DSM for solving operator equations

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Introduction

Motivations:
a) Develop a general method for solving operator eqns, *especially ill-posed*,
b) Develop a general method for constructing convergent iterative processes for solving such eqns.

$$F(u) - f = 0 \quad (1), \quad F : H \to H, \quad \exists y : F(y) - f = 0$$

Original author's assumptions were:

$$\sup_{u \in B(u_0,R)} \|F^{(j)}(u)\| \leq M_j(R), j \leq 2, \quad B(u_0, R) := \{u : \|u - u_0\| \leq R\}.$$

Current progress: in many cases $j \leq 1$ is sufficient.
Well-posed (WP): $\sup_{u \in B(u_0,R)} \|[F'(u)]^{-1}\| \leq m(R)$
Ill-posed (IP): not well-posed.

DSM: $$\begin{cases} \dot{u} &= \Phi(t,u), \\ u(0) &= u_0. \end{cases}$$

$(*)$ $\quad\exists! u(t)$ on $[0,\infty)$; $\quad\exists u(\infty)$; $\quad F(u(\infty)) = f$

For what classes of equations $F(u) = f$ can one find Φ such that $(*)$ holds?
How does one choose Φ?

In general, the solution u *does not exist globally*.
We give sufficient conditions for the global existence of u, among many other things.

Theorem 1. For *any WP eq. (1)* one can find Φ such that $(*)$ holds and

$$\|u(t) - u(\infty)\| \le re^{-c_1 t};$$
$$\|F(u(t)) - f\| \le \|F(u_0) - f\|e^{-c_1 t}. \quad (**)$$

Here $c_1, r > 0$ are constants.
Examples:

a) $\Phi = -[F'(u)]^{-1}[F(u) - f]$,
b) $\Phi = -[F'(u_0)]^{-1}[F(u) - f]$,
c) $\Phi = -T^{-1}A^*[F(u) - f], \quad A := F'(u), \quad T := A^*A$,
d) $\Phi = -A^*[F(u) - f]$.

a) Newton-type method, b) Modified Newton-type method, c) Gauss-Newton-type method, d) gradient-type method.

Theorem 2. For any linear IP equation:

$$Au - f = 0, \qquad (1)$$

where A is a linear, closed, densely defined operator, and equation (1) is solvable, one can find Φ such that $(*)$ holds,

$$u(t) \underset{t\to\infty}{\longrightarrow} y$$

holds for any u_0, and y is the unique minimal-norm element of the set $N := \{u : Au - f = 0\}$.

For instance, one can take (using a Newton-type method):

$$\Phi = -u + T_{\varepsilon(t)}^{-1} A^* f, \quad T = A^* A, \quad T_\varepsilon = T + \varepsilon I,$$

$$0 < \varepsilon(t) \searrow 0, \quad \int^\infty \varepsilon(s) ds = \infty.$$

For unbounded A the element f may not belong to $D(A^*)$. In this case, the element $T_{\varepsilon(t)}^{-1} A^* f$, with $\varepsilon(t) > 0$, can be defined by considering the closure of the operator $T_{\varepsilon(t)}^{-1} A^*$ with the domain $D(A^*)$. This operator is closable, its closure is a bounded operator, defined on all of H, and

$$||T_{\varepsilon(t)}^{-1} A^*|| \leq \frac{1}{2\sqrt{\varepsilon(t)}}, \quad \varepsilon(t) > 0.$$

It is possible to replace element $T^{-1}_{\varepsilon(t)}A^*f$ by the well defined element $A^*Q^{-1}_{\varepsilon(t)}f$, with

$$Q := AA^*.$$

The operator $A^*Q^{-1}_{\varepsilon(t)}$ is a bounded linear operator defined on all of H, and

$$||A^*Q^{-1}_{\varepsilon(t)}|| \leq \frac{1}{2\sqrt{\varepsilon(t)}}, \quad \varepsilon(t) > 0.$$

These assumptions allow one, among other things, to handle differential operators on unbounded domains in the cases when the spectrum of such operators is continuous and contains the point $\lambda = 0$.

An example.

If $A = A^* \geq m > 0$ and

$$\dot{u} = -(Au - f), \quad u(0) = u_0,$$

then

$$u = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} f ds.$$

One has $\lim_{t\to\infty} ||e^{-tA} u_0|| = 0$ and

$$\lim_{t\to\infty} \int_0^t e^{-(t-s)A} f ds = \lim_{t\to\infty} \int_m^\infty dE_\lambda f (1 - e^{-t\lambda})/\lambda = A^{-1} f = y,$$

where $Ay = f$.

Equations with monotone operators.

Theorem 3. *For any eq.* $F(u) = f$ *with* $F' \geq 0$*, one can find* Φ *such that the conclusion of Theorem 2 holds.*

For example, one may take

$$\Phi = -A_{\varepsilon(t)}^{-1}[F(u) - f + \varepsilon(t)u], \qquad A := F'(u), \quad A_\varepsilon := A + \varepsilon I,$$

$$0 < \varepsilon \searrow 0, \quad \frac{|\dot{\varepsilon}|}{\varepsilon} \leq \frac{1}{2}, \quad \frac{|\dot{\varepsilon}|}{\varepsilon} \to 0 \quad as \quad t \to \infty.$$

Another choice (simple iterations):

$$\Phi = -[F(u) + a(t)u - f]$$

Yet another choice (gradient method):

$$\Phi = -(A^* + a(t)I)[F(u) + a(t)u - f],$$

where $A := F'(u)$.

