McKean–Vlasov diffusion and the well-posedness of the Hookean bead-spring chain model

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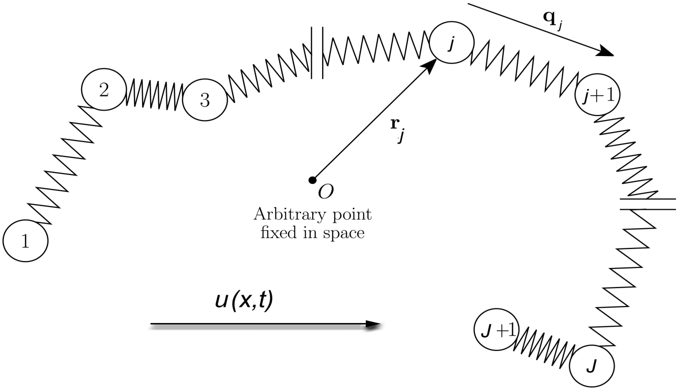
Motivation

The Hookean bead-spring-chain model describes the conformational dynamics of an ideal polymer chain. A polymer molecule is modelled as a linear chain of massless beads connected with Hookean springs, subjected to Brownian noise.



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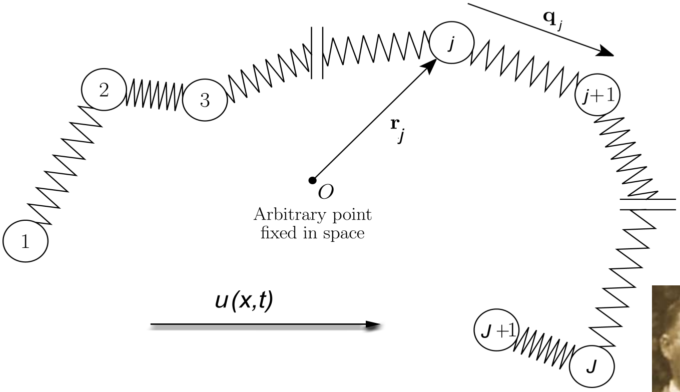

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P. E. Rouse. A Theory of the Linear Viscoelastic Properties of Dilute Solutions of Coiling Polymers, J. Chem. Phys. 21, 1272 (1953).

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E. Süli and G. Yahiaoui. McKean–Vlasov diffusion and the well-posedness of the Hookean bead-spring-chain model for dilute polymeric fluids: small-mass limit and equilibration in momentum space. 85 pages. [cf. arXiv].

Main results

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- If the flow domain Ω is bounded, then the configuration space domain $D^J = \underbrace{D \times \cdots \times D}_{J}$, where $D = \Omega - \Omega = \{r - \hat{r} : r, \hat{r} \in \Omega\}$, is also bounded.

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- Before taking the small-mass limit, the Fokker–Planck equation, posed on $\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0,T] \ni (r, v, t)$, is mixed parabolic-hyperbolic.

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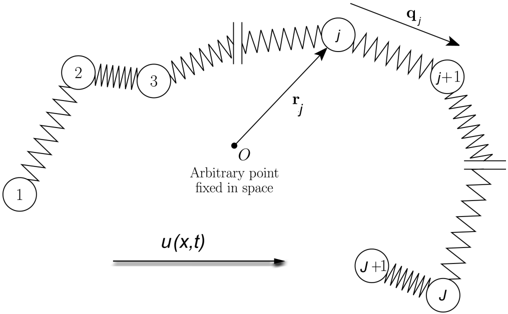
- We rigorously prove an assertion, deduced by Schieber & Öttinger (1988) using formal asymptotics, that:

  passage to the small-mass limit $\Rightarrow$ equilibration in momentum space.

Formulation of the model



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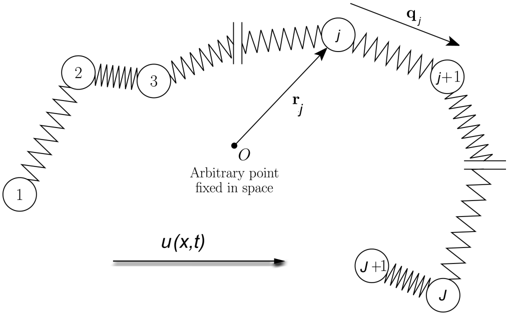

$$r := (r_1^{\mathrm{T}}, \dots, r_{J+1}^{\mathrm{T}})^{\mathrm{T}}, \quad r_j \in \Omega \quad \text{for } j = 1, \dots, J+1,$$

$$v := (v_1^{\mathrm{T}}, \dots, v_{J+1}^{\mathrm{T}})^{\mathrm{T}}, \quad v_j \in \mathbb{R}^d \quad \text{for } j = 1, \dots, J+1,$$

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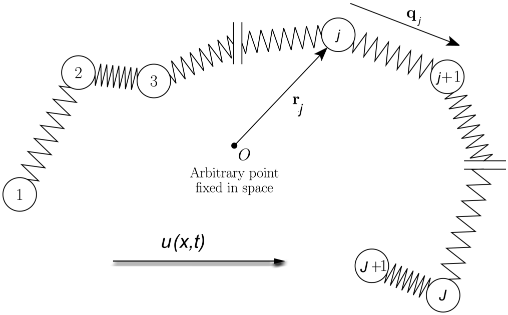

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$$q_j \in D := \Omega - \Omega = \{r - \hat{r} \,:\, r, \hat{r} \in \Omega\}, \qquad \text{for } j = 1, \ldots, J;$$

Condition: $$x := \frac{1}{J+1} \sum_{j=1}^{J+1} r_j.$$

Hydrodynamic interaction

Oseen system on the space-time domain $\overline{\Omega} \times [0, T]$, where Ω is a bounded open convex $\mathcal{C}^2$ domain in $\mathbb{R}^d$, $d \in \{2, 3\}$, $0 \in \Omega$, $b \in L^\infty(\Omega \times (0, T))$, $\nabla \cdot b = 0$:

$$
\begin{aligned}
\partial_t u + (b \cdot \nabla) u - \mu \triangle u + \nabla \pi &= \nabla \cdot \mathbb{K} && \text{for } (x, t) \in \Omega \times (0, T],\\
\nabla \cdot u &= 0 && \text{for } (x, t) \in \Omega \times (0, T],\\
u(x, t) &= 0 && \text{for } (x, t) \in \partial\Omega \times (0, T],\\
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with the non-Newtonian extra-stress tensor (Kramers–Kirkwood stress tensor)

