On the homogenization problem for the linear Boltzmann equation

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Workshop 2020 Partial differential equations

September 22th, 2020

Results in collaboration with H. Hutridurga and O. Mula

Table of contents

The linear Boltzmann equation in the phase space

Ω: bounded domain of $\mathbb{R}^d$ with C^1 boundary

$\mathbb{V} \subseteq \mathbb{R}^d$: velocity space

$f = f(t, x, v)$: population density of neutrons

Two main phenomena:

- Absorption of neutrons with an absorption rate $\sigma \geq 0$
- Scattering and creation of neutrons with transition kernel $\kappa \geq 0$

$$\partial_t f + v \cdot \nabla_x f + \sigma(x, v) f - \int_{\mathbb{V}} \kappa(x, v \cdot v') f(t, x, v') \, \mathrm{d}v' = 0$$

Mathematical study of spatial oscillations

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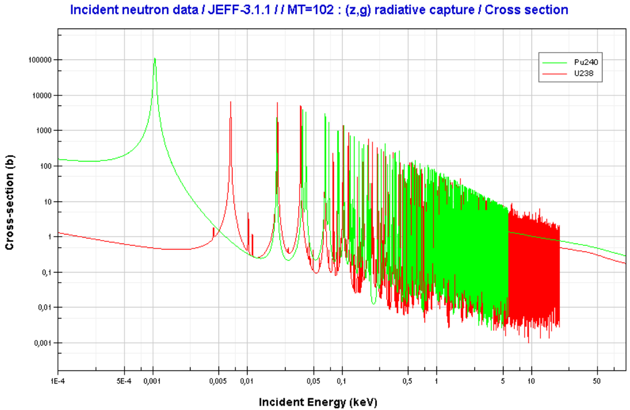
Oscillating behavior in energy



Radiative capture cross sections of ^{238}U and ^{240}Pu

Energy self-shielding

Energy self-shielding is related to the high oscillation of the optical parameters with respect to the energy of the incoming flux.

The simple average of the optical parameters in the linear Boltzmann equation does not allow to obtain accurate results (measured and expected energy dependent neutron fluxes are not in agreement)

Practical strategy: introduce a correction to the linear Boltzmann equation

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Table of contents

The energy description

$\omega = v/|v|$: trajectory angle of the neutron

$E = m|v|^2/2$: kinetic energy of the neutron (m: neutron mass)

New unknown: neutron flux (v expressed via the pair (ω, E))
$\varphi(t, x, \omega, E) = \varphi(t, x, v) := |v| f(t, x, v) \quad E \in [E_{\min}, E_{\max}]$

The linear Boltzmann equation

$$\sqrt{\frac{m}{2E}}\, \partial_t \varphi + \omega \cdot \nabla_x \varphi + \sigma(x, \omega, E)\, \varphi$$

$$- \int_{E_{\min}}^{E_{\max}} \int_{|\omega'|=1} \kappa(x, \omega \cdot \omega', E, E')\, \varphi(x, \omega', E') \mathrm{d}\omega' \, \mathrm{d}E' = 0$$

$\varphi(0, x, \omega, E) = \varphi_{\mathrm{in}}(x, \omega, E)$

$\varphi = 0 \ \forall t, E > 0$ and for $(x, \omega) \in \Gamma_- = \left\{(x, \omega) \in \partial\Omega \times \mathbb{S}^{d-1} : n_x \cdot \omega < 0\right\}$

Table of contents

L. Tartar's example (1979)

For the unknown u^ε, consider the differential equation

$$\partial_t u^\varepsilon + \sigma\left(\frac{x}{\epsilon}\right) u^\epsilon = 0; \qquad\qquad u^\epsilon(0,x) = u_{\mathrm{in}}(x).$$

Notation for the Laplace transform (in the time variable) of a function:

$$\widehat{f}(p) := \int_0^\infty e^{-ps} f(s)\,\mathrm{d}s \qquad \text{for } p > 0.$$

Notation: let $\mathrm{Y} := (0,1)^d$ be the unit cube in $\mathbb{R}^d$; for any $v \in \mathrm{L}^1(\mathrm{Y})$

$$\langle v \rangle = \int_{\mathrm{Y}} v(y)\,\mathrm{d}y$$

denotes the average of v in Y.

