A finite-dimensional reduction of dissipative dynamical systems

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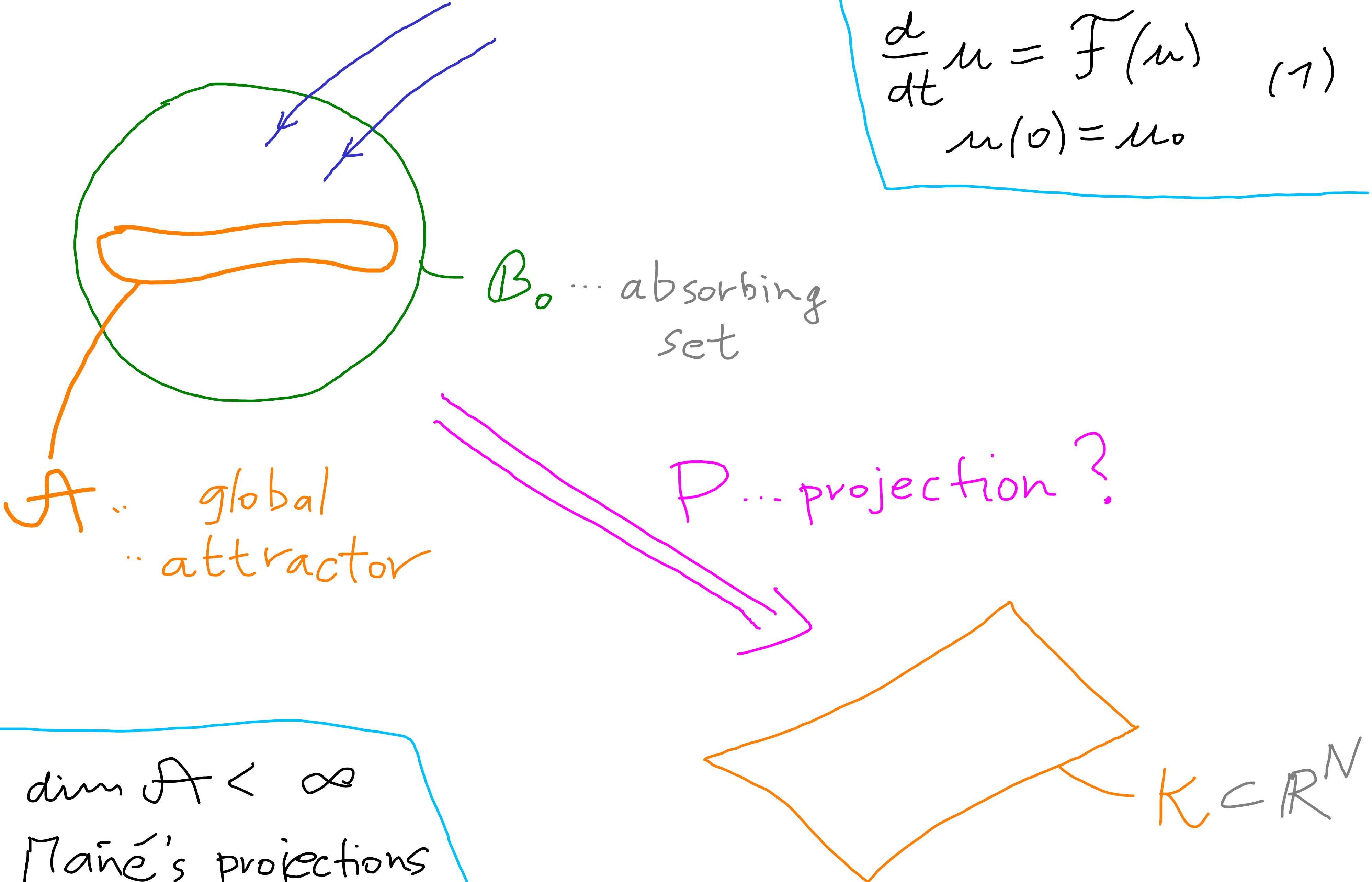
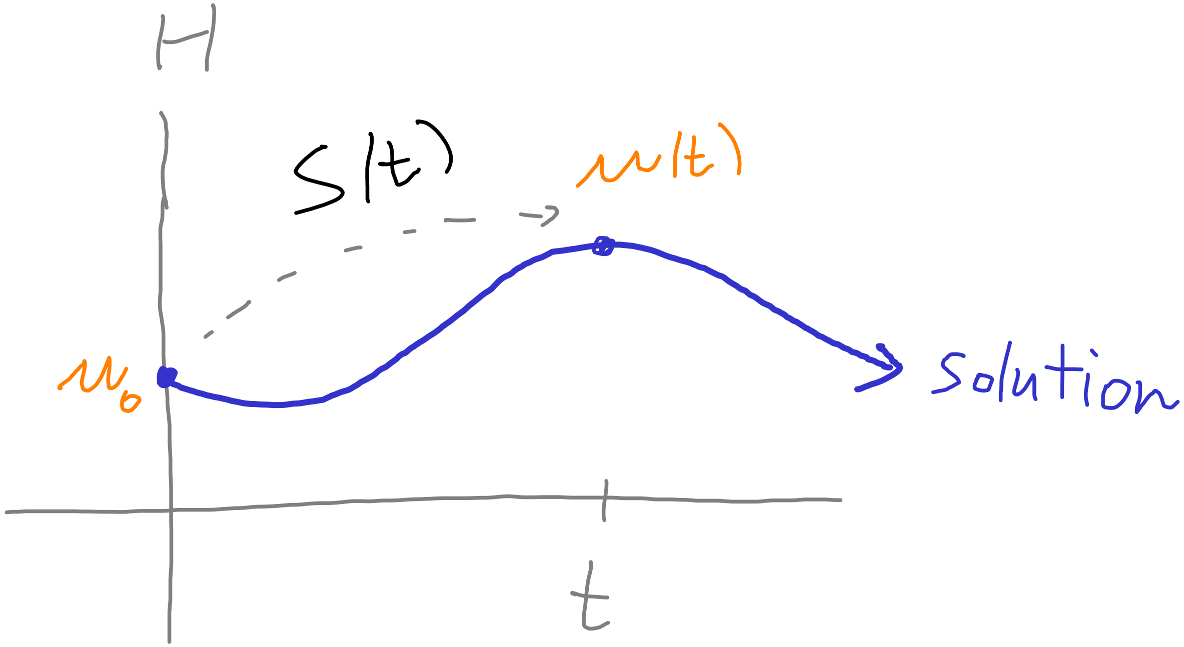
1. Basic problem
2. Dynamics of trajectories
3. Abstract result
4. Application