Noisy data 1

If f_δ is given, $||f_\delta - f|| < \delta$, then one solves the problem:

$$\dot{u}_\delta(t) = -A^{-1}_{\varepsilon(t)}[F(u_\delta(t)) - f_\delta + \varepsilon(t)u_\delta(t)], \quad u_\delta(0) = u_0,$$

sets $u_\delta := u_\delta(t_\delta)$, and finds t_δ from the equation (a discrepancy principle):

$$||F(u_\delta(t_\delta) - f_\delta|| = C\delta^\gamma, \quad C \in (1,2),\ \gamma \in (0,1). \qquad (D)$$

$$||F(u_\delta(t)) - f_\delta|| > C\delta^\gamma, \quad \forall t < t_\delta.$$

Theorem 3'. (a posteriori stopping rule: discrepancy principle)
If $\gamma \in (0,1)$*, and* $||F(u_0) - f_\delta|| > C\delta^\gamma$*, then (D) has a unique solution* t_δ *such that* $t_\delta \to \infty$ *as* $\delta \to 0$*, and* u_δ *converges to* y*, i.e.,* $\lim_{\delta\to 0} ||u_\delta - y|| = 0$*, where* y *is the minimal-norm solution to the eq.* $F(u) = f$*,* $u_\delta := u_\delta(t_\delta)$*.*

Noisy data 2

Theorem 3". (a priori stopping time rule)
If $\lim_{\delta\to 0} \frac{\delta}{\epsilon(t_\delta)} = 0$ *and* $\lim_{\delta\to 0} t_\delta = \infty$, *then* $\lim_{\delta\to 0} ||u_\delta(t_\delta) - y|| = 0$, *where* y *is the minimal-norm solution to the eq.* $F(u) = f$.

Theorem 4. *Consider the equation* $Au = f$. *In Theorem 2 the DSM yields a stable approximation to* y *in the following sense: if* $\|f_\delta - f\| \le \delta$, *and the data are* $\{\delta, f_\delta, A\}$, *then there exists a* t_δ *such that* $\lim_{\delta \to 0} \|u_\delta - y\| = 0$, *where* $u_\delta := u_\delta(t_\delta)$, *and* $u_\delta(t)$ *solves eq. (2) with* f *replaced by* f_δ.

E.g., $\begin{cases} \dot{u}_\delta & = -u_\delta + T^{-1}_{\varepsilon(t)} A^* f_\delta, \\ u_\delta(0) & = u_0, \end{cases}$

where $T := A^*A$, $T_\varepsilon = T + \varepsilon I$.

A priori and a posteriori stopping rules for finding t_δ are found.

Stopping rules

A priori stopping rule (an equation for finding t_δ):

$$\delta^{2q} = \epsilon(t), \quad 0 < q < 1.$$

A posteriori stopping rule (a DP (discrepancy principle)):

$$||Au_\delta(t) - f_\delta|| = C\delta^\gamma, \quad C \in (1,2), \quad \gamma \in (0.9, 1).$$

In both cases the result is

$$\lim_{\delta \to 0} ||u_\delta(t_\delta) - y|| = 0.$$

There is actually *no need to solve the DP equation* for $t = t_\delta$, because the DP equation can be checked as t grows.

New discrepancy principle.

Theorem. *Assume that* A *is a bounded linear operator in a Hilbert space* H, *equation* $Au = f$ *is solvable,* y *is its minimal-norm solution,* $||f_\delta - f|| \leq \delta$, *and* $||f_\delta|| > C\delta$, *where* $C > 1$ *is a constant. Then equation* $||Au_{\delta,\epsilon} - f_\delta|| = C\delta$ $(*)$ *is solvable for* ϵ *for any fixed* $\delta > 0$, *where* $u_{\delta,\epsilon}$ *is any element satisfying inequality* $F(u_{\delta,\epsilon}) \leq m + (C^2 - 1 - b)\delta^2$, $F(u) := ||A(u) - f_\delta||^2 + \epsilon||u||^2$, $m = m(\delta, \epsilon) := inf_u F(u)$, $b = const > 0$, *and* $C^2 > 1 + b$.
If $\epsilon = \epsilon(\delta)$ *solves* $(*)$, *and* $u_\delta := u_{\delta,\epsilon(\delta)}$, *then* $\lim_{\delta \to 0} \|u_\delta - y\| = 0$.

Significance of this result:
One does not need the exact minimizer of the VR functional: an approximate minimizer can be used in the discrepancy principle if it gives to the functional value sufficiently close to the infimum.

Spectral assumption

Assume that the set

$$\{z : |\arg z - \pi| \le \delta < \pi/2, \quad |z| \le \varepsilon_0, \quad \varepsilon_0 = const > 0,$$

consists of the regular points of the operator $A(u) := F'(u)$. Let

$$F(u) + \varepsilon u = f \quad (1),\ F : X \to X, \qquad X \text{ is a Banach space.}$$

(S) Spectral assumption: $\|A_\varepsilon^{-1}\| \le \frac{c}{\varepsilon}, \quad c = const > 0, \quad 0 < \varepsilon < \varepsilon_0,$

$$A = F'(u), \qquad A_\varepsilon := A + \varepsilon I.$$

Solvability theorem

Theorem 6. *If (S) holds and eq. (1) has a solution, then it can be solved by a DSM, that is, (*) holds.*

For example, one can take

$$\Phi = -A_\varepsilon^{-1}(F(u) + \varepsilon u - f).$$

Singular perturbation problem: convergence as $\varepsilon \to 0$.

Theorem 7. *If (S) holds and $F(y) = 0$, then one can choose w such that equation*

$$F(u_\varepsilon) + \varepsilon(u_\varepsilon - w) = 0$$

is solvable for every $\varepsilon \in (0, \varepsilon_0)$, and $\lim_{\varepsilon \to 0} \|u_\varepsilon - y\| = 0$.