Homogenization of Tartar's example

Theorem (Tartar)

Let the coefficient $\sigma(\cdot)$ be a strictly positive, bounded and purely periodic coefficient of period Y. Then the L^∞ weak $*$ limit $u_{\mathrm{hom}}(t,x)$ of the solution family u^ϵ satisfies the following integro-differential equation

$$\begin{cases} \partial_t u_{\mathrm{hom}}(t,x) + \langle\sigma\rangle u_{\mathrm{hom}}(t,x) - \displaystyle\int_0^t \mathcal{M}(t-s) u_{\mathrm{hom}}(s,x)\,\mathrm{d}s = 0 \\ u_{\mathrm{hom}}(0,x) = u_{\mathrm{in}}(x) \end{cases}$$

where the memory kernel $\mathcal{M}(\tau)$ is given in terms of its Laplace transform

$$\widehat{\mathcal{M}}(p) = p + \langle\sigma\rangle - \mathcal{B}(p) = \int_{\mathrm{Y}} \Big(p + \sigma(y) - \mathcal{B}(p)\Big)\,\mathrm{d}y \qquad \forall p > 0,$$

with the constant $\mathcal{B}(p)$ taking the value $\mathcal{B}(p) := \left(\displaystyle\int_{\mathrm{Y}} \frac{\mathrm{d}y}{p+\sigma(y)}\right)^{-1}$.

Table of contents

Tartar's example revisited

$$\begin{cases} \partial_t u^\epsilon(t,x) + \sigma^\epsilon(x) u^\epsilon(t,x) = f^\epsilon(t,x) & (t,x) \in (0,T) \times \Omega \\ u^\epsilon(0,x) = u^\epsilon_{\mathrm{in}}(x) & x \in \Omega \end{cases}$$

$$\sigma^\epsilon(x) := \sigma\left(x, \frac{x}{\epsilon}\right), \quad f^\epsilon(t,x) := f\left(t, x, \frac{x}{\epsilon}\right), \quad u^\epsilon_{\mathrm{in}}(x) := u_{\mathrm{in}}\left(x, \frac{x}{\epsilon}\right),$$

$$\sigma(x,y) \in \mathrm{L}^\infty(\Omega; \mathrm{C}_{\mathrm{per}}(\mathrm{Y}))$$

$$f(t,x,y) \in \mathrm{L}^\infty((0,T) \times \Omega; \mathrm{C}_{\mathrm{per}}(\mathrm{Y})), \quad u_{\mathrm{in}}(x,y) \in \mathrm{L}^2(\Omega; \mathrm{C}_{\mathrm{per}}(\mathrm{Y}))$$

Notation: $\mathrm{C}_{\mathrm{per}}(\mathrm{Y})$ denote Y-periodic continuous functions on $\mathbb{R}^d$

Hypothesis: there exists a positive constant σ_{min} such that

$$\sigma(x,y) \geq \sigma_{\mathsf{min}} \quad \forall (x,y) \in \Omega \times \mathrm{Y}$$

Two-scale convergence

The notion of two-scale convergence is a weak-type convergence as it is given in terms of test functions

Definition

A family of functions $v^\epsilon(x) \subset \mathrm{L}^2(\Omega)$ two-scale converges to a limit $v^0(x,y) \in \mathrm{L}^2(\Omega \times \mathrm{Y})$ if, for any smooth test function $\psi(x,y)$, Y-periodic in the y variable,

$$\lim_{\epsilon \to 0} \int_\Omega v^\epsilon(x)\psi\left(x, \frac{x}{\epsilon}\right) \mathrm{d}x = \int_\Omega \int_Y v^0(x,y)\psi(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

