Řešení příkladů ze cvičení Kalkulus 2, 10.12.2025

$$\boldsymbol{I = \int_M \mathrm{d}x\,\mathrm{d}y}\,, \quad M \text{ omezená křivkami } xy = 1, xy = 3, 2y = x, y = 2x$$

$$u = xy, v = \frac{y}{x}, 1 \le u \le 3, \frac{1}{2} \le v \le 2 \Rightarrow x = \sqrt{\frac{u}{v}}, y = \sqrt{uv}, J(u,v) = \frac{1}{2v}, I = \int_1^3 \,\mathrm{du} \int_{1/2}^2 \frac{1}{2v}\,\mathrm{dv} = 2\log 2$$

$$\boldsymbol{I = \int_M (x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y}\,, \quad M = \{(x,y); 1 \le x^2+y^2 \le 4, y \ge |x|\}$$

$$x = r\cos t, y = r\sin t, J(r,t) = r, 1 \le r \le 2, \frac{\pi}{4} \le t \le \frac{3\pi}{4} \Rightarrow I = \int_{\pi/4}^{3\pi/4} \,\mathrm{dt} \int_1^2 r^3\,\mathrm{dr} = \frac{15}{8}\pi$$

$$\boldsymbol{\int_M e^{-x^2-y^2}\,\mathrm{d}x\,\mathrm{d}y}\,, \quad M \text{ je první kvadrant}$$

$$\text{polární souřadnice, } 0 < r < \infty, 0 < t < \frac{\pi}{2} \Rightarrow I = \int_0^{\pi/2} \,\mathrm{dt} \int_0^\infty re^{-r^2}\,\mathrm{dr} = \frac{\pi}{4} \Rightarrow \int_0^\infty e^{-x^2}\,\mathrm{dx} = \frac{\sqrt{\pi}}{2}$$

$$\boldsymbol{I = \int_M (x+y^2)\,\mathrm{d}x\,\mathrm{d}y}\,, \quad M = \{(x,y); \frac{(x-2)^2}{9} + \frac{(y-1)^2}{4} \le 1, 1 \le y \le 4\}$$

$$x = 2+3r\cos t, y = 1+2r\sin t, J(r,t) = 6r, r \in (0,1), t \in (0,\pi), I = \int_0^1\int_0^\pi (4r^2\sin^2 t + 4r\sin t + 3r\cos t + 3)6r\,\mathrm{dt}\,\mathrm{dr} = 4(3\pi+4)$$

$$\boldsymbol{I = \int_M \sqrt{xy}\,\mathrm{d}x\,\mathrm{d}y}\,, \quad M = \{(x,y); \left(\frac{x}{2}+\frac{y}{3}\right)^4 \le \frac{xy}{6}, x, y \ge 0\}$$

$$x = 2r\cos^2 t, y = 3r\sin^2 t, J(r,t) = 6r\sin(2t), r \in (0, \cos t\sin t), t \in (0, \frac{\pi}{2}), I = 12\sqrt{6}\int_0^{\pi/2} \sin^2(2t)\Big(\int_0^{\sin(2t)/2} r^2\,\mathrm{dr}\Big)\,\mathrm{dt} = \frac{\sqrt{6}}{15}$$

$$\boldsymbol{I = \int_M xz\sqrt{x^2+y^2}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}\,, \quad M = \{(x,y,z); 1 \le x^2+y^2 \le 4, 0 \le z \le 3, x \le 0, y \ge 0\}$$

$$x = r\cos t, y = r\sin t, z = z, J(r,t,z) = r, r \in (0,2), t \in (\frac{\pi}{2}, \pi), z \in (0,3), I = \int_1^2 r^3\,\mathrm{dr} \int_{\pi/2}^\pi \cos t\,\mathrm{dt} \int_0^3 \,\mathrm{dz} = -\frac{135}{8}$$

$$\boldsymbol{I = \int_M \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}\,, \quad M = \{(x,y,z); \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1\}$$

$$x = ar\cos t\cos u, y = br\sin t\cos u, z = cr\sin u, J(r,t,u) = abcr^2\cos u, r \in (0,1), t \in (0, 2\pi), u \in (-\frac{\pi}{2}, \frac{\pi}{2})$$

$$I = abc\int_0^{2\pi} \,\mathrm{dt} \int_{-\pi/2}^{\pi/2} \cos u\,\mathrm{du} \int_0^1 r^2\sqrt{1-r^2}\,\mathrm{dr} = \frac{\pi^2}{4}abc$$