Example of the choice of w:

$$y - w = \tilde{A}v, \quad ||v|| < 2M_2 c(1+c)^{-1}, \quad \tilde{A} := F'(y).$$

Unbounded F. Semilinear elliptic problems

If $F(u) = Lu + g(u)$, L is linear, closed, densely defined operator, and $\|L^{-1}\| \le m$, then equation $F(u) = 0$ is equivalent to

$$u + L^{-1}g(u) = 0. \tag{1'}$$

Example (semilinear elliptic problems):

$$L = -\nabla^2, \quad g(u) = u^3, \quad H = L^2(D).$$

Theorem 8. *Assume that*

$$\sup_{u\in B(u_0,R)} \|[I+L^{-1}g'(u)]^{-1}\| \leq m_1(R),$$

and

$$\|u_0+L^{-1}g(u_0)\| m_1(R) \leq R.$$

Then $(*)$ *holds for the problem:*

$$\begin{cases} \dot{u} = -[I+L^{-1}g'(u)]^{-1}[u+L^{-1}g(u)], \\ u(0)=u_0. \end{cases}$$

Theorem 9. *If F is monotone, i.e., $(F(u)-F(v), u-v) \geq 0$, hemicontinuous, $D(F)=H$, and $F(y)=f$, then $(*)$ holds for the problem:*

$$\begin{cases} \dot{u} = -F(u) - \varepsilon(t)u + f, \\ u(0) = u_0, \end{cases}$$

where $0 < \varepsilon(t) \searrow 0$, $\varepsilon(t) = \dfrac{c_1}{(c_0+t)^b}$, $0 < b < 1$, $c_0, c_1 = const > 0$.

Theorem 10. *If*

$$\sup_{R>0} \frac{R}{m(R)} = \infty,$$

then eq. $F(u) = f$ *is solvable for any* $f \in H$.

Theorem 11. *If*

$$\|[F'(u)]^{-1}\| \leq \psi(\|u\|),$$

where ψ *is a continuous positive function, and* $\displaystyle\int_0^\infty \frac{ds}{\psi(s)} = \infty$*,* *then* F *is a global homeomorphism of* H *onto* H*.*

There are many examples of local homeomorphisms which are not global ones. There are examples of global homeomorphisms F for which F' is compact.

Construction of convergent iterative processes.

$$u_{n+1} = u_n + h_n \Phi(t_n, u_n), \quad t_{n+1} = t_n + h_n.$$

Theorem 12. *Any well-posed eq.* $F(u) = 0$ *can be solved by a convergent iterative process with* $h_n = h = const$ *and* $\Phi = \Phi(u)$*. The process converges at an exponential rate.*

Other iterative schemes can be constructed, e.g., Runge-Kutta's type, et al.

Getting rid of the inversion of the derivative.

Assume that $||A^{-1}|| \leq m$. We want to solve an equation:

$$F(u) = 0.$$

$A = F'(u),\ T = A^*A,\ T_\epsilon = T + \epsilon I.$

$$(2') \qquad \begin{cases} \dot{u} & = -QF(u), \\ \dot{Q} & = -TQ + A^*, \\ u(0) & = u_0, \quad Q(0) = Q_0, \end{cases}$$

Theorem 13. *For problem (2') conclusion* $(*)$ *holds.*

Ill-posed problem:

$$(2'') \qquad \begin{cases} \dot{u} & = -Q[A^*F(u) + \varepsilon(t)(u - z)], \\ \dot{Q} & = -T_{\varepsilon(t)}Q + I, \\ u(0) & = u_0, \quad Q(0) = Q_0. \end{cases}$$

Assume: $0 < \varepsilon(t) \searrow 0$, $0 < \dfrac{|\dot{\varepsilon}|}{\varepsilon} \leq c$, $T(y) \neq 0$, $y - z = T(y)v$, $||v||$ is sufficiently small.

Theorem 14. *Under the above assumptions conditions* $(*)$ *hold for problem* $(2'')$.

Theorem 15A. Let

$$\dot{g}(t) \le -\gamma(t)g(t)+\alpha(t, g(t))+\beta(t), \qquad t \ge t_0, \quad \dot{g} = \frac{dg}{dt}, \quad g \ge 0, \tag{1}$$

$0 \le \alpha(t, y)$ is a nondecreasing function of y on $[0, \infty]$ and $\alpha(t, y), \gamma(t), \beta(t)$ are continuous with respect to t on $[t_0, \infty)$. Suppose there exists a function $\mu(t) > 0$, $\mu \in C^1[t_0, \infty)$, such that

$$\alpha\left(t, \frac{1}{\mu(t)}\right) + \beta(t) \le \frac{1}{\mu(t)}\left[\gamma(t) - \frac{\dot{\mu}(t)}{\mu(t)}\right], \qquad t \ge t_0. \tag{2}$$

Let

$$\mu(t_0)g(t_0) < 1.$$

Then $g(t)$ exists globally and the following estimate holds:

$$0 \leq g(t) < \frac{1}{\mu(t)}, \qquad \forall t \geq t_0.$$

How does one apply this inequality?

$$\dot{u} = f(t,u) = A_1(t)u + A_2(t,u) + A_0(t).$$

$$||u(t)|| := g(t), \qquad ||A_0(t)|| \leq \beta(t), \qquad (A_1 u, u) \leq -\gamma(t) g^2,$$

$$(A_2, u) \leq \alpha(t,g) g.$$

Now we get the basic inequality:

$$\dot{g} \leq -\gamma(t) g + \alpha(t,g) + \beta(t).$$

The choice of $\mu(t)$ is often easy.

How does one apply this inequality to stability theory?

Lyapunov stability known result says: if

$$u' = f(u), \quad u(0) = u_0, \quad (1)$$

$u \in \mathbb{R}^n$, $A = f'(0)$, $(Au, u) \leq -a\|u\|^2$, $a > 0$,
$\|f(u) - Au\| \leq c_2\|u\|^2$, then the solution to (1) is exponentially stable.
Our theory allows one to get this and new results by using Theorem 15A. Let $g(t) := \|u\|$. From (1) we get

$$g' \leq -ag + c_2 g^2.$$

Let $\mu = \mu_0 e^{bt}, b < a$. Conditions of Theorem 15A are:
$c_2/\mu \leq (a - b)$, $\mu_0 g(0) < 1$. These inequalities are satisfied if $c_2/\mu_0 \leq a - b$, $g(0) < 1/\mu_0$. Thus, by Theorem 15A, the solution to (1) exists for all $t \geq 0$ and $\|u(t)\| \leq ce^{-bt}$ for any $b < a$.