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Two-scale convergence: two results

Theorem (Nguetseng, Allaire)

Suppose a family $v^\epsilon(x) \subset \mathrm{L}^2(\Omega)$ is uniformly bounded, i.e.,

$$\|v^\epsilon\|_{\mathrm{L}^2(\Omega)} \leq C$$

with constant C being independent of ϵ. Then, we can extract a sub-sequence (still denoted v^ϵ) such that v^ϵ two-scale converges to some limit $v^0(x,y) \in \mathrm{L}^2(\Omega \times Y)$.

Proposition (Nguetseng, Allaire)

Let v^ϵ be a sequence of functions in $\mathrm{L}^2(\Omega)$ which two-scale converges to a limit $v^0 \in \mathrm{L}^2(\Omega \times \mathrm{Y})$. Then $v^\epsilon(x)$ converges to $\langle v\rangle(x) = \int_{\mathrm{Y}} v^0(x,y)\,\mathrm{d}y$ weakly in $\mathrm{L}^2(\Omega)$, i.e.

$$\lim_{\epsilon\to 0}\int_\Omega v^\epsilon(x)\varphi(x)\,\mathrm{d}x = \int_\Omega \varphi(x)\int_Y v^0(x,y)\,\mathrm{d}y\,\mathrm{d}x \qquad \text{for all } \varphi \in \mathrm{L}^2(\Omega).$$

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Properties of the ODE

For any given $g \in \mathrm{L}^\infty(Y)$, the linear operator

$$\mathcal{L}_g\, v := gv - \langle gv \rangle \quad \forall v \in \mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})$$

is bounded in $\mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})$ as

$$\|\mathcal{L}_g h\|^2_{\mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})} = \int_{\mathrm{Y}} |g(y)h(y) - \langle gh \rangle|^2 \, \mathrm{d}y = \int_{\mathrm{Y}} |g(y)h(y)|^2 \, \mathrm{d}y - \langle gh \rangle^2$$

By Cauchy-Schwarz:

$$|\langle gh \rangle| = \left| \int_{\mathrm{Y}} g(y)h(y) \, \mathrm{d}y \right| \le \left(\int_{\mathrm{Y}} |g(y)h(y)|^2 \, \mathrm{d}y \right)^{\frac{1}{2}}.$$

As a consequence, $\mathcal{L}_g : \mathrm{L}^2_{\mathrm{per}}(\mathrm{Y}) \mapsto \mathrm{L}^2_{\mathrm{per}}(\mathrm{Y})$ is the infinitesimal generator of a uniformly continuous semigroup given by

$$e^{t\mathcal{L}_g} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathcal{L}_g^n.$$

Theorem (H. Hutridurga, O. Mula, FS)

$$u^\epsilon \rightharpoonup u_{\mathrm{hom}} \quad \text{weakly in } \mathrm{L}^2((0,T)\times\Omega)$$

$$\begin{cases} \partial_t u_{\mathrm{hom}}(t,x) + \langle \sigma \rangle(x) u_{\mathrm{hom}}(t,x) - \displaystyle\int_0^t \mathcal{K}(t-s,x) u_{\mathrm{hom}}(s,x)\,\mathrm{d}s = \mathcal{S}(t,x) \\ u_{\mathrm{hom}}(0,x) = \langle u_{\mathrm{in}} \rangle(x) \end{cases}$$

The memory kernel is given by

$$\mathcal{K}(\tau,x) = \int_{\mathrm{Y}} \sigma(x,y) e^{-\tau \mathcal{L}_\sigma} \mathcal{L}_1 \sigma(x,y)\,\mathrm{d}y$$