$$\frac{d}{dt}u = F(u) \qquad (1)$$
$$u(0) = u_0$$

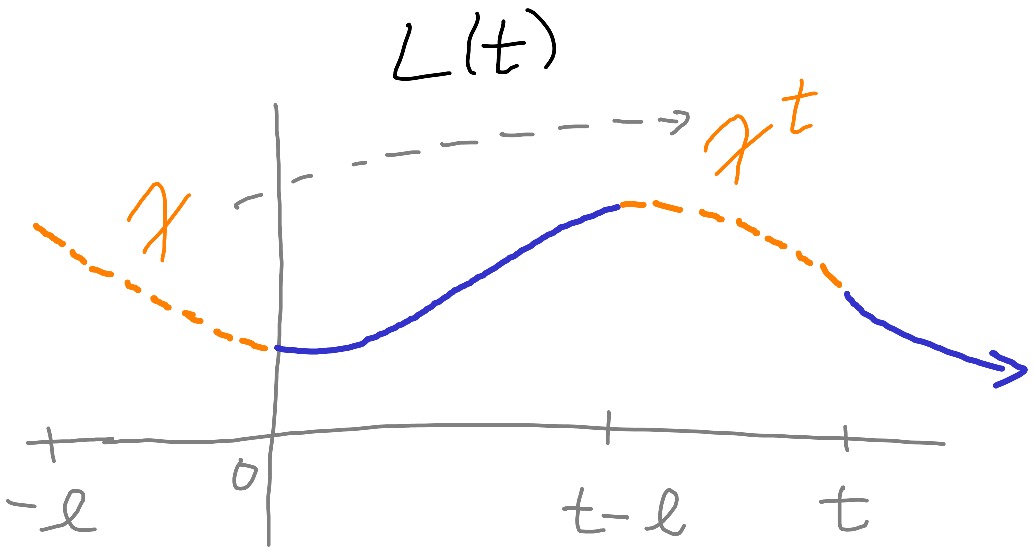


- $\dim \mathcal{A} < \infty$
- Mañé's projections
- inertial manifolds

Dynamics of trajectories



$$\frac{d}{dt} u = \mathcal{F}(u) \qquad (1)$$
$$u(0) = u_0$$



- (semi)infinite traj.

