Approximating the Navier–Stokes equations on $\mathbb{R}^3$ with large periodic domains

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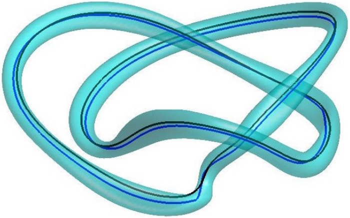
Motivation

Experiments that generate an initial vorticity ($\omega = \operatorname{curl} u$) knotted in a trefoil (Scheeler et al.; PNAS, 2014).



Mathematical model: the Navier–Stokes equations on $\mathbb{R}^3$ with compactly-supported initial vorticity.

But numerical computations (e.g. Kerr; JFM, 2018) use periodic boundary conditions.

Heuristic idea: 'computations in a large enough periodic domain mimic solutions on $\mathbb{R}^3$'.

Consider the Navier–Stokes equations

$$\partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \qquad \nabla \cdot u = 0$$

on the periodic domain $Q_\alpha := (-\alpha, \alpha)^3$ with $u(0) = u_\alpha^0 \in \dot{H}^1_\sigma(Q_\alpha)$.

The dot $\dot{}$ means zero average, i.e. $\displaystyle\int_{Q_\alpha} u_\alpha^0 = 0$.

The σ subscript means divergence free.

Questions

1. If $u_\alpha^0 \to u^0 \in \dot{H}^1_\sigma(\mathbb{R}^3)$, does the solution u_α converge to u?

2. If $u^0 \in C_c^\infty(\mathbb{R}^3)$ gives rise to a smooth solution on $\mathbb{R}^3$ for $t \in [0, T]$, does the same hold on Q_α when α is large enough?

The answer 'YES' to 1. is 'semi-classical' (Heywood, 1988); but the answer YES to 2. depends on sufficiently strong convergence of $u_\alpha \to u$, which is what we will discuss here.

Rescaling & inequalities

Given f_α defined on Q_α, the rescaled function $f(x) = f_\alpha(\alpha x)$ is defined on Q_1 and

$$\|\partial^\gamma f_\alpha\|_{L^p(Q_\alpha)} = \alpha^{(3/p)-k}\|\partial^\gamma f\|_{L^p(Q_1)}, \qquad \text{when} \qquad |\gamma| = k.$$

This can be used to show that certain constants do not depend on α.

Sobolev inequalities:

$$\|u\|_{L^6(Q_\alpha)} \le C\|\nabla u\|_{L^2(Q_\alpha)} \qquad [3/6 = 1/2 = 3/2 - 1]$$

$$\|u\|_{L^\infty(Q_\alpha)} \le C_A\|\nabla u\|_{L^2(Q_\alpha)}^{1/2}\|\Delta u\|_{L^2(Q_\alpha)}^{1/2} \qquad [0 = (1/2)/2 + (-1/2)/2]$$

Calderon–Zygmund: if $-\Delta p = \nabla \cdot [(u \cdot \nabla)u]$ with $\int_{Q_\alpha} p = \int_{Q_\alpha} u = 0$

$$\|p\|_{L^2(Q_\alpha)} \le C_Z\|u\|_{L^4(Q_\alpha)}^2 \qquad [3/2 + 2 = 2 \times (3/4 + 1)].$$

[Rescaling here is $p(x) := \alpha^2 p_\alpha(\alpha x)$ and $u(x) := \alpha u_\alpha(\alpha x)$.]

Reconstructing the velocity from the vorticity

To solve

$$\operatorname{curl} u = \omega, \qquad \nabla \cdot u = 0;$$

take the curl of the first equation to yield

$$-\Delta u = \operatorname{curl} \omega \qquad \Rightarrow \qquad u = (-\Delta)^{-1} \operatorname{curl} \omega.$$

On $\mathbb{R}^3$ we have an explicit solution

$$u = \operatorname{curl}^{-1} \omega := -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \times \omega(y) \, \mathrm{d}y \tag{1}$$

(Biot–Savart Law). On Q_α we can write an explicit solution in terms of Fourier series (it is better to use an integral formulation like (1)).