The source term is given by

$$\mathcal{S}(t,x) = \langle f \rangle(t,x) - \int_0^t \int_{\mathrm{Y}} \sigma(x,y) e^{-(t-s)\mathcal{L}_\sigma} \mathcal{L}_1 f(s,x,y)\,\mathrm{d}y\,\mathrm{d}s$$
$$- \int_{\mathrm{Y}} \sigma(x,y) e^{-t\mathcal{L}_\sigma} \mathcal{L}_1 u_{\mathrm{in}}(x,y)\,\mathrm{d}y.$$

Proof of the theorem

Explicit solution:

$$u^\epsilon(t,x) = u_{\mathrm{in}}\left(x, \frac{x}{\epsilon}\right) e^{-\sigma^\epsilon(x)t} + \int_0^t e^{-\sigma^\epsilon(x)(t-s)} f\left(s, x, \frac{x}{\epsilon}\right)\, \mathrm{d}s.$$

The regularity properties of the initial condition u_{in} and of the source term f, together with the fact that $\sigma^\epsilon \geq 0$, imply that, uniformly in ϵ

$$\|u^\epsilon\|_{\mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega))} \leq C < \infty$$

Nguetseng & Allaire's theorem guarantees the existence of a subsequence u^ϵ which two-scale converges to a function $u^0 \in \mathrm{L}^2((0,T) \times \Omega \times \mathrm{Y})$

Equation satisfied by the limit u^0

Passing to the limit as $\epsilon \to 0$ in the sense of two-scale, we obtain

$$u^0(t,x,y) = u_{\text{in}}(x,y)e^{-\sigma(x,y)t} + \int_0^t e^{-\sigma(x,y)(t-s)} f(s,x,y)\,\mathrm{d}s$$

i.e., u^0 solves

$$\begin{cases} \partial_t u^0(t,x,y) + \sigma(x,y)u^0(t,x,y) = f(t,x,y) & (t,x,y) \in (0,T)\times\Omega\times\mathrm{Y} \\ u^0(0,x,y) = u_{\text{in}}(x,y) & (x,y)\in\Omega\times\mathrm{Y} \end{cases}$$

Decomposition

u^ϵ converges weakly in $\mathrm{L}^2((0,T)\times\Omega)$ to

$$u_{\mathrm{hom}}(t,x) = \langle u^0\rangle(t,x)$$

and we can then decompose the two-scale limit into a homogeneous part and a remainder which is of zero mean over the periodic cell:

$$u^0(t,x,y) = u_{\mathrm{hom}}(t,x) + r(t,x,y) \qquad \text{where} \qquad \langle r\rangle = 0.$$

We have

$$\partial_t u_{\mathrm{hom}} + \sigma(x,y)u_{\mathrm{hom}} + \partial_t r + \sigma(x,y)r = f(t,x,y).$$

Integrating the above equation over the periodicity cell Y yields

$$\partial_t u_{\mathrm{hom}} + \langle\sigma\rangle(x)u_{\mathrm{hom}} = \langle f\rangle(t,x) - \langle\sigma(x,\cdot)r(t,x,\cdot)\rangle$$

as the reminder r is of zero average in the y variable.

The coupled system for $u_{\mathrm{hom}}(t,x)$ and $r(t,x,y)$

Equation for the remainder term:

$$\partial_t r + \sigma(x,y) r - \int_{\mathrm{Y}} \sigma(x,y) r(t,x,y)\,\mathrm{d}y$$

$$= \Big(\langle\sigma\rangle(x) - \sigma(x,y)\Big) u_{\mathrm{hom}} + f(t,x,y) - \langle f\rangle(t,x).$$