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\mathbb{K}(x, t; \varrho) := \sum_{j=1}^{J} \mathbb{E}^x \left(\lambda q_j \otimes q_j \right) \qquad \text{for } (x, t) \in \Omega \times (0, T],\ J \geq 1,
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$$\mathcal{U}(r,t;\varrho) := \left(u(r_1,t;\varrho)^{\mathrm{T}}, \cdots, u(r_{J+1},t;\varrho)^{\mathrm{T}} \right)^{\mathrm{T}}.$$

SDE : $$\epsilon^2 \ddot{r} = \underbrace{\mathcal{L} r}_{\text{Hookean spring force}} + \underbrace{\zeta\big(\mathcal{U}(r,t;\varrho) - \dot{r}\big)}_{\text{Stokes drag force}} + \underbrace{\sqrt{2\beta}\,\dot{W}}_{\text{Brownian force}} .$$

$\epsilon^2 > 0$ is the mass of a bead in the chain, $\beta = k\mathtt{T}\zeta > 0$, where k is the Boltzmann constant, $\mathtt{T}$ is the absolute temperature and ζ is the drag coefficient; $\mathcal{L}$ is the following $(J+1) \times (J+1)$ block-matrix:

$$\mathcal{L} := \lambda \begin{pmatrix} -\mathbb{I} & \mathbb{I} & \mathbb{O} & \dots & \mathbb{O} \\ \mathbb{I} & -2\mathbb{I} & \mathbb{I} & \ddots & \mathbb{O} \\ \mathbb{O} & \mathbb{I} & -2\mathbb{I} & \mathbb{I} & \mathbb{O} \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ \mathbb{O} & \dots & \mathbb{I} & -2\mathbb{I} & \mathbb{I} \\ \mathbb{O} & \dots & \mathbb{O} & \mathbb{I} & -\mathbb{I} \end{pmatrix} \in \mathbb{R}^{(J+1)d \times (J+1)d},$$

where $\lambda > 0$ is a constant stiffness of the Hookean springs. W.l.o.g., we set $\zeta = 1$.

The SDE may then be rewritten as the first-order system

$$\begin{aligned}
\epsilon\dot{r} &= v, \\
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Let

$$\varrho \,:\, (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times [0, T] \mapsto \varrho(r, v, t) \in \mathbb{R}_{\geq 0}$$

be the probability density function of the diffusion process (r, v).

The law of (r, v) depends on ϱ itself through the function $\mathcal{U}$, and it is therefore a McKean–Vlasov diffusion process.

Definition of the polymeric extra stress tensor $\mathbb{K}$

We define

$$\mathbb{E}\Big(\sum_{j=1}^{J} \lambda q_j \otimes q_j\Big)(t) := \int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} \Big(\sum_{j=1}^{J} \lambda q_j(r) \otimes q_j(r)\Big)\, \varrho(r,v,t)\,\mathrm{d}r\,\mathrm{d}v$$

and perform a change of variables, replacing integration over $r \in \Omega^{J+1}$ by integration over $(q,x) \in D^J \times \Omega$ via the linear bijection

$$(q,x) \in D^J \times \Omega \quad\mapsto\quad r = B(q,x) \in \Omega^{J+1}, \qquad |\text{Jacobian}[B(q,x)]| = 1.$$

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Then, for $(x,t)\in\Omega\times(0,T]$, the Kramers–Kirkwood stress tensor is:

$$\mathbb{K}(x,t;\varrho)=\mathbb{E}^x\Big(\sum_{j=1}^{J}\lambda q_j\otimes q_j\Big)(x,t)=\frac{\int_{D^J\times\mathbb{R}^{(J+1)d}}\big(\sum_{j=1}^{J}\lambda q_j\otimes q_j\big)\,\varrho\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}}\varrho\,\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}$$

Fokker–Planck equation for ϱ

$$\partial_t \varrho - \frac{\beta}{\epsilon^2}\left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (v_j\,\varrho) + \beta\,\partial_{v_j}^2 \varrho\right) + \frac{1}{\epsilon}\left(\sum_{j=1}^{J+1} v_j \cdot \partial_{r_j}\varrho + ((\mathcal{L}r)_j + u(r_j,t)) \cdot \partial_{v_j}\varrho\right) = 0$$

for all $(r,v,t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0,T]$,

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Let

$$\partial\Omega^{(j)} := \Omega \times \cdots \times \Omega \times \partial\Omega \times \Omega \times \cdots \times \Omega, \qquad j = 1,\ldots,J+1,$$

with $\partial\Omega$ at the j-th position in this $(J+1)$-fold Cartesian product, and

$$\nu^{(j)}(r) := (0^{\mathrm{T}},\ldots,0^{\mathrm{T}},(\nu(r_j))^{\mathrm{T}},0^{\mathrm{T}},\ldots,0^{\mathrm{T}})^{\mathrm{T}} \in \mathbb{R}^{(J+1)d}.$$

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Specular boundary condition:

$$\varrho(r,v,t) = \varrho(r,v_*^{(j)},t)$$

for all $(r,v,t)\in\partial\Omega^{(j)}\times\mathbb{R}^{(J+1)d}\times(0,T]$, with $v\cdot\nu^{(j)}(r)<0$, where

$$v_*^{(j)} := v - 2(v\cdot\nu^{(j)}(r))\,\nu^{(j)}(r), \qquad j=1,\dots,J+1.$$

Existence of solutions

Define the Maxwellian

$$M(v) := (2\pi\beta)^{-\frac{J+1}{2}} \exp(-|v|^2/2\beta), \qquad v \in \mathbb{R}^{(J+1)d},$$

and rescale:

$$\widehat{\varrho} := \frac{\varrho}{M} \quad \text{and} \quad \widehat{\varrho}_0 := \frac{\varrho_0}{M}.$$

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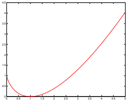
Consider the nonnegative strictly convex function with superlinear growth:

$$\mathcal{F}(s) := s(\log s - 1) + 1, \quad s \in \mathbb{R}_{>0}, \quad \text{with } \mathcal{F}(0) := 1.$$



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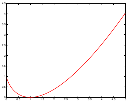

The initial datum $\varrho_0 = \varrho_0(r,v) \geq 0$ is assumed to satisfy

$$\varrho_0 \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}), \qquad \int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} \varrho_0(r,v)\,\mathrm{d}r\,\mathrm{d}v = 1,$$

$$M\mathcal{F}(\widehat{\varrho}_0) \in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d});$$

i.e. $\varrho_0 \geq 0$ is assumed to have finite relative entropy with respect to M.

