DYNAMICAL SYSTEMS METHOD (DSM) FOR SOLVING OPERATOR EQUATIONS

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Abstract

Consider an operator equation $F(u) = 0$ in a Hilbert space H or in a Banach space X. Assume that this equation is solvable, possibly non-uniquely. Let us call the problem of solving this equation ill-posed (IP) if the operator $F'(u)$ is not boundedly invertible, and well-posed (WP) otherwise. This terminology differs from the standard (Hadamard's one). A general method, Dynamical Systems Method (DSM), for solving nonlinear and linear problems, especially ill-posed problems in H, is presented. This method consists of the construction of a dynamical system, that is, a Cauchy problem,

$$\dot{u} = \Phi(t, u), \quad u(0) = u_0,$$

which has the following three properties:

1) it has a unique global solution, i.e., its solution is defined on $[0, \infty)$,

2) there exists the limit $\lim_{t\to\infty} u(t) = u(\infty)$,

and

3) $F(u(\infty)) = 0$.

The choices of $\Phi(t, u)$ are proposed and the DSM is justified for wide classes of operator equations.

a) arbitrary solvable linear equations of the form $Au = f$ with densely defined closed linear operator A,

b) for any well-posed nonlinear equations with twice Fréchet differentiable operator F,

c) for ill-posed nonlinear equations with monotone operators,

d) for ill-posed nonlinear equations with non-monotone operators such that $F'(y) \neq 0$, where $F(y) = 0$,

d) for operators such that $A := F'(u)$ satisfies the spectral assumption:

$$||(A + sI)^{-1}|| \leq c/s,$$

where $c > 0$ is a constant, and $s \in (0, s_0), s_0 > 0$ is a fixed number, arbitrarily small, c does not depend on s and u,

e) for monotone operators which are not Fréchet differentiable, but only hemicontinuous and defined on all of H,

k) in Newton-type schemes the main difficulty is to invert the derivative of the operator. A novel scheme, based on the DSM, allows one to avoid this inversion.

l) universality of the Newton's method is established in the following sense: if $F(y) = 0$, $||[F'(y)]^{-1}|| \leq m$, $m = const > 0$, and $||F'(u) - F'(v)|| \leq \omega(||u - v||)$ for all $u, v \in B(y, R)$, $B(y, R) := \{u : ||u - y|| \leq R\}$, where $\omega(r) \geq 0$ is a continuous, strictly monotone growing function, $\omega(0) = 0$, $m\omega(R) < 0.5$, then for any $w \in B(y, R)$ the Newton method $u_{n+1} = u_n - [F'(u_n)]^{-1}F(u_n),\ u_0 = w$, converges to y, and the DSM Newton method $\dot{u} = -[F'(u)]^{-1}F(u),\ u(0) = w$ converges to y in the sense that the above conclusions 1)-3) hold.

A global convergence theorem is obtained for the regularized continuous analog of Newton's method for monotone operators. Global convergence means that convergence is established for an arbitrary initial approximation, not necessarily the one which is sufficiently close to the solution.

A general approach to constructing convergent iterative schemes for solving well-posed nonlinear operator equations is described and convergence theorems are obtained for such schemes.

Stopping rules for stable solution of ill-posed problems with noisy data are given.

DSM can be used for proving theoretical results, e.g.,

i) sufficient conditions for a nonlinear map to be a global homeomorphism;

ii) a hard implicit function theorem.

Theoretical basis for the DSM is developed in monograph [9], and many numerical examples of applications of the DSM are included in the monograph [25] joint with my student, Professor N.S.Hoang. In [25] many theoretical and numerical results on various versions of the discrepancy principle are given.

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