbf{u} - \overline{T_k(\varrho)}\,\mathrm{div}\,\mathbf{u}_\delta)\mathrm{d}x\mathrm{d}t \\ &+ \int_0^\tau \int_\Omega (\overline{T_k(\varrho)} - T_k(\varrho_\delta))\,\mathrm{div}\,\mathbf{u}_\delta \mathrm{d}x\mathrm{d}t\end{aligned} \quad (50)$$

for all $\tau \in \bar{I}$.

Strong convergence of densities III

Using the effective viscous flux identity and Hypothesis (H4) we get

$$\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(\tau,\cdot)\mathrm{d}x = \int_0^\tau \int_\Omega (T_k(\varrho) - \overline{T_k(\varrho)})\,\mathrm{div}\,\mathbf{u}\mathrm{d}x\mathrm{d}t$$
$$+ \frac{1}{2\mu+\lambda}\int_0^\tau \int_\Omega \left(\overline{\overline{\mathcal{P}(\varrho,s)}}\;\overline{T_k(\varrho)} - \overline{\overline{\mathcal{P}(\varrho,s)\,T_k(\varrho)}}\right)\mathrm{d}x\mathrm{d}t \qquad (51)$$
$$+ \frac{1}{2\mu+\lambda}\int_0^\tau \int_\Omega \left(\overline{\overline{\mathcal{R}(\varrho,s)}}\;\overline{T_k(\varrho)} - \overline{\overline{\mathcal{R}(\varrho,s)\,T_k(\varrho)}}\right)\mathrm{d}x\mathrm{d}t$$

for all $\tau \in \overline{I}$. The first term on the rhs is bounded by (for $\gamma + \gamma_{BOG} > 2$ we can proceed without the oscillation defect measure estimate)

$$\|\overline{T_k(\varrho)} - T_k(\varrho)\|_{L^2(Q_T)}\|\,\mathrm{div}\,\mathbf{u}\|_{L^2(Q_T)} \leq c \limsup_{\delta\to 0} \|T_k(\varrho_\delta) - T_k(\varrho)\|_{L^1(Q_T)}^{\frac{\gamma-1}{2\gamma}},$$

the second term is non-positive and the third term can be estimated by

$$c\Lambda(1+R_0)\int_0^\tau \int_\Omega \left(\overline{\varrho\ln\varrho} - \varrho\ln\varrho\right)\mathrm{d}x\,\mathrm{d}t$$

for Λ sufficiently large, where R_0 is connected with the support of $\mathfrak{R}$.

Strong convergence of densities IV

Since

$$\| T_k(\varrho) - \overline{T_k(\varrho)} \|_{L^1(Q_T)} \to 0, \qquad k \to \infty,$$

we may let $k \to \infty$ to get

$$\int_\Omega (\overline{\varrho \ln \varrho} - \varrho \ln \varrho)(\tau, \cdot)\mathrm{d}x \leq Cc\Lambda(1 + R_0) \int_0^\tau \int_\Omega \left(\overline{\varrho \ln \varrho} - \varrho \ln \varrho \right) \mathrm{d}x\, \mathrm{d}t.$$

By Gronwall lemma and strict convexity of $\varrho \mapsto \varrho \ln \varrho$ on $[0, \infty)$

$$\varrho_\delta \to \varrho \text{ a.a. in } Q_T.$$

Hence by our previous Propositions

$$Z_\delta \to Z \text{ a.a. in } Q_T$$

and we also show the energy inequality. The theorem is proved.

Oldroyd-B: Existence of a solution I

We have to show that the proof applies also to the reformulated problem for the Oldroyd-B fluid.

Hypothesis (H1).

$$(\eta_0, \varrho_0, \tau_0) \in \mathcal{O}_{\underline{\vec{a}}} := \big\{ (Z_0, Z_1, Z_2) \in \mathbb{R}^3 | Z_0 \in [0, \infty), \\ \underline{a}_i Z_0 < Z_i < \overline{a}_i Z_0 \big\}, i = 1, 2 \tag{52}$$

where $0 \leq \underline{a}_i < \overline{a}_i < \infty$, $i = 1, 2$.