Existence of solutions to FP, for u fixed

STEP 1.

We perform a parabolic regularization of FP equation by adding $-\alpha\Delta_r\varrho$ to it, with homogeneous Neumann boundary condition on $\partial\Omega^{J+1}$.

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STEP 2.

The assumption on the initial datum is strengthened by assuming *additionally* that

$$\widehat{\varrho}_0 \in L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}).$$

STEP 3.

For $u \in L^2(0,T;W_0^{1,\sigma}(\Omega)^d)$ for some $\sigma > d$, fixed, use linear parabolic theory based on Galerkin approximation to prove the existence of a unique weak solution to the α-regularized FP eq.:

$$\widehat{\varrho}_\alpha \in \mathcal{C}([0,T];L^2_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))\cap L^2(0,T;W^{1,2}_{*,M}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})),$$

$$\widehat{\varrho}_\alpha \in W^{1,2}(0,T;(W^{1,2}_{*,M}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))').$$

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$$\widehat{\varrho}_\alpha \in W^{1,2}(0,T;(W^{1,2}_{*,M}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))').$$

STEP 4.

We then define $\varrho_\alpha := M\widehat{\varrho}_\alpha$, and prove that

$$\int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} \varrho_\alpha(r,v,t)\,\mathrm{d}r\,\mathrm{d}v = 1 \qquad \forall\, t\in[0,T].$$

Using Stampacchia's cut-off method, we prove that $\varrho_\alpha(r,v,t) \geq 0$.

STEP 5.

As $\alpha \to 0_+$, parabolic energy estimates and relative entropy estimates yield uniform bounds on $\widehat{\varrho}_\alpha$, which imply that

$$
\begin{aligned}
\widehat{\varrho}_\alpha &\rightharpoonup \widehat{\varrho} && \text{weakly* in } L^\infty(0,T;L^2_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})),\\
\nabla_v\widehat{\varrho}_\alpha &\rightharpoonup \nabla_v\widehat{\varrho} && \text{weakly in } L^2(0,T;L^2_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})),\\
\alpha\nabla_r\widehat{\varrho}_\alpha &\to 0 && \text{strongly in } L^2(0,T;L^2_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})),\\
M\,\partial_t\widehat{\varrho}_\alpha &\rightharpoonup M\,\partial_t\widehat{\varrho} && \text{weakly in } L^2(0,T;(W^{s,2}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))'),\\
v_j\,\widehat{\varrho}_\alpha &\rightharpoonup v_j\,\widehat{\varrho} && \text{weakly in } L^2(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})),\\
((\mathcal{L}r)_j+u(r_j,\tau))\,\widehat{\varrho}_\alpha &\rightharpoonup ((\mathcal{L}r)_j+u(r_j,\tau))\,\widehat{\varrho} && \text{weakly in } L^2(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})),
\end{aligned}
$$

for $j = 1,\ldots,J+1$ and $s > (J+1)d+1$. Furthermore,

$$\varrho := M\widehat{\varrho} \geq 0 \quad \text{and} \quad \int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} \varrho(r,v,t)\,\mathrm{d}r\,\mathrm{d}v = 1 \qquad \forall\, t\in[0,T].$$

STEP 6.

$\widehat{\varrho}$ solves FP and by weak lower-semicontinuity satisfies the energy inequality:

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, \mathcal{F}(\widehat{\varrho}(t))\, \mathrm{d}v\, \mathrm{d}r + \frac{2\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, |\partial_{v_j} \sqrt{\widehat{\varrho}}|^2\, \mathrm{d}v\, \mathrm{d}r\, \mathrm{d}\tau$$
$$\leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, \mathcal{F}(\widehat{\varrho}_0)\, \mathrm{d}v\, \mathrm{d}r + \frac{16d}{\beta}\, (J+1)\, [\mathsf{diam}(\Omega)]^2\, T + \frac{J+1}{\beta}\, \|u\|^2_{L^2(0,T;L^\infty(\Omega))}$$

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Note:

We supposed that $\widehat{\varrho}_0 \in L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})$, but the bound only depends on the $L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})$ norm of $\mathcal{F}(\widehat{\varrho}_0)$, the $L^2(0,T; L^\infty(\Omega)^d)$ norm of u, and the constants d, β, J, L, T, all of which are independent of ϵ.

Existence of solutions to the Oseen–Fokker–Planck system

STEP 7.

We formulate an iterative process, by defining the sequence of functions $(u^{(k)}, \widehat{\varrho}^{(k)})$, for $k = 1, 2, \ldots$, and let $k \to \infty$.

Existence of solutions to the Oseen–Fokker–Planck system

STEP 7.

We formulate an iterative process, by defining the sequence of functions $(u^{(k)}, \widehat{\varrho}^{(k)})$, for $k = 1, 2, \ldots$, and let $k \to \infty$.

We set $u^{(1)} \equiv 0$. Given a divergence-free $u^{(k)} \in L^2(0, T; W_0^{1,\sigma}(\Omega)^d)$, for some $k \geq 1$ and $\sigma > d$, we define $\widehat{\varrho}^{(k)}$ as the weak solution of the FP eq.:

$$
\begin{aligned}
M\partial_t \widehat{\varrho}^{(k)} &- \frac{\beta^2}{\epsilon^2}\left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (M\partial_{v_j} \widehat{\varrho}^{(k)})\right) \\
&+ \frac{1}{\epsilon}\left(\sum_{j=1}^{J+1} M v_j \cdot \partial_{r_j} \widehat{\varrho}^{(k)} + ((\mathcal{L}r)_j + u^{(k)}(r_j, t)) \cdot \partial_{v_j}(M\widehat{\varrho}^{(k)})\right) = 0, \\
&\text{for all } (r, v, t) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T], \\
\widehat{\varrho}^{(k)}(r, v, 0) &= \widehat{\varrho}_0^{(k)}(r, v) \qquad \text{for all } (r, v) \in \Omega^{J+1} \times \mathbb{R}^{(J+1)d},
\end{aligned}
$$

subject to a (weakly imposed) specular boundary condition w.r.t. r.