Young's convolution inequality yields bounds on u: since $\frac{x}{|x|^3} \in L^{3/2,\infty}$, if $\omega \in L^p$, $p \in (1, 3)$, then

$$\|u\|_{L^q} \le C_p \|\omega\|_{L^p}, \qquad \frac{1}{q} = \frac{1}{p} - \frac{1}{3}.$$

This bound is valid for every Q_α and on $\mathbb{R}^3$ $[3/q = 3/p - 1]$.

On $\mathbb{R}^3$ and Q_α we also have $\|\nabla u\|_{L^2} = \|\omega\|_{L^2}$.

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Since $\omega_i = \epsilon_{ijk}\partial_j u_k$ and $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$,

$$\begin{aligned}\int |\omega|^2 &= \int \epsilon_{ijk}(\partial_j u_k)\epsilon_{ilm}(\partial_l u_m) \\ &= \int [\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}](\partial_j u_k)(\partial_l u_m) \\ &= \int (\partial_j u_k)(\partial_j u_k) - (\partial_j u_k)(\partial_k u_j) = \int \sum_{j,k} |\partial_j u_k|^2,\end{aligned}$$

integrating by parts twice in the final term and using the fact that u is divergence free.

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We have a family u_α defined on Q_α. How do we talk about 'convergence' to a solution on $\mathbb{R}^3$?

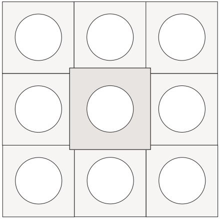
Extension of the function u_α from Q_α to $\mathbb{R}^3$

$$\text{Set} \qquad \tilde{u}_\alpha(x) := \psi_\alpha(x) u_\alpha^{\mathrm{p}}(x),$$

where $\psi_\alpha \in C_c^\infty(\mathbb{R}^3)$ with $0 \le \psi_\alpha \le 1$,

$$\psi_\alpha(x) = \begin{cases} 1 & x \in (-\alpha, \alpha)^3 \\ 0 & x \notin (-(\alpha+1), \alpha+1)^3, \end{cases}$$

$|\nabla \psi_\alpha| \le M_1$, and $|\nabla^2 \psi_\alpha| \le M_2$, uniformly in α.



For $\alpha \ge 1$: $\qquad \|\tilde{u}_\alpha\|_{L^2(\mathbb{R}^3)} \le e_1 \|u_\alpha\|_{L^2(Q_\alpha)}$,

$$\|\nabla \tilde{u}_\alpha\|_{L^2(\mathbb{R}^3)} \le e_2 \|u_\alpha\|_{H^1(Q_\alpha)}, \qquad \|\tilde{u}_\alpha\|_{H^2(\mathbb{R}^3)} \le e_3 \|u_\alpha\|_{H^2(Q_\alpha)}.$$

Convergence of corresponding velocities

Suppose that $\omega \in \dot{L}^2_\sigma(\mathbb{R}^3)$ has compact support; set $u = \mathrm{curl}^{-1}\omega$. For every α sufficiently large that $\mathrm{supp}(\omega) \subset Q_\alpha$ define $u_\alpha = \mathrm{curl}_\alpha^{-1}\omega$.

Then

$$\|u_\alpha\|_{L^2} \le C\|\omega\|_{L^{6/5}}, \qquad \|\nabla u_\alpha\|_{L^2} = \|\omega\|_{L^2},$$

$\tilde{u}_\alpha \rightharpoonup u$ weakly in $H^1(\mathbb{R}^3)$, and $\tilde{u}_\alpha \to u$ strongly in $L^2(K)$ for every compact subset K of $\mathbb{R}^3$.