Assumption:

$$\varrho_0 \leq \overline{C} \eta_0, \quad \tau_0 \leq \overline{C} \eta_0 \quad \text{a.e. in } \Omega \text{ for some } \overline{C} > 0. \tag{53}$$

Hypothesis (H1) is fulfilled with $\underline{a}_1 = \underline{a}_2 = 0$ and $\overline{a}_1 = \overline{a}_2 = \overline{C}$. Note that this choice of the domination corresponds to the fact that the estimated quantity is η for which we have quadratic growth, i.e. it corresponds to the choice $\gamma = 2$ in the theorem by Novotný–Pokorný. If the growth of the gas pressure is faster than quadratic, we just use as the main quantity the density ϱ and modify appropriately the domination condition.

Oldroyd-B: Existence of a solution II

Hypothesis (H2).

$$\begin{aligned} &\eta_0 \in L^{\gamma}(\Omega),\ \varrho_0 \in L^{\beta_1}(\Omega), \tau_0 \in L^{\beta_2} \ \text{ if } \beta_i > \gamma, \\ &\mathbf{m}_0 \in L^1(\Omega;\mathbb{R}^3),\ \varrho_0|\mathbf{u}_0|^2 \in L^1(\Omega). \end{aligned} \tag{54}$$

This hypothesis is fulfilled by our assumption on the initial data. Recall that momentum and kinetic energy contains only ϱ

Oldroyd-B: Existence of a solution III

We denote the total pressure

$$h(\eta,\varrho,\tau) := q(\eta) + p(\varrho) - \tau = \delta\,\eta^2 + k(L-1)\eta + a\varrho^\gamma - \tau. \qquad (55)$$

Hypothesis (H3). Function $h \in C(\overline{\mathcal{O}_{\vec{a}}}) \cap C^1(\mathcal{O}_{\vec{a}})$ and

$$\forall \varrho \in (0,1), \quad \sup_{s \in \Pi_{i=1}^{2}[\underline{a}_i, \overline{a}_i]} |h(\eta, \eta s_1, \tau s_2)| \leq C\varrho^\alpha \text{ with some } C > 0 \text{ and } \alpha > 0, \qquad (56)$$

and

$$\underline{C}(\eta^\gamma + \varrho^{\beta_1} + \tau^{\beta_2} - 1) \leq h(\eta,\varrho,\tau) \leq \overline{C}(\eta^\gamma + \varrho^{\beta_1} + \tau^{\beta_2} + 1) \text{ in } \mathcal{O}_{\vec{a}} \qquad (57)$$

with $\gamma \geq \frac{9}{5}$, $\beta_i > 0$, $i = 1, 2$. We moreover assume

$$|\partial_\varrho h(\eta,\varrho,\tau)| + |\partial_\tau h(\eta,\varrho,\tau)| \leq C(\eta^{-\underline{\Gamma}} + \eta^{\overline{\Gamma}-1}) \text{ in } \mathcal{O}_{\vec{a}} \qquad (58)$$

with some $0 \leq \underline{\Gamma} < 1$, and with some $0 < \overline{\Gamma} < \gamma + \gamma_{BOG}$ if $\underline{a}_i = 0$,
$0 < \overline{\Gamma} < \max\{\gamma + \gamma_{BOG}, \beta_i + (\beta_i)_{BOG}\}$ if $\underline{a}_i > 0$.

We denote

$$S := \{(\eta,\varrho,\tau)\in\mathbb{R}^3 : 0<\varrho<\overline{C}\eta,\quad 0<\tau<\overline{C}\eta\} \tag{59}$$

and

$$\overline{S} := \{(\eta,\varrho,\tau)\in\mathbb{R}^3 : 0\le\varrho\le\overline{C}\eta,\quad 0\le\tau\le\overline{C}\eta\}. \tag{60}$$