STEP 8.

Given $\widehat{\varrho}_0$ as in our *original* assumption, consider

$$G_k(s) := \frac{s}{1 + k^{-\frac{1}{4}}\sqrt{s}}\,, \qquad s \in [0, \infty),$$

and define the renormalized initial condition

$$\widehat{\varrho}_0^{(k)} := G_k(\widehat{\varrho}_0), \qquad k = 1, 2, \ldots.$$

STEP 8.

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and define the renormalized initial condition

$$\widehat{\varrho}_0^{(k)} := G_k(\widehat{\varrho}_0), \qquad k = 1, 2, \ldots .$$

Then:

$$\begin{aligned}
\widehat{\varrho}_0^{(k)} &\in L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}) && \text{for each fixed } k \geq 1,\\
\widehat{\varrho}_0^{(k)} &\in L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), && \text{for each fixed } k \geq 1,\\
M\mathcal{F}(\widehat{\varrho}_0^{(k)}) &\in L^1(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0}), && \text{for each fixed } k \geq 1,\\
\int_{\Omega^{J+1} \times \mathbb{R}^{(J+1)d}} M(v)\, \widehat{\varrho}_0^{(k)}\, \mathrm{d}r\, \mathrm{d}v &\leq 1, && \text{for each fixed } k \geq 1,\\
\widehat{\varrho}_0^{(k)} \to \widehat{\varrho}_0 & \quad \text{strongly in } L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \text{ as } k \to \infty,\\
\mathcal{F}(\widehat{\varrho}_0^{(k)}) \to \mathcal{F}(\widehat{\varrho}_0) & \quad \text{strongly in } L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \text{ as } k \to \infty.
\end{aligned}$$

STEP 9.

By STEP 6, the sequence $\widehat{\varrho}^{(k)}$ satisfies the following energy inequality:

$$\begin{aligned}\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}^{(k)}(t))\,\mathrm{d}v\,\mathrm{d}r \\ + \frac{2\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\left|\partial_{v_j} \sqrt{\widehat{\varrho}^{(k)}}\,\right|^2 \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau \\ \leq \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\,\mathcal{F}(\widehat{\varrho}_0^{(k)})\,\mathrm{d}v\,\mathrm{d}r \\ + \frac{16d}{\beta}\,(J+1)\,[\mathrm{diam}(\Omega)]^2\,T + \frac{J+1}{\beta}\,\|u^{(k)}\|^2_{L^2(0,T;L^\infty(\Omega))},\end{aligned}$$

and the sequence $u^{(k)}$ will be shown (in STEP 13) to satisfy:

$$\|u^{(k)}\|_{L^2(0,T;L^\infty(\Omega))} \leq C,$$

where C is a positive constant, independent of k.

STEP 10.

Define $(u^{(k+1)}, \pi^{(k+1)})$, with

$$u^{(k+1)} \in L^\infty(0,T; L^2(\Omega)^d) \cap L^2(0,T; W_0^{1,2}(\Omega)^d),$$
$$\pi^{(k+1)} \in \mathcal{D}'(0,T; L^2(\Omega)/\mathbb{R}),$$

as the weak solution of the Oseen system:

$$\begin{aligned} \partial_t u^{(k+1)} + (b\cdot\nabla)u^{(k+1)} - \mu\triangle u^{(k+1)} + \nabla\pi^{(k+1)} &= \nabla\cdot\mathbb{K}^{(k)} && \text{for } (x,t)\in\Omega\times(0,T],\\ \nabla\cdot u^{(k+1)} &= 0 && \text{for } (x,t)\in\Omega\times(0,T],\\ u^{(k+1)}(x,0) &= u_0(x) && \text{for } x\in\Omega, \end{aligned}$$

$u_0 \in W_0^{1-2/z,z}(\Omega)^d$, with $z = d+\vartheta$, $\vartheta\in(0,1)$, is divergence-free, and

$$\mathbb{K}^{(k)}(x,t) := \frac{\int_{D^J\times\mathbb{R}^{(J+1)d}} \left(\sum_{j=1}^J \lambda q_j\otimes q_j\right) M\,\widehat{\varrho}^{(k)}\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}^{(k)}\,\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}.$$

STEP 11.

Clearly,

$$\|\mathbb{K}^{(k)}\|_{L^\infty(0,T;L^\infty(\Omega))} \leq \lambda\, d \max_{q\in D^J} \|q\|^2 =: C,$$

where C is a positive constant, independent of k. Thus, there exists a $\mathbb{K} \in L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{symm}}))$ (to be identified), such that

$$\mathbb{K}^{(k)} \to \mathbb{K} \qquad \text{weak}^* \text{ in } L^\infty(0,T;L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{symm}})) \text{ as } k \to \infty.$$

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Clearly,

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We would like to show that

$$\mathbb{K}(x,t) := \frac{\int_{D^J\times\mathbb{R}^{(J+1)d}} \big(\sum_{j=1}^J \lambda q_j \otimes q_j\big)\, M\, \widehat{\varrho}\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}\,\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}$$

but this is far from trivial. [We shall return to this in STEPS 15–17.]

STEP 12.

As $W_0^{1-2/z,z}(\Omega)^d \hookrightarrow L^2(\Omega)^d$ for $z = d + \vartheta$ and some $\vartheta \in (0,1)$, there exists a unique weak solution $(u^{(k+1)}, \pi^{(k+1)})$ to the Oseen system with

$$\|u^{(k+1)}\|_{L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;W^{1,2}(\Omega))} \leq C(1 + \|u_0\|_{L^2(\Omega)}),$$

where C is independent of k.

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where C is independent of k. Hence, by function space interpolation,

$$\|u^{(k+1)}\|_{L^{2+\frac{4}{d}}(Q_T)} \leq C$$

where $Q_T := \Omega \times (0,T)$.

