SMOOTH APPROXIMATIONS

ANTOINE MOTTET
CHARLES UNIVERSITY

JOINT WORK WITH M. PINSKER

REMINDER
OLIGOMORPHIC CLONES

- $\mathcal{C}, \mathcal{D}$ CLONES ON COUNTABLE SET A
  - POINTWISE-CONVERGENCE TOPOLOGY
  - CONTAIN OLIGOMORPHIC GROUP
  - CLOSED

$$\rightsquigarrow \mathrm{Pol}(\mathbb{A}) = \left\{ f : A^n \to A \;\middle|\; n \geq 1\ \forall R^{\mathbb{A}}\ \forall \bar{a}_1, \ldots, \bar{a}_n \in R^{\mathbb{A}}\ f(\bar{a}_1, \ldots, \bar{a}_n) \in R^{\mathbb{A}} \right\}$$

$\mathbb{A}$ ω-CATEGORICAL

- $\mathcal{F}$ CLONE ON FINITE SET
  ($\Rightarrow$ $\mathcal{F}$ DISCRETE SPACE)

- $\mathcal{P}$ CLONE OF PROJECTIONS ON $\{0,1\}$
  $\forall \mathcal{C} : \mathcal{P} \subseteq \mathcal{C}$

DEF $\xi : \mathcal{C} \longrightarrow \mathcal{D}$ ARITY-PRESERVING

MINION HOM.	CLONE HOM.
• $\xi(f \circ (g_1, \dots, g_n)) = \xi(f) \circ (g_1, \dots, g_n)$	• $\xi(g) = g \quad \forall g \in \mathcal{P}$
$\forall f \in \mathcal{C}^{(n)}$	• $\xi(f \circ (g_1, \dots, g_n)) = \xi(f) \circ (\xi(g_1), \dots, \xi(g_n))$
$\forall g_1, \dots, g_n \in \mathcal{P}$	$\forall f \in \mathcal{C}^{(n)}$
$f(x,y) = f(y,x)$	$\forall g_1, \dots, g_n \in \mathcal{C}^{(m)}$
$\xi(f)(x,y) \approx \xi(f)(y,x)$	

QUESTIONS:

• (UNIFORM) CONTINUITY:

$\mathcal{C} \curvearrowright A \qquad \mathcal{D} \curvearrowright B$

$\xi : \mathcal{C} \to \mathcal{D}$ UNIF. CONT.

$\Updownarrow$

$\forall n \geq 1\ \forall Y \subseteq B$ FINITE $\exists X \subseteq A$ FINITE

$\forall f, g \in \mathcal{C}^{(n)} \left(f|_X = g|_X \Rightarrow \xi(f)|_Y = \xi(g)|_Y \right)$

$\hookrightarrow$ IS EVERY $\xi : \mathcal{C} \to \mathcal{D}$ CONTINUOUS?

$\hookrightarrow \quad \mathcal{C} \to \mathcal{D} \Rightarrow \mathcal{C} \to \mathcal{D}$?

- EQUATIONS/IDENTITIES

$\exists f \in \mathcal{C}^{(2)} : f(x,y) \approx f(y,x)$

$\Rightarrow \nexists \xi : \mathcal{C} \to \mathcal{P}$

BY DEFINITION:

- $\nexists\, \xi : \mathcal{C} \to \mathcal{P}$

- $\exists\, \Sigma$ FIN. SET OF IDENTITIES SAT. IN $\mathcal{C}$ BUT NOT SAT. IN $\mathcal{P}$.

CAN ONE SAY MORE ABOUT Σ?

- COMPLEXITY

$$\mathcal{C} = \mathrm{Pol}(\mathbb{A}) \not\to \mathcal{P}$$

$$\mathrm{CSP}(\mathbb{A}) = \{ \mathbb{X} \text{ finite} \mid \mathbb{X} \to \mathbb{A} \}$$

<u>THEOREM:</u> (BULATOV, ZHUK)

$\mathbb{A}$ FINITE $\quad \mathrm{Pol}(\mathbb{A})$ IDEMPOTENT

$\mathrm{Pol}(\mathbb{A}) \not\to \mathcal{P}$

THEN $\mathrm{CSP}(\mathbb{A})$ IS IN P.