Discrete version.

Let

$$\frac{g_{n+1}-g_n}{h_n} \leq -\gamma_n g_n + \alpha(n, g_n) + \beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1,$$

so

$$g_{n+1} \leq (1-\gamma_n h_n)g_n + h_n\alpha(n, g_n) + h_n\beta_n, \quad n \geq 0, \qquad 0 < \gamma_n h_n < 1,$$

holds, where g_n, β_n and γ_n are positive sequences of real numbers.

. Assume that

$$\frac{g_{n+1}-g_n}{h_n} \leq -\gamma_n g_n + \alpha(n, g_n) + \beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1, \tag{3}$$

or, equivalently,

$$g_{n+1} \leq g_n(1 - h_n\gamma_n) + h_n\alpha(n, g_n) + h_n\beta_n, \qquad h_n > 0, \tag{4}$$

where $0 < h_n\gamma_n < 1$.
If there is a sequence of positive numbers $(\mu_n)_{n=1}^{\infty}$, such that the following conditions hold:

$$\alpha(n, \frac{1}{\mu_n}) + \beta_n \le \frac{1}{\mu_n}\left(\gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_n h_n}\right), \tag{5}$$

$$g_0 \le \frac{1}{\mu_0}, \tag{6}$$

then

$$0 \le g_n \le \frac{1}{\mu_n} \qquad \forall n \ge 0. \tag{7}$$

Therefore, if $\lim_{n\to\infty} \mu_n = \infty$, then $\lim_{n\to\infty} g_n = 0$.
Remark. The result holds with $h_n = 1$ and $0 < \gamma_n < 1$.

Theorem 16. If $Q(t)$, $G(t)$ and $T(t)$ are linear operator-functions from $[0,\infty) \to H$, where H is a Hilbert space, and

$$\begin{cases} \dot{Q} & = -T(t)Q + G(t), \\ Q(0) & = Q_0, \end{cases}$$

where $(Th,h) \geq \varepsilon(t)\|h\|^2$, $\varepsilon(t) \geq 0$, then, with $a(t) := e^{\int_0^t \varepsilon(s)ds}$, one has:

$$\|Q(t)\| \leq a^{-1}(t)\,\|Q_0\| + a^{-1}(t) \int_0^t a(s)\|G(s)\|ds.$$

$$F(u) = 0 \quad (1), \quad \begin{cases} \dot{u} & = \Phi(t, u), \\ u(0) & = u_0. \end{cases} \quad (2)$$

Theorem 17. If

1) $(F'\Phi, F) \le -c_1 \|F\|^2$, $\forall u \in H$, $c_1 = const > 0$
2) $\|\Phi\| \le c_2 \|F\|$,
3) $r \le R$,

where

$r := \frac{c_2}{c_1} \|F_0\|. \qquad F_0 = F(u_0),$

then $(*)$ and $(**)$ hold, where

$$(**) \qquad \|u(t) - u(\infty)\| \le re^{-c_1 t}, \quad \|F(u(t))\| \le \|F_0\| e^{-c_1 t}.$$

Proof. Let $g(t) := \|F(u(t))\|$.
Then
$g\dot{g} = (F'\Phi, F) \leq -c_1 g^2$.
Thus

$$g(t) \leq g(0)e^{-c_1 t} = \|F_0\|e^{-c_1 t},$$

$$\|\dot{u}\| \leq \|\Phi\| \leq c_2\|F_0\|e^{-c_1 t}.$$

So, with $r := \frac{c_2}{c_1}||F_0||$, $r \leq R$, one gets:
$\|u(t) - u(\infty)\| \leq re^{-c_1 t}$,
$\|u(t) - u(0)\| \leq r \leq R$.
Theorem 17 is proved. $\square$

Examples

a) $\Phi = -[F'(u)]^{-1}F \Rightarrow c_1 = 1,\ c_2 = m,\ \boxed{m\|F_0\| \leq R.}$

b) $\Phi = -[F'(u_0)]^{-1}F \Rightarrow c_2 = m,$
$-((F'(u) - F'(u_0) + F'(u_0))[F'(u_0)]^{-1}F, F) \leq$
$-\|F\|^2 + mM_2R\|F\|^2,$
$c_1 = 1 - mM_2R, \quad c_2 = m, \quad m\|F_0\|/(1 - mM_2R) \leq R.$
If $R = \frac{1}{2mM_2}$ then $\boxed{4m^2M_2\|F_0\| \leq 1.}$

c) $\Phi = -T^{-1}A^*F \Rightarrow c_1 = 1,\ c_2 = m^2M_1,\ \boxed{m^2M_1\|F_0\| \leq R.}$

d) $\Phi = -A^*F \Rightarrow c_1 = m^{-2},\ c_2 = M_1,\ \boxed{m^2M_1\|F_0\| \leq R.}$

Linear Ill-posed Problems.

$$\begin{cases} \dot{u} &= -u + T^{-1}_{\varepsilon(t)} A^* f, \quad T = A^*A. \\ u(0) &= u_0 \end{cases}$$

$$0 < \varepsilon(t) \searrow 0, \ \int^{\infty} \varepsilon ds = \infty.$$

$$u = u_0 e^{-t} + \int_0^t e^{-(t-s)} T^{-1}_{\varepsilon(s)} T y \, ds$$

Lemma 1. $\lim\limits_{t\to\infty} \int_0^t e^{-(t-s)} h(s) ds = h(\infty)$ (if $\exists h(\infty)$.)