Coupled system for $u_{\mathrm{hom}}(t,x)$ and $r(t,x,y)$

$$\begin{cases} \partial_t u_{\mathrm{hom}} + \langle\sigma\rangle(x) u_{\mathrm{hom}} = \langle f\rangle(t,x) - \langle \sigma(x,\cdot) r(t,x,\cdot)\rangle \\ \partial_t r + \mathcal{L}_\sigma r = -u_{\mathrm{hom}} \mathcal{L}_1 \sigma + \mathcal{L}_1 f \\ u_{\mathrm{hom}}(0,x) = \langle u_{\mathrm{in}}(x)\rangle \\ r(0,x,y) = \mathcal{L}_1 u_{\mathrm{in}}. \end{cases}$$

The decoupled equation for $u_{\mathrm{hom}}(t,x)$

Solve for the remainder term $r(t,x,y)$ in terms of u_{hom}

$$r(t,x,y) = e^{-t\mathcal{L}_\sigma}\mathcal{L}_1 u_{\mathrm{in}}(x,y) + \int_0^t e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 f(s,x,y)\,\mathrm{d}s$$

$$-\int_0^t e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1\sigma(x,y)u_{\mathrm{hom}}(s,x)\,\mathrm{d}s$$

Substitute this expression for the remainder in the evolution for u_{hom}

$$\partial_t u_{\mathrm{hom}} + \langle\sigma\rangle(x)u_{\mathrm{hom}} = \langle f\rangle(t,x)$$

$$+\int_0^t\int_{\mathrm{Y}}\sigma(x,y)e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1\sigma(x,y)u_{\mathrm{hom}}(s,x)\,\mathrm{d}y\,\mathrm{d}s$$

$$-\int_0^t\int_{\mathrm{Y}}\sigma(x,y)e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 f(s,x,y)\,\mathrm{d}y\,\mathrm{d}s$$

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The decoupled equation for $u_{\mathrm{hom}}(t,x)$

Solve for the remainder term $r(t,x,y)$ in terms of u_{hom}

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$$- \int_0^t \int_{\mathrm{Y}} \sigma(x,y) e^{-(t-s)\mathcal{L}_\sigma}\mathcal{L}_1 f(s,x,y)\,\mathrm{d}y\,\mathrm{d}s$$
$$- \int_{\mathrm{Y}} \sigma(x,y) e^{-t\mathcal{L}_\sigma}\mathcal{L}_1 u_{\mathrm{in}}(x,y)\,\mathrm{d}y$$

Table of contents

The rapidly oscillating problem for the linear Boltzmann equation ($0 < \varepsilon \ll 1$)

$$\sqrt{\frac{m}{2E}}\,\partial_t \varphi^\varepsilon + \omega \cdot \nabla_x \varphi^\varepsilon + \sigma^\varepsilon\left(x, \omega, E\right)\varphi^\varepsilon$$

$$- \int_{E_{\min}}^{E_{\max}} \int_{|\omega'|=1} \kappa^\varepsilon\left(x, \omega \cdot \omega', E, E'\right)\varphi^\varepsilon(x, \omega', E')\mathrm{d}\omega'\,\mathrm{d}E' = 0$$

$$\sigma^\varepsilon(x, \omega, E) = \sigma\left(x, \omega, E, \frac{E}{\varepsilon}\right)$$

$$\kappa^\varepsilon(x, \omega \cdot \omega', E, E') = \kappa\left(x, \omega \cdot \omega', E, E', \frac{E'}{\varepsilon}\right)$$

$\sigma(x, \omega, E, y)$ and $\kappa\left(x, \omega \cdot \omega', E, E', y'\right)$ are assumed to be periodic in the y and y' variables respectively.