An abstract problem

$$\boxed{\frac{d}{dt}u + A(u) + B(u) = 0 \qquad u(0) = u_0} \tag{1}$$

$u : [0,T] \to H$ the unknown

$A : V \longrightarrow V'$ nonlinear, elliptic

$B : V \longrightarrow V'$ lower-order

Gelfand triple

$$\boxed{V \hookrightarrow \hookrightarrow H \hookrightarrow V'}$$

Hilbert

Typical assumptions

$$A(0) = 0\,,$$
$$\|A(u) - A(v)\|_{V'} \leq c_1 \|u - v\|_V\,,$$
$$\langle A(u) - A(v), u - v\rangle \geq a\|u - v\|_V^2\,,$$

2-elliptic

lower order

$$\|B(u) - B(v)\|_{V'} \leq b\|u - v\|_H^{\alpha} \|u - v\|_V^{1-\alpha}\,, \qquad \alpha \in (0, 1]\,,$$
$$\langle B(u), u\rangle \geq -\varepsilon\|u\|_V^2 - K_\varepsilon\,, \qquad \varepsilon > 0\,.$$

dissipativity

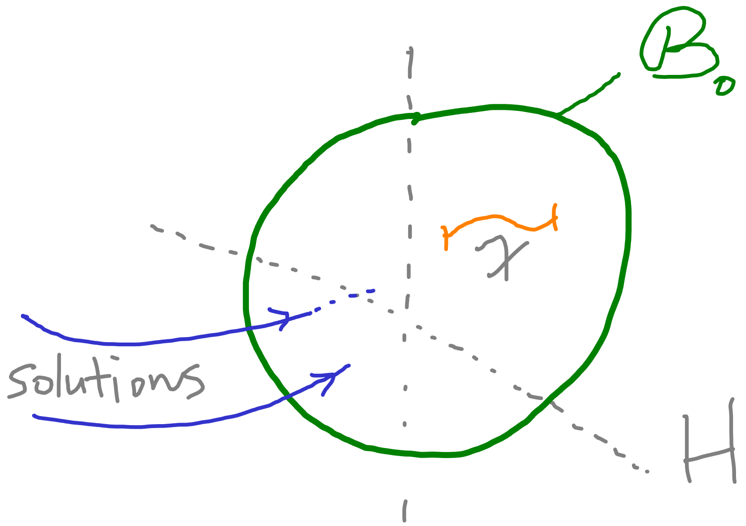
Theorem 1. *Let $u_0 \in H$ and $T > 0$ be given. Then there exists a weak solution*

$$\begin{aligned} u &\in L^\infty(0, T; H) \cap L^2(0, T; V)\,, \\ d_t u &\in L^2(0, T; V')\,, \end{aligned} \tag{4}$$

where $u(0) = u_0$ in the sense of a representative $u \in C([0, T]; H)$.

Well-posedness

Lemma 3. *For $R > 0$ large enough, the ball $\mathcal{B}_0 = \{u \in H;\ \|u\|_H^2 \le R\}$ is uniformly absorbing and positively invariant.*



Set of ℓ-trajectories (in $\mathcal{B}_0$)

$$\mathcal{B}_\ell := \{\chi : [-\ell, 0] \to H;\ \chi \text{ is weak solution, } \chi(-\ell) \in \mathcal{B}_0\}$$