Lemma 2. $\lim\limits_{\varepsilon\to 0} T_\varepsilon^{-1} T y = y$ if $y \perp N(T) = N(A)$.

Otherwise the limit is $y - P_{N(T)} y$.

Stopping rules 1.

Assume $\|f_\delta - f\| \le \delta$. Then

$$\|u_\delta(t_\delta) - y\| \le \|u_\delta(t_\delta) - u(t_\delta)\| + \|u(t_\delta) - y\|.$$

$$\lim_{t_\delta \to \infty} \|u(t_\delta) - y\| = 0$$

$$\|u_\delta(t_\delta) - u(t_\delta)\| \le \| \int_0^{t_\delta} e^{-(t_\delta - s)} T^{-1}_{\varepsilon(s)} A^*(f_\delta - f)\|$$

$$\le \frac{\delta}{2\sqrt{\varepsilon(t_\delta)}}$$

Rule 1: If $\lim_{\delta \to 0} \dfrac{\delta}{\sqrt{\varepsilon(t_\delta)}} = 0$ and $\lim_{\delta \to 0} t_\delta = \infty$,

then $\lim_{\delta \to 0} \|u_\delta(t_\delta) - y\| = 0$.

New discrepancy principle.

Theorem. Assume that A is a bounded linear operator in a Hilbert space H, equation $Au = f$ is solvable, y is its minimal-norm solution, $||f_\delta - f|| \leq \delta$, and $||f_\delta|| > C\delta$, where $C > 1$ is a constant. Then equation $||Au_{\delta,\epsilon} - f_\delta|| = C\delta$ $(*)$ is solvable for ϵ for any fixed $\delta > 0$, where $u_{\delta,\epsilon}$ is any element satisfying inequality $F(u_{\delta,\epsilon}) \leq m + (C^2 - 1 - b)\delta^2$, $F(u) := ||A(u) - f_\delta||^2 + \epsilon||u||^2$, $m = m(\delta, \epsilon) := inf_u F(u)$, $b = const > 0$, and $C^2 > 1 + b$. If $\epsilon = \epsilon(\delta)$ solves $(*)$, and $u_\delta := u_{\delta,\epsilon(\delta)}$, then $\lim_{\delta\to 0} \|u_\delta - y\| = 0$.

The point: One does not need the exact minimizer of the VR functional: an approximate minimizer can be used in the discrepancy principle if it gives to the functional value sufficiently close to the infimum.

Nonlinear operator equations with monotone operators.

$$\dot{u} = -A^{-1}_{a(t)}[F(u) + a(t)u - f], \quad u(0) = u_0.$$

$$z := F(u) + a(t)u - f,$$

$$\dot{z} = -z + \dot{a}u = -z + \frac{\dot{a}}{a}a(u - V) + \dot{a}V.$$

$$F(V) + aV - f = 0. \qquad ||z|| := g, \qquad u - V := h, \qquad a||h|| \leq g.$$

$$\dot{g} \leq -g(1 - \frac{|\dot{a}|}{a}) + c|\dot{a}|, \qquad c = max_{t\geq 0}||V||.$$

$$\lim_{t\to\infty} V(t) = y.$$

We prove:

$$\lim_{t\to\infty} g = 0, \qquad lim_{t\to\infty}\frac{g}{a} = 0.$$

Global convergence theorem.

Theorem. If F is a monotone, continuously Fréchet differentiable operator in H, equation $F(y) = f$ has a solution, y is its (unique) minimal-norm solution, and $a(t) > 0$ is a monotonically decaying function such that

$$\lim_{t\to\infty} a(t) = 0, \qquad \lim_{t\to\infty} \frac{|\dot{a}|}{a} = 0, \qquad \frac{|\dot{a}|}{a} < 1/2,$$

then

$$\lim_{t\to\infty} u(t) = y.$$

The convergence is global: it holds for any initial element u_0.

Nonlinear operator equations with monotone operators/earlier version.

$$\dot{u} = -A_{a(t)}^{-1}[F(u) + a(t)u], \quad u(0) = u_0.$$

Assume that $a(t) > 0$ decays monotonically to zero as $t \to \infty$, $|\dot{a}|/a < 1/2$, and $|\dot{a}|/a^2 \leq 1$.
Let $F(v) + a(t)v = 0$. This eq. is uniquely solvable, and we prove:

$$||v|| \leq ||y||, \quad ||\dot{v}|| \leq ||y|||\dot{a}|/a, \quad ||v(t) - y|| \to 0,$$

as $t \to \infty$.
We want to prove that $u(t)$ exists on $[0, \infty)$ and $||u(t) - y|| \to 0$ as $t \to \infty$.
It is sufficient to prove $||w(t)|| \to 0$ as $t \to \infty$, where $w := u(t) - v(t)$.

One has:

$$\dot{w} = -\dot{v} - A^{-1}_{a(t)}[F(u) - F(v) + a(t)w].$$

Let $g = g(t) := ||w||$. Then one derives the inequality:

$$\dot{g} \leq -g + \frac{c_0}{a(t)} g^2 + c_1 \frac{|\dot{a}|}{a(t)}.$$

Choose $\mu = \frac{c}{a(t)}$, $c = const > 0$, and check conditions of the basic lemma.

$$c_1 \frac{|\dot{a}|}{a(t)} + \frac{c_0}{a(t)} \mu^{-2} \leq \mu^{-1}(1 - \frac{\dot{\mu}}{\mu}),$$

$$g(0)\mu(0) < 1.$$

These inequalities are satisfied if $\mu = ca^{-1}(t)$ and $a := a(t)$ is chosen so that

$$cg(0)a^{-1}(0) < 1,$$

and

$$c_1 c \frac{|\dot{a}|}{a^2(t)} + c_0 c^{-1} a^2(t) + \frac{|\dot{a}|}{a(t)} \leq 1.$$