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where C is independent of k. Hence, by function space interpolation,

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where $Q_T := \Omega \times (0,T)$. Therefore, also,

$$\|b \otimes u^{(k+1)}\|_{L^{2+\frac{4}{d}}(Q_T)} \leq C,$$

whereby

$$\|\mathbb{K}^{(k)} - b \otimes u^{(k+1)}\|_{L^{2+\frac{4}{d}}(Q_T)} \leq C.$$

STEP 13.

By maximal regularity theory for the Stokes system [Koch–Solonnikov (2001)], there is a positive constant $C = C_\sigma$, independent of k, s.t.

$$\|u^{(k+1)}\|_{W^{1,\frac{1}{2}}_\sigma(Q_T)} \le C\left(\|\mathbb{K}^{(k)} - b \otimes u^{(k+1)}\|_{L^\sigma(Q_T)} + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}\right),$$

where $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$, with $z = d + \vartheta$ for some $\vartheta \in (0,1)$,

$$W^{1,\frac{1}{2}}_\sigma(Q_T) := L^\sigma(0,T; W^{1,\sigma}_0(\Omega)^d) \cap W^{1/2,\sigma}(0,T; L^\sigma(\Omega)^d).$$

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As $W^{1,\frac{1}{2}}_\sigma(Q_T) \hookrightarrow L^2(0,T; W^{1,\sigma}_0(\Omega)^d)$, it follows that by STEP 12 that

$$\|u^{(k+1)}\|_{L^2(0,T;W^{1,\sigma}(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}), \qquad \sigma > d,$$

STEP 13.

By maximal regularity theory for the Stokes system [Koch–Solonnikov (2001)], there is a positive constant $C = C_\sigma$, independent of k, s.t.

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As $W^{1,\frac{1}{2}}_\sigma(Q_T) \hookrightarrow L^2(0,T; W^{1,\sigma}_0(\Omega)^d)$, it follows that by STEP 12 that

$$\|u^{(k+1)}\|_{L^2(0,T;W^{1,\sigma}(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}), \qquad \sigma > d,$$

and therefore, by Sobolev embedding,

$$\|u^{(k+1)}\|_{L^2(0,T;L^\infty(\Omega))} \leq C(1 + \|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}).$$

STEP 14.

We deduce that

$$
\begin{aligned}
&u^{(k)} \to u \qquad \text{weakly in } L^2(0,T;W_0^{1,\sigma}(\Omega)^d) \text{ as } k \to \infty, \qquad \sigma > d,\\
&u^{(k)} \to u \qquad \text{weakly in } W^{1,2}(0,T;W^{-1,\sigma}(\Omega)^d) \text{ as } k \to \infty,\\
&u^{(k)} \to u \qquad \text{strongly in } L^2(0,T;\mathcal{C}^{0,\gamma}(\overline{\Omega})^d) \text{ as } k \to \infty, \qquad 0 < \gamma < 1 - \tfrac{d}{\sigma}, \quad \sigma > d,
\end{aligned}
$$

where the last result follows, via the Aubin–Lions lemma, thanks to the compact embedding of $W_0^{1,\sigma}(\Omega)^d$ into $\mathcal{C}^{0,\gamma}(\overline{\Omega})^d$ for $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$.

STEP 14.

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$$u^{(k)} \to u \qquad \text{weakly in } L^2(0,T;W_0^{1,\sigma}(\Omega)^d) \text{ as } k \to \infty, \qquad \sigma > d,$$
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where the last result follows, via the Aubin–Lions lemma, thanks to the compact embedding of $W_0^{1,\sigma}(\Omega)^d$ into $\mathcal{C}^{0,\gamma}(\overline{\Omega})^d$ for $0 < \gamma < 1 - \frac{d}{\sigma}$, $\sigma > d$.

It is now straightforward to pass to the limit in the Oseen system:

$$\begin{aligned} \partial_t u + (b\cdot\nabla)u - \mu\triangle u + \nabla\pi &= \nabla\cdot\mathbb{K} && \text{for } (x,t) \in \Omega \times (0,T], \\ \nabla\cdot u &= 0 && \text{for } (x,t) \in \Omega \times (0,T], \\ u(x,t) &= 0 && \text{for } (x,t) \in \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x) && \text{for } x \in \Omega. \end{aligned}$$

STEP 15.

It remains to identify the weak* limit $\mathbb{K}$ of the sequence $(\mathbb{K}^{(k)})_{k\geq 0}$ in terms of the limit $\widehat{\varrho}$ of the sequence $(\widehat{\varrho}^{(k)})_{k\geq 0}$.

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We rewrite FP equation as

$$M\partial_t \widehat{\varrho}^{(k)} - \frac{\beta^2}{\epsilon^2}\left(\sum_{j=1}^{J+1} \partial_{v_j} \cdot (M\partial_{v_j} \widehat{\varrho}^{(k)})\right) + \frac{1}{\epsilon}\left(\sum_{j=1}^{J+1} M v_j \cdot \partial_{r_j} \widehat{\varrho}^{(k)}\right)$$

$$= -\frac{1}{\epsilon}\sum_{j=1}^{J+1}\Big(((\mathcal{L}r)_j + u^{(k)}(r_j,t)) \cdot \partial_{v_j}(M\widehat{\varrho}^{(k)})\Big)$$

$$\text{in } \mathcal{D}'(\Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0,T)),$$

$$\widehat{\varrho}^{(k)}(r,v,0) = \widehat{\varrho}_0^{(k)}(r,v) = G_k(\widehat{\varrho}_0(r,v)),$$

and note that the differential operator on the left-hand side is hypoelliptic.

STEP 16.

(A) Boundedness of the sequence of initial data:

$$\sup_{k\geq 1} \|M\, \widehat{\varrho}_0^{(k)}\|_{L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})} \leq C.$$

STEP 16.