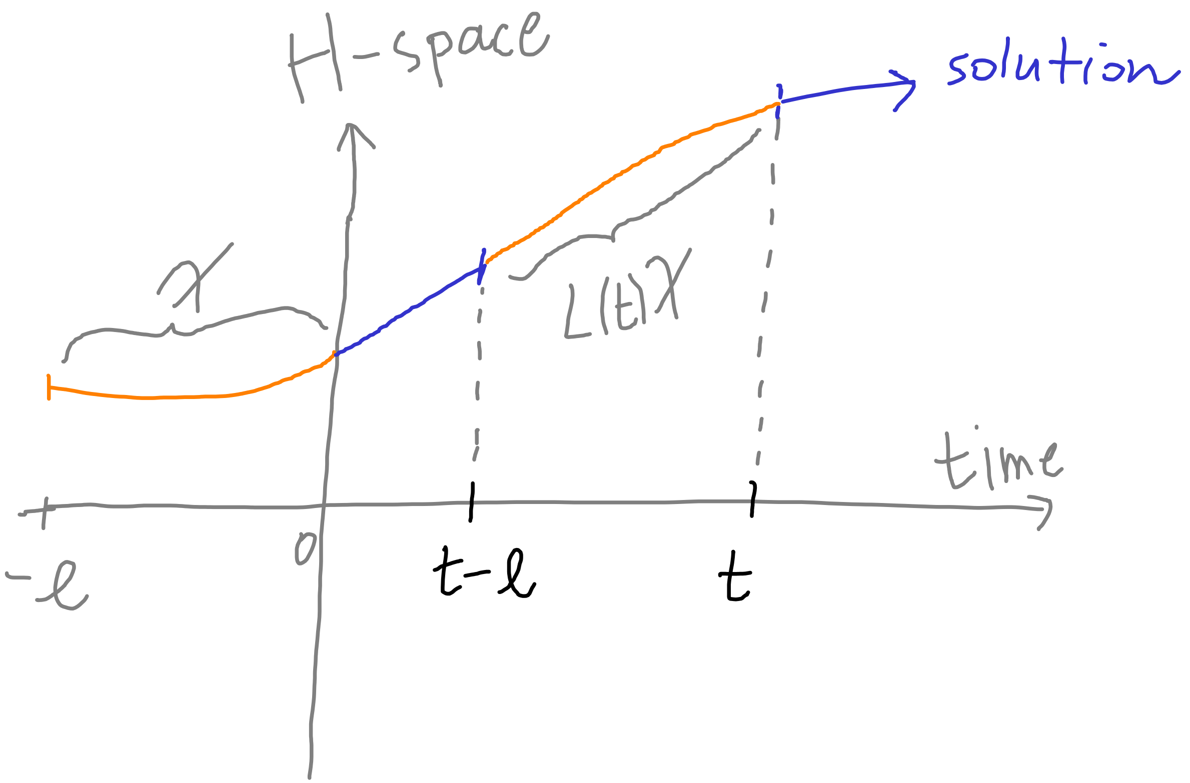
$$H_\ell := L^2(-\ell, 0; H)$$ metric

Lemma 4. *The set $\mathcal{B}_\ell$ is compact in H_ℓ.*

Lemma 5. *(i) $L(t) : H_\ell \to H_\ell$ is Lipschitz continuous on $\mathcal{B}_\ell$.*
(ii) $L(\ell) : H_\ell \to W_\ell$ is Lipschitz continuous on $\mathcal{B}_\ell$, where

$$\|\chi\|_{W_\ell}^2 = \int_{-\ell}^{0} \|\chi\|_V^2 + \|d_t\chi\|_{V'}^2 \, .$$

"Smoothing" property



Results

1) $\exists$ global attractor (with $\dim < \infty$)

$\exists$ exponential attractor

$$\boxed{\begin{array}{c}(L(t), \mathcal{B}_e) \\ \Updownarrow \\ (S(t), H)\end{array}}$$

2) $\exists$ ODE-like reduction ($H \longrightarrow \mathbb{R}^N$)

ad2) $\mathcal{A}_\ell$ ... global attractor (for $L(t)$)

$$\mathcal{A}^{\leq 0} = \{\chi^t : (-\infty, 0] \to \mathcal{B}_\ell \text{, solves (1), } \chi^0 \in \mathcal{A}_\ell\}$$

$$\mathcal{T} = \{p(t) : (-\infty, 0] \to \mathbb{R}^N \text{, } p(t) = P\chi^t \text{, } \chi^t \in \mathcal{A}_\ell\}$$

finite-dim. projection

Lemma 6. *For any $p \in \mathcal{T}$, there exists unique $\chi^0 \in \mathcal{A}^{\leq 0}$ such that $p(t) = P\chi^t$, $t \leq 0$. The mapping $E : p \mapsto \chi^0$ is Lipschitz from X to W_ℓ, where X is given by*

$$\|p\|_X = \underbrace{\sup_{s \leq 0} \mathrm{e}^{\gamma s} \|p(s)\|_{\mathbb{R}^N}}_{X\text{-norm}}$$

with suitable $\gamma > 0$.

ODE-like reduction

(1) $\frac{d}{dt}u + A(u) + B(u) = 0$ (PDE)

$$\Updownarrow$$

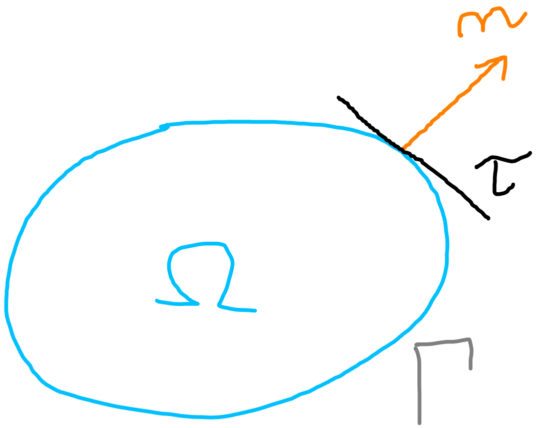
(2) $\frac{d}{dt}x(t) + F(x^{\leq t}) = 0$ (delayed ODE)

(with $F : X \to \mathbb{R}^N$ Lipschitz)

Application: nonlinear fluids & dynamic b.c.

$$\left.\begin{aligned} \partial_t u + (u\cdot\nabla)u - \operatorname{div} S + \nabla\pi &= f \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } \Omega$$

$$\left.\begin{aligned} u\cdot n &= 0 \\ \alpha u + \beta\,\partial_t u + (Sn)_\tau &= 0 \end{aligned}\right\} \text{ on } \Gamma$$



$$S \sim \nu_0 D + \nu_1 |D|^{r-2} D$$

$$(r \geq 2)$$

Thank you!