The equation is complemented with zero incoming flux condition on the boundary and initial condition $\varphi_{\text{in}} \in \mathrm{L}^2(\Omega \times \mathbb{S}^{d-1} \times (E_{\min}, E_{\max}))$

Further hypotheses on the optical parameters

Let

$$\bar{\kappa}^\varepsilon(x,\omega,E) = \int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \kappa^\varepsilon(x,\omega\cdot\omega',E,E')\,\mathrm{d}\omega'\,\mathrm{d}E'$$

$$\tilde{\kappa}^\varepsilon(x,\omega,E) = \int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \kappa^\varepsilon(x,\omega\cdot\omega',E',E)\,\mathrm{d}\omega'\,\mathrm{d}E'.$$

Assume that there exists $\alpha > 0$ such that for all $\varepsilon > 0$,

$$\sigma^\varepsilon(x,\omega,E) - \bar{\kappa}^\varepsilon(x,\omega,E) \geq \alpha \quad \text{and} \quad \sigma^\varepsilon(x,\omega,E) - \tilde{\kappa}^\varepsilon(x,\omega,E) \geq \alpha$$

Hypothesis on the kernel structure

κ^ε exhibits separation in the E and E' variables:

$$\kappa^\varepsilon(x,\omega\cdot\omega',E,E') := \kappa_1(x,\omega\cdot\omega',E)\kappa_2\left(x,\omega\cdot\omega',E',\frac{E'}{\varepsilon}\right)$$

with $\kappa_2\,(x,\omega\cdot\omega',E',y')$ being periodic in the y' variable.

Table of contents

Theorem (H. Hutridurga, O. Mula, FS)

$\varphi^\epsilon \rightharpoonup \varphi_{\mathrm{hom}}$ in $L^2((0,T)\times\Omega\times\mathbb{S}^{d-1}\times[E_{\min},E_{\max}])$, solution of

$$\partial_t\varphi_{\mathrm{hom}} + \sqrt{E}\,\omega\cdot\nabla_x\varphi_{\mathrm{hom}} + \sqrt{E}\left(\int_0^1 \sigma(\omega,E,y')\mathrm{d}y'\right)\varphi_{\mathrm{hom}}$$

$$-\int_{E_{\min}}^{E_{\max}}\int_{\mathbb{S}^{d-1}}\sqrt{E}\,\kappa_1(\omega\cdot\omega',E)\left(\int_0^1\kappa_2(\omega\cdot\omega',E',y')\mathrm{d}y'\right)\varphi_{\mathrm{hom}}\mathrm{d}\omega'\mathrm{d}E' =$$

$$\int_{E_{\min}}^{E_{\max}}\int_{\mathbb{S}^{d-1}}\sqrt{E}\,\kappa_1(\omega\cdot\omega',E)\int_0^1\kappa_2(\omega\cdot\omega',E',y')\times$$

$$\left[e^{-t\sqrt{E'}\mathcal{L}_\sigma}\mathcal{L}_1\varphi_{\mathrm{in}} - \int_0^t e^{-(t-s)\sqrt{E'}\mathcal{L}_\sigma}\sqrt{E'}\mathcal{L}_1\sigma(\omega',E',y')\varphi_{\mathrm{hom}}\,\mathrm{d}s\right]\mathrm{d}y'\mathrm{d}\omega'\mathrm{d}E'$$

$$-\sqrt{E}\int_0^1\sigma(\omega,E,y)\left[e^{-t\sqrt{E}\mathcal{L}_\sigma}\mathcal{L}_1\varphi_{\mathrm{in}} - \int_0^t e^{-(t-s)\sqrt{E}\mathcal{L}_\sigma}\sqrt{E}\mathcal{L}_1\sigma(\omega,E,y)\varphi_{\mathrm{hom}}\mathrm{d}s\right]\mathrm{d}y$$

with initial condition $\varphi_{\mathrm{hom}}(0,x,\omega,E) = \langle\varphi_{\mathrm{in}}(x,\omega,E,\cdot)\rangle$ and zero absorption condition at the in-flux phase-space boundary.