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(B) Equiboundedness of the sequence of initial data:

$$\sup_{k\geq 1} \|\chi_{|v|\geq R}(\cdot)\, M\, \widehat{\varrho}_0^{(k)}\|_{L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})} \leq \frac{C}{R^2} \qquad \forall\, R\geq 1.$$

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(B) Equiboundedness of the sequence of initial data:

$$\sup_{k\geq 1} \|\chi_{|v|\geq R}(\cdot)\,M\,\widehat{\varrho}_0^{(k)}\|_{L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})} \leq \frac{C}{R^2} \qquad \forall\, R \geq 1.$$

(C) Boundedness of the sequence of right-hand sides:

$$\sup_{k\geq 1} \int_0^T \Big(1 + \|u^{(k)}(\cdot,t)\|_{L^\infty(\Omega)}\Big)\,\|M\,\partial_{v_j}\,\widehat{\varrho}^{(k)}(\cdot,\cdot,t)\|_{L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})}\,\mathrm{d}t \leq C.$$

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$$\sup_{k\geq 1} \|\chi_{|v|\geq R}(\cdot)\,M\,\widehat{\varrho}_0^{(k)}\|_{L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})} \leq \frac{C}{R^2} \qquad \forall\, R\geq 1.$$

(C) Boundedness of the sequence of right-hand sides:

$$\sup_{k\geq 1}\int_0^T \Big(1+\|u^{(k)}(\cdot,t)\|_{L^\infty(\Omega)}\Big)\,\|M\,\partial_{v_j}\widehat{\varrho}^{(k)}(\cdot,\cdot,t)\|_{L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})}\,\mathrm{d}t \leq C.$$

(D) Equiboundedness of the sequence of right-hand sides:

$$\lim_{R\to\infty}\sup_{k\geq 1}\|\chi_{|v|\geq R}(\cdot)\,((\mathcal{L}r)_j+u^{(k)})\cdot\partial_{v_j}(M\widehat{\varrho}^{(k)})\|_{L^1(0,T;L^1(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))} = 0.$$

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By an argument of R. DiPerna & P.-L. Lions (1988), (A)–(D) imply that

$$\widehat{\varrho}^{(k)} \to \widehat{\varrho} \qquad \text{strongly in } L^1(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})) \quad \text{as } k \to \infty.$$

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This then (eventually, after a further technical argument,) implies that

$$\mathbb{K}^{(k)}(x,t) := \frac{\int_{D^J\times\mathbb{R}^{(J+1)d}} \big(\sum_{j=1}^{J} \lambda q_j\otimes q_j\big)\, M\,\widehat{\varrho}^{(k)}\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}^{(k)}\,\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}$$

converges to

$$\mathbb{K}(x,t) := \frac{\int_{D^J\times\mathbb{R}^{(J+1)d}} \big(\sum_{j=1}^{J} \lambda q_j\otimes q_j\big)\, M\,\widehat{\varrho}\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}{\int_{D^J\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}\,\big(B(q,x),v,t\big)\,\mathrm{d}q\,\mathrm{d}v}$$

weakly* in $L^\infty(\Omega\times(0,T))$.

STEP 17.

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This then (eventually, after a further technical argument,) implies that

$$\mathbb{K}^{(k)}(x,t) := \frac{\int_{D^J \times \mathbb{R}^{(J+1)d}} (\sum_{j=1}^{J} \lambda q_j \otimes q_j)\, M\, \widehat{\varrho}^{(k)}\big(B(q,x), v, t\big)\, \mathrm{d}q\, \mathrm{d}v}{\int_{D^J \times \mathbb{R}^{(J+1)d}} M\, \widehat{\varrho}^{(k)}\, \big(B(q,x), v, t\big)\, \mathrm{d}q\, \mathrm{d}v}$$

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weakly* in $L^\infty(\Omega \times (0,T))$. Thus we have shown the existence of a weak solution to the coupled Oseen–Fokker–Planck system with the bead-mass $\epsilon > 0$ fixed. □

Small-mass limit and equilibration in momentum space

We showed the existence of functions $u = u_\epsilon$ and $\widehat{\varrho} = \widehat{\varrho}_\epsilon$, such that

$$u_\epsilon \in \mathcal{C}([0,T]; L^\sigma(\Omega)^d) \cap L^2(0,T; W_0^{1,\sigma}(\Omega)^d) \cap W^{1,2}(0,T; W^{-1,\sigma}(\Omega)^d),$$

with $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$, is a weak solution to the Oseen system,

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with $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$, is a weak solution to the Oseen system, and $\widehat{\varrho}_\epsilon$ with

$$\mathcal{F}(\widehat{\varrho}_\epsilon) \in L^\infty(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}; \mathbb{R}_{\geq 0})),$$

$$\nabla_v \sqrt{\widehat{\varrho}_\epsilon} \in L^2(0,T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

$$\nabla_v \widehat{\varrho}_\epsilon \in L^2(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})),$$

$$M\, \partial_t \widehat{\varrho}_\epsilon \in L^2(0,T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \quad s > (J+1)d + 1,$$

satisfies the following weak form of the Fokker–Planck equation:

$$
\begin{aligned}
&\int_0^t \left\langle M\, \partial_\tau \widehat{\varrho}_\epsilon(\cdot,\cdot,\tau), \varphi(\cdot,\cdot,\tau)\right\rangle \mathrm{d}\tau \\
&\quad + \frac{\beta^2}{\epsilon^2} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, \partial_{v_j} \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi \,\mathrm{d}v \,\mathrm{d}r \,\mathrm{d}\tau \right) \\
&\quad - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, v_j \widehat{\varrho}_\epsilon \cdot \partial_{r_j} \varphi \,\mathrm{d}v \,\mathrm{d}r \,\mathrm{d}\tau \right) \\
&\quad - \frac{1}{\epsilon} \left(\sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, ((\mathcal{L}r)_j + u_\epsilon(r_j,\tau))\, \widehat{\varrho}_\epsilon \cdot \partial_{v_j} \varphi \,\mathrm{d}v \,\mathrm{d}r \,\mathrm{d}\tau \right) = 0
\end{aligned}
$$

$$
\forall\, \varphi \in L^2(0,T; W^{1,2}_{*,M}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}) \cap W^{s,2}_{*}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \quad s > (J+1)d + 1.
$$

$$
\begin{aligned}
&\int_0^t \big\langle M\,\partial_\tau \widehat{\varrho}_\epsilon(\cdot,\cdot,\tau), \varphi(\cdot,\cdot,\tau)\big\rangle \,\mathrm{d}\tau \\
&\quad + \frac{\beta^2}{\epsilon^2}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,\partial_{v_j}\widehat{\varrho}_\epsilon \cdot \partial_{v_j}\varphi \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) \\
&\quad - \frac{1}{\epsilon}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,v_j\widehat{\varrho}_\epsilon \cdot \partial_{r_j}\varphi \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) \\
&\quad - \frac{1}{\epsilon}\left(\sum_{j=1}^{J+1}\int_0^t \int_{\Omega^{J+1}}\int_{\mathbb{R}^{(J+1)d}} M(v)\,((\mathcal{L}r)_j + u_\epsilon(r_j,\tau))\,\widehat{\varrho}_\epsilon \cdot \partial_{v_j}\varphi \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau\right) = 0 \\
&\forall\,\varphi \in L^2(0,T; W^{1,2}_{*,M}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}) \cap W^{s,2}_{*}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \quad s > (J+1)d+1.
\end{aligned}
$$