Clearly, there are many $a(t)$ which satisfy the above inequalities. If $a(t)$ satisfies these inequalities, then

$$g < \frac{a(t)}{c}$$

and

$$\lim_{t\to\infty} u(t) = y.$$

In the earlier papers I took $\mu = \frac{\lambda}{a(t)}, \quad \lambda = const > 0$, and checked the conditions of the earlier lemma. First condition:

$$\frac{c_0}{a(t)} \leq \frac{\lambda}{2a(t)}(1 - 1/2) = \frac{\lambda}{4a(t)}.$$

This holds if $\lambda = 4c_0$.
Second condition:

$$c_1|\dot{a}|/a \leq \frac{a}{4\lambda}.$$

The scaling transformation: $a \to \nu a$ allows one to satisfy the above inequality. Here $\nu > 0$ is a constant.
Third condition: $g(0)\mu(0) < 1$ holds if ν is sufficiently large.
We have proved that $||w|| < a(t)/\lambda$. Thus $||u(t) - y|| \to 0$ as $t \to \infty$.

Nonlinear operator equations without monotonicity assumptions.

Theorem. If $\widetilde{A} := F'(y) \neq 0$, then for the problem

$$\begin{cases} \dot{u} & = -T^{-1}_{\varepsilon(t)}(A^*F + \varepsilon(u-z)), \\ u(0) & = u_0, \end{cases}$$

conclusions $(*)$ hold, where z is suitably chosen.

Proof.

$$u-y=w, \|w\|=g,\ F(u)-F(y)=Aw+K,$$

$$\|K\|\leq \frac{M_2}{2}g^2,\ u-z=u-y+y-z$$

$$\dot{w}=-T_\varepsilon^{-1}(A^*Aw+\varepsilon w+A^*K+\varepsilon(y-z))$$

$$=-w-T_\varepsilon^{-1}A^*K-\varepsilon T_\varepsilon^{-1}(y-z)$$

$$\dot{w}=-w-T_\varepsilon^{-1}A^*K-\varepsilon T_\varepsilon^{-1}\widetilde{T}v,\ \|v\|\ll 1;$$

$$\widetilde{T}v=y-z \text{ if } \widetilde{T}\neq 0.$$

$$g\dot{g}\leq -g^2+\frac{c_0g^3}{\sqrt{\varepsilon(t)}}+\varepsilon(T_\varepsilon^{-1}-\widetilde{T}_\varepsilon^{-1}+\widetilde{T}_\varepsilon^{-1})\widetilde{T}v.$$

$$\varepsilon\|\widetilde{T}_\varepsilon^{-1}\widetilde{T}v\|\leq\varepsilon\|v\|,$$

$$\varepsilon\|T_\varepsilon^{-1}(A^*A-\widetilde{A}^*A)\widetilde{T}_\varepsilon^{-1}\widetilde{T}\|\ \|v\|,$$

$$\leq 2M_2M_1g\|v\|;\ 2M_1M_2\|v\|=\frac{1}{2}.$$

Thus

$$\dot g \le -\frac{1}{2} g + \frac{c_0 g^2}{\sqrt{\varepsilon(t)}} + \varepsilon \|v\|; \quad \mu = \frac{\lambda}{\sqrt{\varepsilon(t)}}, \quad \frac{\dot\mu}{\mu} = \frac{1}{2} \frac{|\dot\varepsilon|}{\varepsilon} \le \frac{1}{4.}$$

1) $$\frac{c_0}{\sqrt{\varepsilon(t)}} \le \frac{\lambda}{2\sqrt{\varepsilon}} \, \frac{1}{4}; \quad \lambda = 8c_0.$$

2) $$\varepsilon(t)\|v\| \le \frac{\sqrt{\varepsilon(t)}}{2\lambda} \, \frac{1}{4}; \qquad 8\lambda\|v\| \sqrt{\varepsilon(t)} \le 1$$

3) $$g(0)\frac{\lambda}{\sqrt{\varepsilon(0)}} < 1$$

If $\varepsilon(0) > 8g(0)c_0$, then condition 3) holds.
If $\|v\| < \frac{1}{8\lambda\sqrt{\varepsilon(0)}}$, then 2) holds.
Theorem is proved. $\square$

Getting rid of the inversion of the derivative operator.

$F(u) = 0 \quad (1) \qquad \|[F'(u)]^{-1}\| \le m, \quad F(y) = 0.$

Theorem. *Let*

$$\begin{cases} \dot{u} & = -QF, \qquad\qquad (2) \\ \dot{Q} & = -TQ + A^*, \quad u(0) = u_0, \quad Q(0) = Q_0. \end{cases}$$

If u_0 *and* Q_0 *are properly chosen then* $(*)$ *holds, and* $\lim_{t\to\infty} ||Q(t) - \tilde{A}^{-1}|| = 0.$

Here, as earlier, $T := A^*A$, $A := F'(u)$, $\tilde{A} := F'(y)$.

Proof. $T \geq \varepsilon > 0$, $\varepsilon = const$. Thus

$$\|Q(t)\| \leq \|Q_0\|e^{-\varepsilon t} + e^{-\varepsilon t}\int_0^t e^{\varepsilon s} M_1 ds$$
$$\leq \|Q_0\| + \frac{M_1}{\varepsilon} := c_0.$$
$$u - y = w, \|w\| = g(t);\ F(u) - F(y) = \widetilde{A}w + K,$$
$$\widetilde{A} = F'(y), \|K\| \leq \frac{M_2}{2}g^2,$$
$$\dot{w} = -w + w - Q(F(u) - F(y)) = -w + \Lambda w - QK,$$
$$\Lambda = I - Q\widetilde{A}.$$
$$g\dot{g} \leq -g^2 + (\Lambda w, w) + c_1 g^3, \qquad c_1 = \frac{c_0 M_2}{2},$$

Lemma.