Proof: uniform bounds on $\tilde{u}_\alpha$ + compactness arguments + take limits in $\langle \nabla u_\alpha, \nabla\phi\rangle = \langle \mathrm{curl}\,\omega, \phi\rangle$ [weak form of weak form of $-\mathrm{curl}\,u_\alpha = \omega$].

If $\omega \in H^1(\mathbb{R}^3)$ then $u_\alpha \in H^2(\mathbb{R}^3)$ and $\tilde{u}_\alpha \to u$ strongly in $H^1(\mathbb{R}^3)$.

Proof: uniform bounds on $\tilde{u}_\alpha$ + compactness + 'Leray Lemma': uniform decay of u_α as $x \to \infty$

—— Bounds + compactness:

$$-\Delta u_\alpha = \mathrm{curl}\,\omega \qquad \Rightarrow \qquad \|u_\alpha\|_{H^2(Q_\alpha)} \le C\|\omega\|_{H^1(Q_\alpha)}:$$

$\tilde{u}_\alpha \to u$ weakly in $H^2(\mathbb{R}^3)$ and strongly in $H^1(K)$ for every $K \subset\subset \mathbb{R}^3$.

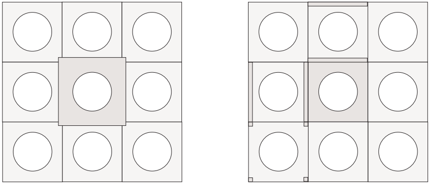
Leray Lemma

Take $\{f_\alpha\}_{\alpha \geq \alpha_0}, f \in L^2(\mathbb{R}^3)$. If

(i) $f_\alpha \to f$ strongly in $L^2(K)$ for every compact subset K of $\mathbb{R}^3$; and
(ii) for every $\eta > 0$ there exist $R(\eta)$ and $\beta(\eta)$ such that

$$\int_{|x| \geq R} |f_\alpha|^2 < \eta \qquad \text{for all } \alpha \geq \beta,$$

then $f_\alpha \to f$ in $L^2(\mathbb{R}^3)$.



Note that if $u_\alpha \in L^2(Q_\alpha)$ and $R < \alpha - 1$ then

$$\int_{|x| \geq R} |\tilde{u}_\alpha|^2 \, \mathrm{d}x \leq 27 \int_{x \in Q_\alpha \colon \, |x| \geq R} |u_\alpha|^2 \, \mathrm{d}x. \tag{2}$$

To obtain the strong convergence in $H^1(\mathbb{R}^3)$, take $\phi = u_\alpha \varrho|_{Q_\alpha}$ as the test function in

$$\langle \nabla u_\alpha, \nabla \phi \rangle = \langle \operatorname{curl} \omega, \phi \rangle,$$

where

$$\varrho = \begin{cases} 0 & |x| < r \\ \frac{|x|-r}{R-r} & r \le |x| \le R \\ 1 & |x| > R. \end{cases} \quad \text{so that} \quad |\nabla \varrho| = \begin{cases} 0 & |x| < r \\ \frac{1}{R-r} & r < |x| < R \\ 0 & |x| > R. \end{cases}$$

Therefore

$$\int_{Q_\alpha} |\nabla u_\alpha|^2 \varrho_\alpha = -\int_{Q_\alpha} (\nabla u_\alpha) \cdot (\nabla \varrho_\alpha) u_\alpha + \int_{Q_\alpha} (\operatorname{curl} \omega) u_\alpha \varrho_\alpha,$$

and taking r sufficiently large that $\operatorname{supp}(\omega) \subset B(0, r)$ yields

$$\begin{aligned} \int_{x \in Q_\alpha\colon\ |x| \ge R} |\nabla u_\alpha|^2 &\le \frac{1}{R-r} \|\nabla u_\alpha\|_{L^2(Q_\alpha)} \|u_\alpha\|_{L^2(Q_\alpha)} \\ &\le \frac{K}{R-r} \|\omega\|_{L^{6/5}} \|\omega\|_{L^2}. \end{aligned}$$

The Leray Lemma now guarantees that $\nabla \tilde{u}_\alpha \to \nabla u$ in $L^2(\mathbb{R}^3)$.