Treatment of the integral term

♠ Method of characteristics

♠ Change of variable $x - \sqrt{E}\omega t = r$

♠ Multiply the integral term by a test function $g\left(E, \frac{E}{\epsilon}\right)$ and integrate w.r.t. E:

$$\int_{E_{\min}}^{E_{\max}} \int_0^t \int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \kappa^\varepsilon\left(\omega\cdot\omega', E, E'\right) e^{-(t-s)\sigma^\varepsilon(\omega,E)} \psi^\varepsilon\left(s, r, \omega', E'\right) \times g\left(E, \frac{E}{\epsilon}\right) \mathrm{d}\omega' \, \mathrm{d}E' \, \mathrm{d}s \, \mathrm{d}E = \int_{E_{\min}}^{E_{\max}} \int_0^t \int_{\mathbb{S}^{d-1}} \sqrt{E}\, \kappa_1(\omega\cdot\omega', E) \times e^{-(t-s)\sigma^\varepsilon(\omega,E)} w^\varepsilon(s, r, \omega, \omega') g\left(E, \frac{E}{\epsilon}\right) \mathrm{d}\omega' \, \mathrm{d}s \, \mathrm{d}E,$$

where

$$w^\varepsilon(s, r, \omega, \omega') := \int_{E_{\min}}^{E_{\max}} \kappa_2\left(\omega\cdot\omega', E', \frac{E'}{\epsilon}\right) \psi^\varepsilon\left(s, r, \omega', E'\right) \mathrm{d}E'.$$

♠ For every $(s, r, \omega, \omega') \in (0, T) \times \Omega \times \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$,

$$w^\varepsilon(s, r, \omega, \omega') \to w^0(s, r, \omega, \omega') =$$

$$\int_{E_{\min}}^{E_{\max}} \int_0^1 \kappa_2(\omega \cdot \omega', E', y') \psi^0(s, r, \omega', E', y') \mathrm{d}y' \mathrm{d}E'$$

pointwise as $\varepsilon \to 0$, where ψ^0 is the two-scale limit of ψ^ε.

♠ Consequence: two-scale convergence

$$\int_{E_{\min}}^{E_{\max}} \int_0^t \int_{\mathbb{S}^{d-1}} \sqrt{E}\, \kappa_1(\omega \cdot \omega', E) e^{-(t-s)\sigma^\varepsilon(\omega, E)} w^\varepsilon(s, y, \omega, \omega') g\left(E, \frac{E}{\epsilon}\right) \mathrm{d}\omega' \mathrm{d}s \mathrm{d}E$$

$$\rightharpoonup \int_{E_{\min}}^{E_{\max}} \int_0^t \int_{\mathbb{S}^{d-1}} \int_0^1 \sqrt{E}\, \kappa_1(\omega \cdot \omega', E) e^{-(t-s)\sqrt{E}\sigma(\omega, E, y)} w^0(s, r, \omega, \omega') \times$$

$$g(E, y) \mathrm{d}y \mathrm{d}\omega \mathrm{d}s \mathrm{d}E$$

As a result, passing to the limit in the sense of two-scale

$$\psi^0(t,r,\omega,E,y) = \varphi_{\rm in}(r,\omega,E,y)e^{-t\sqrt{E}\sigma(\omega,E,y)} + \int_0^t \int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \int_0^1 \sqrt{E}\kappa_1(\omega\cdot\omega',E)e^{-(t-s)\sqrt{E}\sigma(\omega,E,y)}\kappa_2(\omega\cdot\omega',E',y') \times \psi^0(s,r,\omega',E',y')\,\mathrm{d}y'\mathrm{d}\omega'\mathrm{d}E'\mathrm{d}s$$

which implies that ψ^0 solves the two-scale evolution

$$\begin{cases} \partial_t\psi^0(t,r,\omega,E,y) + \sqrt{E}\sigma(\omega,E,y)\psi^0(t,r,\omega,E,y) \\ = \displaystyle\int_{E_{\min}}^{E_{\max}} \int_{\mathbb{S}^{d-1}} \sqrt{E}\kappa_1(\omega\cdot\omega',E)\int_0^1 \kappa_2(\omega\cdot\omega',E',y')\psi^0(s,r,\omega',E',y')\mathrm{d}y\mathrm{d}\omega\mathrm{d}E' \\ \psi^0(0,r,\omega,E,y) = \varphi_{\rm in}(r,\omega,E,y). \end{cases}$$