Then $h(\eta,\varrho,\tau)\in C(\overline{S})$ and $h(\eta,\varrho,\tau)\in C^1(S)$. Moreover, for all $\eta\in(0,1)$ and for all $(\eta,\varrho,\tau)\in\overline{S}$

$$|h(\eta,\varrho,\tau)|\le\delta\eta^2+k|L-1|\eta+a\overline{C}^\gamma\eta^\gamma+\overline{C}\eta\le C(\eta+\eta^\gamma)\le C\eta. \tag{61}$$

Next, in S,

$$C_1(\eta^2+\varrho^\gamma-\tau-1)\le h(\eta,\varrho,\tau)\le C_2(\eta^2+\varrho^\gamma+\tau+1) \tag{62}$$

for some positive constants C_1, C_2. Using the domination assumption and the resulted domination for all times

$$C_1(\eta^2+\varrho^\gamma+\tau-1)\le h(\eta,\varrho,\tau)\le C_2(\eta^2+\varrho^\gamma+\tau+1). \tag{63}$$

Oldroyd-B: Existence of a solution V

Moreover,

$$C_1(\eta^2 + \varrho^\gamma + \tau|\log\tau| - 1) \leq H(\eta,\varrho,\tau) \leq C_2(\eta^2 + \varrho^\gamma + \tau|\log\tau| + 1), \quad (64)$$

which follows from the form of our Helmholtz free energy (note that it is different in comparison to [NP]). Furthermore, for each $(\eta,\varrho,\tau) \in S$,

$$|\partial_\tau h(\eta,\varrho,\tau)| = 1, \quad |\partial_\varrho h(\eta,\varrho,\tau)| = |a\gamma\varrho^{\gamma-1}|. \quad (65)$$

For $\gamma \geq 1$ it implies that (58) is fulfilled for the choice $\underline{a}_1 = 0$. However, for $\gamma \in (0,1)$ we cannot fulfil this assumption for $\underline{a}_1 = 0$, as $\gamma - 1 < 0$ and we need to control the function ϱ by η from below. However, this condition is in fact in our case not needed and we have an alternative way how to overcome its use. It is connected with much easier form of the pressure than the general case assumed in [NP]. Thus, the main part of Hypothesis (H3) is satisfied, the rest can be overcome.

Oldroyd-B: Existence of a solution VI

Hypothesis (H4). We assume

$$h(\eta, \eta s_1, \eta s_2) = \mathcal{P}(\eta, s_1, s_2) - \mathcal{R}(\eta, s_1, s_2), \tag{66}$$

where $[0, \infty) \ni \varrho \mapsto \mathcal{P}(\eta, s_1, s_2)$is non decreasing for any $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2$, and $\varrho \mapsto \mathcal{R}(\varrho, s_1, s_2)$ is for any $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2$ a non-negative C^2-function in $[0, \infty)$ with uniformly bounded C^2-norm with respect to $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2$ and with compact support uniform with respect to $s_i \in [\underline{a}_i, \overline{a}_i]$, $i = 1, 2$. Here, $\underline{a}_i, \overline{a}_i$ are the constants from relation (16).

Oldroyd-B: Existence of a solution VII

For each $(\eta, \varrho, \tau) \in \overline{S}$, we define the following functions

$$s_\varrho := \begin{cases} \dfrac{\varrho}{\eta}, & \text{if } \eta > 0, \\ 0, & \text{if } \eta = 0, \end{cases} \qquad s_\tau := \begin{cases} \dfrac{\tau}{\eta}, & \text{if } \eta > 0, \\ 0, & \text{if } \eta = 0. \end{cases} \tag{67}$$

Clearly $s_\varrho, s_\tau \in [0, \overline{C}]$ for all $(\eta, \varrho, \tau) \in \overline{S}$. Then for each $(\eta, \varrho, \tau) \in \overline{S}$, we can write

$$h(\eta, \varrho, \tau) = h(\eta, \eta s_\varrho, \eta s_\tau) = \delta, \eta^2 + k(L-1)\eta + a\eta^\gamma s_\varrho^\gamma - \eta s_\tau, \quad s_\varrho, s_\tau \in [0, \overline{C}]. \tag{68}$$