Convergence of Navier–Stokes solutions

Suppose that $u^0 \in H^1_\sigma(\mathbb{R}^3)$, $u^0_\alpha \in \dot{H}^1_\sigma(Q_\alpha)$, and $\tilde{u}^0_\alpha \to u^0$ in $H^1(\mathbb{R}^3)$, with $\|u^0_\alpha\|^2_{H^1(Q_\alpha)} \le M$ for all $\alpha \ge \alpha_0$.

Then there exists a time $T(M)$ such that the solutions u_α (on Q_α) and u (on $\mathbb{R}^3$) are strong on $[0,T]$ and

$$\tilde{u}_\alpha \to u \qquad \text{in} \qquad L^r(0,T;H^1(\mathbb{R}^3)), \qquad r \in [1,4).$$

Key point: strong convergence in $H^1(\mathbb{R}^3)$.

Particular cases: $u^0_\alpha = u^0 \in C^\infty_c(\mathbb{R}^3)$ or $u^0_\alpha = \mathrm{curl}^{-1}_\alpha \omega$, $\omega \in C^\infty_c(\mathbb{R}^3)$.

[Compactly supported initial velocity or vorticity.]

Proof: standard uniform energy estimates + compactness + Leray Lemma for strong convergence in $L^p(0,T;L^2(\mathbb{R}^3))$, $p \in [1,\infty)$ + interpolation

Leray approach in L^2

Take the inner product [in $L^2(Q_\alpha)$] of

$$\partial_t u_\alpha - \Delta u_\alpha + (u_\alpha \cdot \nabla) u_\alpha + \nabla p_\alpha = 0$$

with $\varrho_\alpha u_\alpha$, where $\varrho_\alpha = \varrho|_{Q_\alpha}$.

Then (cf. proof of Proposition 14.3 in Robinson et al., 2016) an integration by parts yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{Q_\alpha} \varrho_\alpha |u_\alpha|^2 + \int_{Q_\alpha} \varrho_\alpha |\nabla u_\alpha|^2$$
$$= -\int_{Q_\alpha} (\partial_j u_{\alpha,i}) u_{\alpha,i} (\partial_j \varrho_\alpha) + \int_{Q_\alpha} |u_\alpha|^2 (u_\alpha \cdot \nabla)\varrho_\alpha + \int_{Q_\alpha} p_\alpha (u_\alpha \cdot \nabla) \varrho_\alpha .$$

Integrating from 0 to t and using the definition of ϱ_α yields

$$\frac{1}{2}\int_{x\in Q_\alpha :\ |x|>R} |u_\alpha(t)|^2 \leq \frac{1}{2}\int_{x\in Q_\alpha :\ |x|>r} |u_\alpha^0|^2$$
$$+ \frac{1}{R-r}\int_0^t \int_{Q_\alpha} |\nabla u_\alpha||u_\alpha| + |u_\alpha|^3 + |p_\alpha||u_\alpha| .$$

After using various inequalities, it follows that for all $t \in [0, T]$,

$$\frac{1}{2}\int_{x\in Q_\alpha:\ |x|>R}|u_\alpha(t)|^2 \leq \frac{1}{2}\int_{x\in Q_\alpha:\ |x|>r}|u_\alpha^0|^2 + \frac{\|u_\alpha^0\|_{L^2(Q_\alpha)}}{R-r}\Bigg[T^{1/2}\int_0^T \|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^2\,\mathrm{d}s + 2C_6^{3/2}\|u_\alpha^0\|_{L^2(Q_\alpha)}^{1/2}T^{1/4}\left(\int_0^T \|\nabla u_\alpha(s)\|_{L^2(Q_\alpha)}^2\,\mathrm{d}s\right)^{3/4}\Bigg];$$

or

$$\int_{x\in Q_\alpha:\ |x|>R}|u_\alpha(t)|^2 \leq \int_{x\in Q_\alpha:\ |x|>r}|u_\alpha^0|^2 + \frac{\Gamma}{R-r},$$

where Γ can be chosen to be independent of α.