Furthermore $\widehat{\varrho}_\epsilon(\cdot,\cdot,0) = \widehat{\varrho}_0(\cdot,\cdot)$,

$$
\int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}_\epsilon(r,v,t)\,\mathrm{d}r\,\mathrm{d}v = \int_{\Omega^{J+1}\times\mathbb{R}^{(J+1)d}} M\,\widehat{\varrho}_0(r,v)\,\mathrm{d}r\,\mathrm{d}v = 1 \qquad \forall\,t \in (0,T].
$$

In addition, $\widehat{\varrho}_\epsilon$ satisfies the following energy inequality:

$$\int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, \mathcal{F}(\widehat{\varrho}_\epsilon(t))\, \mathrm{d}v\, \mathrm{d}r + \frac{2\beta^2}{\epsilon^2} \sum_{j=1}^{J+1} \int_0^t \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, |\partial_{v_j} \sqrt{\widehat{\varrho}_\epsilon}\,|^2 \,\mathrm{d}v\, \mathrm{d}r\, \mathrm{d}\tau$$
$$\leq C\left[1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, \mathcal{F}(\widehat{\varrho}_0)\, \mathrm{d}v\, \mathrm{d}r\right],$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}, \|b\|_{L^\infty(0,T;L^\infty(\Omega))})$, $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$; C is independent of $\epsilon > 0$.

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$$\leq C\Big[1 + \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, \mathcal{F}(\widehat{\varrho}_0)\, \mathrm{d}v\, \mathrm{d}r\Big],$$

where $C = C(\|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}, \|b\|_{L^\infty(0,T;L^\infty(\Omega))})$, $\sigma = \min(\hat{\sigma}, z) > d$, $\hat{\sigma} := 2 + \frac{4}{d}$ and $z = d + \vartheta$ for some $\vartheta \in (0,1)$; C is independent of $\epsilon > 0$.

Hence,

$$\begin{aligned} (\mathcal{F}(\widehat{\varrho}_\epsilon))_{\epsilon>0} \quad & \text{is bounded in } L^\infty(0,T; L^1_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ (\nabla_v \sqrt{\widehat{\varrho}_\epsilon})_\epsilon \quad & \text{is bounded in } L^2(0,T; L^2_M(\Omega^{J+1} \times \mathbb{R}^{(J+1)d})), \\ (M\, \partial_t \widehat{\varrho}_\epsilon)_{\epsilon>0} \quad & \text{is bounded in } L^2(0,T; (W^{s,2}(\Omega^{J+1} \times \mathbb{R}^{(J+1)d}))'), \end{aligned}$$

for $s > (J+1)d + 1$.

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$$
\begin{aligned}
\widehat{\varrho}_\epsilon \rightharpoonup \widehat{\varrho}_{(0)} &\quad \text{weakly in } L^p(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})) \qquad \forall\, p\in[1,\infty),\\
M\,\partial_t\widehat{\varrho}_\epsilon \rightharpoonup M\,\partial_t\widehat{\varrho}_{(0)} &\quad \text{weakly in } L^2(0,T;(W^{s,2}(\Omega^{J+1}\times\mathbb{R}^{(J+1)d}))'), \quad s>(J+1)d+1,\\
v_j\,\widehat{\varrho}_\epsilon \rightharpoonup v_j\,\widehat{\varrho}_{(0)} &\quad \text{weakly in } L^2(0,T;L^1_M(\Omega^{J+1}\times\mathbb{R}^{(J+1)d})), \quad j=1,\dots,J+1.
\end{aligned}
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\end{aligned}
$$

Also, because

$$
\|u_\epsilon\|_{L^2(0,T;W^{1,\sigma}(\Omega))\cap W^{1,2}(0,T;W^{-1,\sigma}(\Omega))} \le C\big(1+\|u_0\|_{W^{1-\frac{2}{\sigma},\sigma}(\Omega)}\big),
$$

with $\sigma > d$, whereby

$$
u_\epsilon \rightharpoonup u_{(0)} \quad \text{weakly in } L^2(0,T;W^{1,\sigma}(\Omega))\cap W^{1,2}(0,T;W^{-1,\sigma}(\Omega)),
$$

$$
u_\epsilon \to u_{(0)} \quad \text{strongly in } L^2(0,T;\mathcal{C}^{0,\gamma}(\overline{\Omega})^d) \text{ as } \epsilon\to 0_+, \qquad 0<\gamma<1-\tfrac{d}{\sigma}.
$$

STEP 19.

Furthermore,

$$\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \, |\partial_{v_j} \sqrt{\widehat{\varrho}_{(0)}}\,|^2 \, \mathrm{d}v \, \mathrm{d}r \, \mathrm{d}\tau \leq 0.$$

STEP 19.

Furthermore,

$$\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v) \, |\partial_{v_j} \sqrt{\widehat{\varrho}_{(0)}}\,|^2 \, \mathrm{d}v \, \mathrm{d}r \, \mathrm{d}\tau \leq 0.$$

Hence,

$$\widehat{\varrho}_{(0)}(r, v, t) = \eta(r, t) \qquad \text{a.e. in } \Omega^{J+1} \times \mathbb{R}^{(J+1)d} \times (0, T)$$

with $\eta \in L^\infty(0, T; L^1(\Omega^{J+1}))$ to be determined.

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with $\eta \in L^\infty(0, T; L^1(\Omega^{J+1}))$ to be determined.

Thus, we have equilibration in momentum space:

$$\varrho_{(0)} := M \, \widehat{\varrho}_{(0)} = M \, \eta,$$

with $\eta \in L^\infty(0, T; L^1(\Omega^{J+1}))$, to be deteremined.