Lemma:

$$|(\Lambda w, w)| \leq q\|w\|^2, \quad 0 < q < 1.$$

Assume:

$$\dot{g} \leq -\gamma g + c_1 g^2, \; 0 < \gamma := 1 - q < 1, \quad c_1 g(0) < 1.$$

Then

$$g(t) \leq c_2 e^{-\gamma t}, \; c_2 = \frac{g(0)}{1 - c_1 g(0)}, \quad g(0) = ||u_0 - y||.$$

Theorem is proved. $\square$

Proof of the Lemma.

$$\dot{\Lambda} = -\dot{Q}\widetilde{A} = TQ\widetilde{A} - A^*\widetilde{A} = -T\Lambda + A^*(A - \widetilde{A}).$$

$$\|\Lambda\| \le \|\Lambda_0\|e^{-\varepsilon t} + e^{-\varepsilon t}\int_0^t e^{\varepsilon s} M_1 M_2 c_2 e^{-\gamma s} ds$$
$$\le \|\Lambda_0\| + c_3\|u_0 - y\| < q < 1,$$

provided that $\Lambda_0 = I - Q_0\tilde{A}$ and $\|u_0 - y\|$ are sufficiently small.
The Lemma is proved. $\square$

Iterative processes.

$$F(u) = 0 \tag{1}$$

$$u_{n+1} = u_n + h\Phi(u_n) \tag{2}$$

there exists y such that $F(y) = 0$, and

$$\|\Phi(u) - \Phi(v)\| \leq L_2\|u - v\|.$$

Theorem. If

a) $(F'\Phi, F) \leq -c_1\|F\|^2$,

b) $\|\Phi\| \leq c_2\|F\|$, and

c) $r \leq R$,

where
$r = \frac{c_2}{c_1}\|F_0\|$ and $F_0 = F(u_0)$,
then
$\|u_n - y\| \leq re^{-chn}$, $\|F(u_n)\| \leq \|F_0\|e^{-chn}$,
where $0 < c < c_1$.

Proof.

Let $\begin{cases} \dot{w}_{n+1}(t) & = \Phi(w_{n+1}) \\ w_{n+1}(t_n) & = u_n, \ t_n = hn. \end{cases}$

Then

$$\begin{aligned} \|w_{n+1}(t) - y\| &\leq \frac{c_2}{c_1}\|F_n\|e^{-c_1(t-t_n)} \\ &\leq re^{-chn-c_1(t-t_n)}, \quad t > hn, \end{aligned}$$

$$\|u_{n+1} - y\| \leq \|u_{n+1} - w_{n+1}\| + \|w_{n+1} - y\|,$$

$$\begin{aligned} \|u_{n+1} - w_{n+1}\| &\leq \int_{t_n}^{t_n+h} \|\Phi(u_n) - \Phi(w_{n+1}(s))\|ds \\ &\leq L_1 \int_{t_n}^{t_n+h} \|u_n - w_{n+1}(s)\|ds \\ &\leq L_1 h \int_{t_n}^{t_n+h} \|\Phi(w_{n+1}(s))\|ds \\ &\leq L_1 h c_2 \int_{t_n}^{t_n+h} \|F(w_{n+1}(s))\|ds \\ &\leq L_1 h^2 c_2 \|F(u_n)\| \leq L_1 c_2 h^2 \|F_0\| e^{-chn} \\ &= L_1 c_1 h^2 r e^{-chn}. \end{aligned}$$

Thus $\|u_{n+1} - y\| \leq re^{-chn}(e^{-c_1 h} + L_1 c_1 h^2) \leq re^{-ch(n+1)}$ provided that
$c < c_1$ and h is such that $e^{-c_1 h} + L_1 c_1 h^2 < e^{ch}$.

$$\begin{aligned}
\|F(u_{n+1})\| &\leq \|F(u_{n+1}) - F(w_{n+1}(t_{n+1}))\| \\
&\quad + \|F(w_{n+1}(t_{n+1}))\|; \\
\|F(w_{n+1}(t_{n+1}))\| &\leq \|F(u_n)\| e^{-c_1 h} \leq \|F_0\| e^{-chn - c_1 h}; \\
\|F(u_{n+1}) - F(w_{n+1}(t_{n+1}))\| &\leq M_1 L_1 c_1 h^2 r e^{-chn} \\
&= \|F_0\| e^{-chn} M_1 L_1 c_2 h^2.
\end{aligned}$$

Thus

$$\begin{aligned}
\|F(u_{n+1})\| &\leq \|F_0\| e^{-chn}(e^{c_1 h} + M_1 L_1 c_2 h^2) \\
&\leq \|F_0\| e^{-ch(n+1)}.
\end{aligned}$$

Theorem is proved. $\square$

Proof of Theorem 15A

Lemma. *Let* $f(t, w)$, $g(t, u)$ *be continuous in the region* $[0, T) \times D$ *(*$D \subset R$, $T \leq \infty$*) and* $f(t, w) \leq g(t, u)$ *if* $w \leq u$, $t \in (0, T)$, $w, u \in D$. *Assume that* $g(t, u)$ *is such that the Cauchy problem*

$$\dot{u} = g(t, u), \quad u(0) = u_0, \quad u_0 \in D,$$

has a unique solution. If

$$\dot{w} \leq f(t, w), \quad w(0) = w_0 \leq u_0, \quad w_0 \in D,$$

then $u(t) \geq w(t)$ *for all* t *for which* $u(t)$ *and* $w(t)$ *are defined.*

Proof of Theorem 15A

Inequality in Theorem 15A can be written as

$$\frac{d(1/\mu)}{dt} \geq -\gamma(t)(1/\mu) + \alpha(t, 1/\mu) + \beta(t),$$

and

$$0 \leq g(t_0) \leq 1/\mu(t_0).$$

Thus, $1/\mu(t)$ is an upper solution to the Cauchy problem

$$\dot{w} = -\gamma(t)w + \alpha(t, w) + \beta(t), \qquad w(t_0) = g_0,$$

while $g(t)$ is its lower solution. If the above Cauchy problem has at most one solution, then

$$0 \leq g(t) \leq 1/\mu(t) \qquad \forall t \geq t_0.$$

Theorem 15A is proved. $\square$

Results for a numerical experiment from [15].
$\int_0^t k(t-s)u(s)ds = f(t)$, $k(t) = (2\pi^{1/2})^{-1}t^{-1.5}e^{-\frac{1}{4t}}$.