It follows that for any $\eta > 0$ there exist $R(\eta)$ and $\beta(\eta)$ such that

$$\int_{x\in Q_\alpha:\ |x|>R(\eta)}|u_\alpha(t)|^2 \leq \eta \qquad \text{for } \alpha \geq \beta(\eta),\ t \in [0, T],$$

with $\beta(\eta) > R(\eta) + 1$; therefore ((2)) we have

$$\int_{|x|>R(\eta)}|\tilde{u}_\alpha(t)|^2 \leq 27\eta \qquad \text{for } \alpha \geq \beta(\eta).$$

Since $\tilde{u}_\alpha$, u, are bounded in $L^\infty(0,T;L^2)$ we obtain convergence of $\tilde{u}_\alpha$ to u in $L^p(0,T;L^2(\mathbb{R}^3))$ for any $p \in [1,\infty)$ by DCT.

The strong convergence in H^1 now comes 'for free'.

Since $\tilde{u}_\alpha \to u$ strongly in $L^p(0,T;L^2(\mathbb{R}^3))$ and $\tilde{u}_\alpha$ is uniformly bounded in $L^2(0,T;H^2(\mathbb{R}^3))$, the Sobolev interpolation inequality

$$\|f\|_{H^1(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}^{1/2}\|f\|_{H^2(\mathbb{R}^3)}^{1/2}$$

implies that

$$\begin{aligned}\int_0^T \|\tilde{u}_\alpha - u\|_{H^1(\mathbb{R}^3)}^r \,\mathrm{d}t &\leq C\int_0^T \|\tilde{u}_\alpha - u\|_{L^2(\mathbb{R}^3)}^{r/2}\|\tilde{u}_\alpha - u\|_{H^2(\mathbb{R}^3)}^{r/2}\,\mathrm{d}t \\ &\leq \left(\int_0^T \|\tilde{u}_\alpha - u\|_{L^2(\mathbb{R}^3)}^{4/(4-r)}\,\mathrm{d}t\right)^{1-(r/4)}\left(\int_0^T \|\tilde{u}_\alpha - u\|_{H^2(\mathbb{R}^3)}^{2}\,\mathrm{d}t\right)^{r/4}.\end{aligned}$$

For $r \in [1,4)$ this implies that $\tilde{u}_\alpha \to u$ in $L^r(0,T;H^1(\mathbb{R}^3))$ as claimed.

‘Transfer of regularity’ from $\mathbb{R}^3$ to Q_α

Theorem

Suppose that $u_\alpha^0 \in \dot{H}^1_\sigma(Q_\alpha)$ *and* $u^0 \in \dot{H}^1_\sigma(\mathbb{R}^3)$, *with* $\tilde{u}_\alpha^0 \to u^0$ *in* $H^1(\mathbb{R}^3)$. *Suppose in addition that there exists* $T^* > 0$ *such that the equations on* $\mathbb{R}^3$ *with initial condition* u^0 *admit a solution*

$$u \in L^\infty([0, T^*]; H^1(\mathbb{R}^3)) \cap L^2(0, T^*; H^2(\mathbb{R}^3)).$$

Then for α *sufficiently large the equations on the periodic domain* Q_α *with initial data* u_α^0 *have a smooth solution*

$$u_\alpha \in L^\infty(0, T^*; H^1(Q_\alpha)) \cap L^2(0, T^*; H^2(Q_\alpha))$$

and $\tilde{u}_\alpha \to u$ *in* $L^r(0, T^*; H^1)$, $r \in [1, 4)$, *as* $\alpha \to \infty$.