We write the total pressure as

$$h(\eta, \eta s_\varrho, \eta s_\tau) = \mathcal{P}(\eta, s_\varrho, s_\tau) - \mathcal{R}(\eta, s_\varrho, s_\tau), \tag{69}$$

with

$$\begin{aligned} \mathcal{P}(\eta, s_\varrho, s_\tau) &= \delta\, \eta^2 + kL\eta + a\eta^\gamma s_\varrho^\gamma - \big(1 - \chi(\eta)\big)\big(k\eta + \eta s_\tau\big), \\ \mathcal{R}(\eta, s_\varrho, s_\tau) &= \chi(\eta)\big(k\eta + \eta s_\tau\big). \end{aligned} \tag{70}$$

By choosing $\overline{R}_1$ (and thus also $\overline{R}$) large enough, it is straightforward to check that the decomposition (69)–(70) satisfies Hypothesis (H4).

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Hypothesis (H5). Functions $\eta \mapsto h(\eta, \varrho, \tau)$, $\varrho > 0$, $\tau > 0$ resp. $(\varrho, \tau) \mapsto \partial_\varrho h(\eta, \varrho, \tau)$, and $(\varrho, \tau) \mapsto \partial_\tau h(\eta, \varrho, \tau)$ $\eta > 0$, are Lipschitz on $(\varrho/\overline{C}, \infty) \times (\tau/\overline{C}, \infty) \cap (\underline{r}, \infty)^2$ resp. $(0, \overline{C}\eta) \times (0, \overline{C}\eta) \cap (\underline{r}, \infty)^2$ for all $\underline{r} > 0$ with Lipschitz constants

$$\widetilde{L}_P \leq C(\underline{r})(1 + (|\varrho| + |\tau|)^A) \ \text{ resp. } \widetilde{L}_P \leq C(\underline{r})(1 + \eta^A) \tag{71}$$

with some non negative number A. Number $C(\underline{r})$ may diverge to $+\infty$ as $\underline{r} \to 0^+$.

Hypothesis (H5) is used in the construction of the approximate problem, is connected with the form of the Helmholtz free energy and it yields that $|\nabla^2_{\eta,\varrho,\tau} H(\eta, \varrho, \tau)| \leq C(r)(1 + \eta^A)$ in the set $\{\eta^2 + \varrho^2 + \tau^2 > r^2\} \cap \overline{S}$. Hence, for our choice of the Helmholtz energy, we only need that

$$|\nabla^2_\eta q(\eta)| + |\nabla^2_\varrho p(\varrho)| + |1/\tau| \leq C(r)(1 + \eta^A)$$

in the set $\{\eta^2 + \varrho^2 + \tau^2 > r^2\} \cap \overline{S}$. However, it follows directly with the choice $A = 0$ from the form of the pressure. The modified Hypothesis (H5) is fulfilled.

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We are left with modifications caused by different form of the transport equation for τ. First, by standard properties of the transport equation it follows

$$\inf_{x\in\Omega} \tau_0(x) \exp\left(- \int_0^t \| \operatorname{div}_x \mathbf{u} \|_{L^\infty(\Omega)} \, \mathrm{d}t' - \frac{t}{2\lambda} \right) \leq \tau(t,x)$$

yielding non-negativity of τ provided the initial condition is so. Next by a similar argument we get

$$0 \leq \varrho(t,x) \leq \overline{C}\eta(t,x), \quad 0 \leq \tau(t,x) \leq \overline{C}\eta(t,x), \quad \text{for all } t,x \in \overline{Q}_T. \qquad (72)$$

Finally, to show that the energy inequality holds (i.e., to pass to the limit in the energy inequality for the approximate problem) we have to employ the renormalized continuity equation in order to show

$$\varrho_n^\gamma \to \varrho^\gamma, \;\; \eta_n \log \eta_n \to \eta \log \eta, \;\; \tau_n \log \tau_n \to \tau \log \tau \;\text{ in }\; C_w([0,T], L^1(\Omega)).$$

Other details are standard. The theorem is proved.

THANK YOU VERY MUCH

FOUR YOUR ATTENTION!