Table: Numerical results for $\delta_{rel} = 0.05$, $n = 10i$, $i = \overline{1, 10}$.

n	DSM n_{iter}	DSM $\frac{\|u_\delta - y\|_2}{\|y\|_2}$	VR$_i$ n_{iter}	VR$_i$ $\frac{\|u_\delta - y\|_2}{\|y\|_2}$	VR$_n$ n_{iter}	VR$_n$ $\frac{\|u_\delta - y\|_2}{\|y\|_2}$
10	3	0.1971	1	0.2627	5	0.2117
20	4	0.3359	1	0.4589	5	0.3551
30	4	0.3729	1	0.4969	5	0.3843
40	4	0.3856	1	0.5071	5	0.3864
50	5	0.3158	1	0.4789	6	0.3141
60	6	0.2892	1	0.4909	6	0.3060
70	7	0.2262	1	0.4792	8	0.2156
80	6	0.2623	1	0.4809	7	0.2600
90	5	0.2856	1	0.4816	7	0.2715
100	7	0.2358	1	0.4826	7	0.3405

Bibliography

[1] A. G. Ramm, **Dynamical Systems Method (DSM) for solving operator equations**, Elsevier, Amsterdam, 2007.

[2] ________, Dynamical systems method for solving nonlinear operator equations, International Jour.
of Applied Math. Sci., 1, N1, (2004), 97-110.

[3] ________, Dynamical systems method for solving operator equations, Communic. in Nonlinear Sci. and Numer. Simulation, 9, N2, (2004), 383-402.

[4] ________, Dynamical systems method and surjectivity of nonlinear maps, Communic. in Nonlinear Sci. and Numer. Simulation, 10, N8, (2005),931-934.

[5] ________, DSM for ill-posed equations with monotone operators, Comm. in Nonlinear Sci. and Numer. Simulation, 10, N8, (2005),935-940.

[6] ________, Dynamical systems method (DSM) for unbounded operators, Proc. Amer. Math. Soc., 134, N4, (2006), 159-1063.

[7] ________, Discrepancy principle for the dynamical systems method, Communic. in Nonlinear Sci. and Numer. Simulation, 10, N1, (2005), 95-101.

[8] ________, Dynamical systems method and a homeomorphism theorem, Amer. Math. Monthly,113, N10, (2006), 98-933.

[9] ________, Dynamical systems method for nonlinear equations in Banach spaces, Communic. in Nonlinear Sci. and Numer. Simulation, 11, (2006), 306-310.

[10] ________, Dynamical systems method (DSM) and nonlinear problems, in the book: Spectral Theory and Nonlinear Analysis, World Scientific Publishers, Singapore, 2005. , pp. 201-228 (ed J. Lopez-Gomez).

[11] ________, **Inverse problems**, Springer, New York, 2005.

[12] ________, Dynamical systems and discrete methods for solving nonlinear ill-posed problems, Appl. Math. Reviews, vol. 1, Ed. G. Anastassiou, World Sci. Publishers, 2000, pp. 491-536. isbn 981-02-4339-1. (with R. Airapetyan)

[13] ________, Solving ill-conditioned linear algebraic systems by the dynamical systems method (DSM), Inverse Problems in Sci. and Engineering,16, N5, (2008), 617-630.
(with N.S.Hoang)

[14] ________, Dynamical systems gradient method for solving ill-conditioned linear algebraic systems, Acta Appl. Math., (2010), (with N.S. Hoang)

[15]________, Dynamical systems method for solving linear finite-rank operator equations, Ann. Polon. Math., 95, N1, (2008), 77-93.
(with N.S. Hoang).

[16]________, Dynamical systems method for solving nonlinear equations with monotone operators, Math. of Computations, 79, 269, (2010), 239-258.

(with N.S. Hoang).

[17] ________, Convergence rates of the continuous regularized Gauss-Newton method, Jour. Inv. Ill-Posed Probl., 10, N3, (2002), 261-280. (with B. Kaltenbacher and A. Neubauer).

[18]________, Global convergence for ill-posed equations with monotone operators: the dynamical systems method, J.Phys. A, 36, (2003), L249-254.

[19]________, On stable numerical differentiation, Australian J. Math. Anal. Appl., 5, N1, (2008), Article 5, pp.1-7. (with Hoang. N.S)

[20]________, Dynamical Systems Method (DSM) for solving equations with monotone operators without smoothness assumptions on $F'(u)$, J. Math. Anal. Appl., 367, N2, (2010), 508-515. (with N.S.Hoang)

[21]________, The Dynamical Systems Method for solving nonlinear equations with monotone operators, Asian Europ. Math. Journ., 3, N1, (2010), 57-105. (with N.S.Hoang)

[22]________, DSM of Newton-type for solving operator equations $F(u) = f$ with minimal smoothness assumptions on F, International Journ. Comp.Sci. and Math. (IJCSM), 3, N1/2, (2010), 3-55. (with N.Hoang)

[23]________, Justification of the Dynamical Systems Method (DSM) for global homeomorphisms, Eurasian Math. Journ (EMJ), 1, N4, (2010), 116-123.

[24]________, **Dynamical systems method and applications. Theoretical developments and numerical experiments**, Wiley, Hoboken, 2012. (with N.Hoang)
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