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$$\sum_{j=1}^{J+1} \int_0^T \int_{\Omega^{J+1}} \int_{\mathbb{R}^{(J+1)d}} M(v)\, |\partial_{v_j} \sqrt{\widehat{\varrho}_{(0)}}\,|^2 \,\mathrm{d}v\,\mathrm{d}r\,\mathrm{d}\tau \leq 0.$$

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J. Schieber & H. C. Öttinger, The effects of bead inertia on the Rouse model. J. Chem. Phys. 89, no. 11, (1988).

STEP 20.

The small-mass limit of the coupled Oseen–Fokker–Planck system satisfies

$$
\begin{aligned}
\partial_t u_{(0)} + (b\cdot\nabla)u_{(0)} - \mu\triangle u_{(0)} + \nabla\pi_{(0)} &= \nabla\cdot\mathbb{K}_{(0)} && \text{for } (x,t)\in\Omega\times(0,T],\\
\nabla\cdot u_{(0)} &= 0 && \text{for } (x,t)\in\Omega\times(0,T],\\
u_{(0)}(x,t) &= 0 && \text{for } (x,t)\in\partial\Omega\times(0,T],\\
u_{(0)}(x,0) &= u_0(x) && \text{for } x\in\Omega.
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u_{(0)}(x,0) &= u_0(x) && \text{for } x \in \Omega.
\end{aligned}$$

The identification of $\mathbb{K}_{(0)}$ (via the DIV-CURL Lemma) is (again) technical:

$$\mathbb{K}_{(0)}(x,t) := \frac{\int_{D^J} \sum_{j=1}^{J} (F(q_j) \otimes q_j)\, \eta\big(B(q,x),t\big)\, \mathrm{d}q}{\int_{D^J} \eta\big(B(q,x),t\big)\, \mathrm{d}q} \qquad \text{for } (x,t) \in \Omega \times (0,T],$$

where $B(x,q) = r$,

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$$

where $B(x,q)=r$, and $\eta\geq 0$, with $\int_{\Omega^{J+1}}\eta(r,t)\,\mathrm{d}r = 1$ for all $t\in[0,T]$, solves

$$
\partial_t\eta + \sum_{j=1}^{J+1}\partial_{r_j}\cdot\Big(\eta\,((\mathcal{L}r)_j + u_{(0)}(r_j,\cdot))\Big) - \beta\sum_{j=1}^{J+1}\partial^2_{r_j}\eta = 0 \qquad \text{in } \Omega^{J+1}\times(0,T],
$$

$$
\eta(\cdot,0) = \widehat{\varrho}_0 \in L^2(\Omega^{J+1};\mathbb{R}_{\geq 0}) \quad + \begin{cases} \text{zero-flux boundary condition on} \\ \partial\Omega^{(j)}\times(0,T] \text{ for } j=1,\dots,J+1. \end{cases}
$$

Change variables in FP from $r \in \Omega^{J+1}$ to $(x, q) \in \Omega \times D^J$

Hence, $\psi(x, q, t) := \eta(B(q, x), t) = \eta(r, t)$ solves on $\Omega \times D^J \times [0, T]$:

$$\partial_t \psi + \nabla_x \cdot (u_{(0)} \psi) + \sum_{j=1}^{J} \partial_{q_j} \cdot ((\nabla_x u_{(0)}) q_j \psi)$$

$$- \beta \sum_{i,j=1}^{J} \partial_{q_j} \cdot \left[\mathcal{R}_{ij} \, \mathfrak{M}(q) \, \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta_x \psi = 0,$$

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Hence, $\psi(x, q, t) := \eta(B(q, x), t) = \eta(r, t)$ solves on $\Omega \times D^J \times [0, T]$:

$$\partial_t \psi + \nabla_x \cdot (u_{(0)}\psi) + \sum_{j=1}^{J} \partial_{q_j} \cdot ((\nabla_x u_{(0)}) q_j \psi)$$

$$- \beta \sum_{i,j=1}^{J} \partial_{q_j} \cdot \left[\mathcal{R}_{ij}\, \mathfrak{M}(q)\, \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) \right] - \frac{\beta}{J+1} \Delta_x \psi = 0,$$

with the initial condition $\psi(x, q, 0) = \psi_0(x, q) := \hat{\varrho}_0(B(q, x))$, and the bdry. cond.

$$\nabla_x \psi(x, q, t) \cdot n_x(x) = 0 \qquad \text{for all } (x, q, t) \in \partial\Omega \times D^J \times (0, T],$$

where n_x is the unit outward normal vector to $\partial\Omega$, and

$$\sum_{i=1}^{J} \left[\beta\, \mathcal{R}_{ij}\, \mathfrak{M}(q)\, \partial_{q_i} \left(\frac{\psi}{\mathfrak{M}(q)} \right) - (\nabla u_{(0)}) q_j \psi \right] \cdot n_{q_j} = 0$$

for all $(x, q, t) \in \Omega \times (D \times \cdots \times \partial D \times \cdots \times D) \times (0, T]$, $j = 1, \ldots, J$, where n_{q_j} is the unit outward normal vector to ∂D for the jth copy of D; and

$$\mathfrak{M}(q) := (2\pi\beta)^{-\frac{1}{2}Jd} \exp\left(-|q|^2/2\beta\right), \quad q \in D^J.$$

Note: If the initial datum ψ_0 is such that, for some constant $n > 0$,

$$\int_{D^J} \psi_0(x,q)\,\mathrm{d}q = n^{-1} \qquad \text{for a.e. } x \in \Omega,$$

then it follows that

$$\int_{D^J} \eta(B(q,x),t)\,\mathrm{d}q = \int_{D^J} \psi(x,q,t)\,\mathrm{d}q = n^{-1} \qquad \text{for a.e. } (x,t) \in \Omega \times [0,T],$$

so the expression for $\mathbb{K}_{(0)}$ simplifies to the Kramers' expression:

$$\mathbb{K}_{(0)} = n \int_{D^J} \sum_{j=1}^{J} (F(q_j) \otimes q_j)\,\psi(x,q,t)\,\mathrm{d}q.$$

The number $n > 0$ is called the *polymer number density per unit volume.*

M. Dostalík, J. Málek, V. Průša, and E. Süli. A simple approach to thermodynamically consistent modelling of non-isothermal flows of dilute compressible polymeric fluids. Fluids 2020, Volume 5, Issue 3, 29 pp.; DOI:10.3390/fluids5030133.