The simplest particular cases of the theorem are when $u_\alpha^0 \equiv u^0 \in \dot{H}^1_\sigma(\mathbb{R}^3)$ for all α sufficiently large or when $u_\alpha^0 = \mathrm{curl}_\alpha^{-1}\omega_0$ for some $\omega_0 \in \dot{H}^1_\sigma(\mathbb{R}^3)$.

Since $u \in L^\infty([0,T^*]; H^1(\mathbb{R}^3))$ there exists $M > 0$ such that $\|u(t)\|^2_{H^1(\mathbb{R}^3)} \leq M$ for all $t \in [0,T^*]$.

There exists a uniform time τ such that any solution with $u(0) = v_0$, where $\|v_0\|^2_{H^1(\mathbb{R}^3)} \leq 2M$, exists at least on the time interval $[0,\tau]$.

Set $N = 2T^*/\tau$.

We have shown that $\tilde{u}_\alpha \to u$ in $L^r(0,\tau; H^1(\mathbb{R}^3))$ as $\alpha \to \infty$: so $\tilde{u}_\alpha(t) \to u(t)$ in $H^1(\mathbb{R}^3)$ for almost every $t \in (0,\tau)$; choose one such t with $t > \tau/2$ and call this t_1.

Choose α_1 such that $\|\tilde{u}_\alpha(t_1)\|_{H^1(\mathbb{R}^3)} \leq 2M$ for all $\alpha \geq \alpha_1$. Since

$$\|u_\alpha(t_1)\|_{H^1(Q_\alpha)} \leq \|\tilde{u}_\alpha(t_1)\|_{H^1(\mathbb{R}^3)},$$

this bound is enough to ensure that, uniformly for $\alpha \geq \alpha_1$, the solutions on Q_α starting from $u_\alpha(t_1)$ exist on the time interval $[t_1, t_1 + \tau] \supset [\tau, 3\tau/2]$.

We now repeat the argument.

Since $\tilde{u}_\alpha(t_1) \to u(t_1)$ in $H^1(\mathbb{R}^3)$, we know that $\tilde{u}_\alpha \to u$ in $L^r(t_1, t_1+\tau; H^1(\mathbb{R}^3))$ as $\alpha \to \infty$.

Again, the convergence in $H^1(\mathbb{R}^3)$ for almost-every time means that there exists $t_2 \in (t_1, t_1+\tau)$ with $t_2 > t_1 + \tau/2 > \tau$ such that $\tilde{u}_\alpha(t_2) \to u(t_2)$ in $H^1(\mathbb{R}^3)$; in particular, there exists $\alpha_2 \geq \alpha_1$ such that $\|u_\alpha(t_2)\|_{H^1(Q_\alpha)} \leq 2M$ for all $\alpha \geq \alpha_2$.

Continue in this way, noting that at each step the interval of existence of the solutions on Q_α (for $\alpha \geq \alpha_n$) increases by at least $\tau/2$. After N steps the entire interval $[0, T^*]$ has been covered, showing that the solution on Q_α starting at u_α^0 is strong on $[0, T^*]$ for all $\alpha \geq \alpha_N$.

Conclusion

The results here justify the use of 'large periodic boxes' to compute solutions on $\mathbb{R}^3$.

- What about expanding Dirichlet domains? (Main issue is the pressure.)

- What about error bounds, i.e. rates of convergence? [A hard problem! Interesting to look at simpler problems, e.g. $-\Delta u = f$.]

- Can the 'transfer of regularity' results be strengthened? [The results proved here are u^0-by-u^0.] Ideal results would relate 'well-posedness' in different settings: e.g. does well-posedness on the torus imply well-posedness on $\mathbb{R}^3$ (cf. Tao, 2013)?

[Would regularity of the Euler equations imply regularity of the Navier–Stokes equations? Constantin, 1986, obtains a 'transfer of regularity' result like that here (for sufficiently small viscosity), but his includes error bounds.]