Concluding remarks on the assumption on the optical parameters σ^ε and κ^ε

- The assumption of separability

$$\kappa^\varepsilon(\omega\cdot\omega', E, E') := \sqrt{E}\,\kappa_1(\omega\cdot\omega', E)\kappa_2\left(\omega\cdot\omega', E', \frac{E'}{\varepsilon}\right)$$

simplifies the computations in the proof. It also lead to a relatively simpler homogenized model.

- In the above separable structure, we can further allow the factor κ_1 to oscillate in the E-variable. The proof of the main theorem can be reworked in this case, at the price of arriving at a more complex two-scale system. The memory structure remains the same but with additional terms.
- It is apparent from the proof of the main theorem that the energy oscillations in σ, not those in the scattering kernel, resulted in the memory effects in the homogenized limit.

Table of contents

Numerical illustration of the homogenization limit

$$\begin{cases} \partial_t \varphi^\epsilon(t,E) + \sigma\left(\dfrac{E}{\epsilon}\right)\varphi^\epsilon(t,E) = \displaystyle\int_{E_{\min}}^{E_{\max}} \kappa\left(\dfrac{E'}{\epsilon}\right)\varphi^\epsilon(t,E')\,\mathrm{d}E' \\ \varphi^\epsilon(0,E) = \varphi_{\mathrm{in}}(E) \end{cases}$$

for $(t,E) \in [0,10] \times (E_{\min}, E_{\max}) = (0,1)$.

Strategy: family of orthogonal Legendre polynomials in $\mathrm{L}^2(E_{\min}, E_{\max})$ denoted by $\{\ell_k\}_{k\geq 0}$

Define the modes

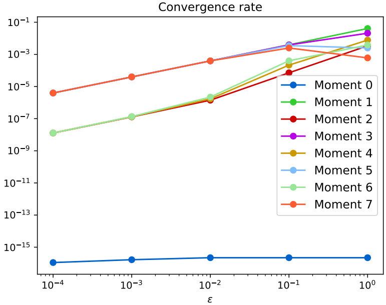
$$m_k^\epsilon(t) = (\varphi^\epsilon(t,\cdot), \ell_k(\cdot))_{\mathrm{L}^2([E_{\min},E_{\max}])}\,, \quad k \geq 0$$

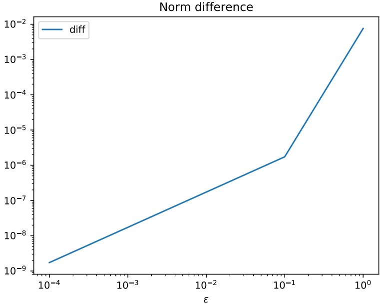
of the solution for $t \in [0,T]$. Likewise,

$$m_k^{\mathrm{hom}}(t) := (\varphi_{\mathrm{hom}}(t,\cdot), \ell_k(\cdot))_{\mathrm{L}^2([E_{\min},E_{\max}])}\,, \quad k \geq 0$$

are the modes of the homogenized solution φ_{hom}.

Numerical simulations for $\sigma(y) = 2 + \frac{1}{2}\sin(2\pi y)$, $\kappa(y') = 1 + \frac{1}{2}\sin(2\pi y')$, $\varphi_{\rm in}(y) = 1 + \sin(2\pi y)$





Convergence rates in ϵ:

error $e_k^\epsilon = \max_{t\in[0,T]} |m_k^\epsilon(t) - m_k^{\mathrm{hom}}(t)|$ (left)

norm difference $|\|\varphi^\epsilon\|_{\mathrm{L}^2} - \|\varphi^0\|_{\mathrm{L}^2}|$ (right)

THANK YOU FOR YOUR ATTENTION!