CANONICAL
CLONES

$G \curvearrowright A \qquad f : A^n \longrightarrow A \qquad m \geq 1$

DEFINITION:

f m-CANONICAL IF $f \curvearrowright A^m / G$:

$$\forall \bar{a}_1, \dots, \bar{a}_n \in A^m \; \forall \alpha_1, \dots, \alpha_n \in G \; \exists \beta \in G$$

$$f(\alpha_1 \bar{a}_1, \dots \alpha_n \bar{a}_n) = \beta f(\bar{a}_1, \dots, \bar{a}_n)$$

CANONICAL = $\forall m$, m-CANONICAL

EQUIVALENTLY

$$\bar{a} \equiv_{\mathcal{G}}^{m} \bar{b} \;:\Leftrightarrow\; \mathcal{G} \cdot \bar{a} = \mathcal{G} \cdot \bar{b}$$

f m-CANONICAL $\Leftrightarrow f \in \mathrm{Pol}(\mathbb{A}, \equiv_{\mathcal{G}}^{m})$.

$\rightsquigarrow \{ f : A^n \to A \mid n \geq 0,\ f\ \mathcal{G}\text{-CANONICAL} \}$ CLONE.

1) CONTINUITY

THEOREM (BODIRSKY PINSKER PONGRÁCZ)

$\mathcal{C}$ CANONICAL WRT $\mathrm{Aut}(\mathbb{A})$
$\mathbb{A}$ FINITELY HOMOGENEOUS

$\mathcal{D}$ IDEMPOTENT

THEN EVERY $\xi : \mathcal{C} \to \mathcal{D}$ IS CONT.

PROOF m LARGE ($\geq$ ARITY RELATIONS OF $\mathbb{A}$)

$$f \curvearrowright A^m/\mathcal{G} = g \curvearrowright A^m/\mathcal{G}$$

$$\Rightarrow \exists e \in \overline{\mathcal{G}} : f = e \cdot g .$$

$$\xi(f) = \xi(e \cdot g) = \xi(e)\xi(g) = \xi(g).$$

$\square$

2) IDENTITIES

THEOREM: (BODIRSKY-PINSKER-PONGRÁCZ + BARTO-KOZIK + SIGGERS ...)

$\mathcal{C}$ CANONICAL WRT $Aut(\mathbb{A})$

$\mathbb{A}$ FIN. HOMOGENEOUS

$$\overline{Aut(\mathbb{A})} = \mathcal{C}^{(1)}$$

- $\mathcal{C} \not\to \mathcal{P}$
- $\exists f \in \mathcal{C}^{(n)}\ u, v \in \mathcal{C}^{(1)}$

  $$u \cdot f(x_1, \ldots, x_n) = v \cdot f(x_2, \ldots, x_n, x_1)$$
- $\exists f \in \mathcal{C}^{(n)}\ \exists u_1, \ldots, u_n \in \mathcal{C}^{(1)}$

  $$u_1 f(y, x, \ldots, x) \approx \cdots \approx u_n f(x, \ldots, x, y)$$
- $\exists f \in \mathcal{C}^{(6)}\ \exists u, v \in \mathcal{C}^{(1)}$:

  $$u \cdot f(x\,y\,x\,z\,y\,z) \approx v\, f(y\,x\,z\,x\,z\,y)$$

FALSE IN GENERAL

OPEN

TRUE (BARTO-PINSKER)

3) COMPLEXITY

THEOREM

$\mathcal{C} = \mathrm{Pol}(\mathbb{A})$ $\quad \mathcal{C}$ CANONICAL WRT $\mathrm{Aut}(\mathbb{B})$

$\mathbb{B}$ FIN. HOMOGENEOUS
FIN. BOUNDED

$\overline{\mathrm{Aut}(\mathbb{B})} = \mathcal{C}^{(1)}$.

IF $\mathcal{C} \not\to \mathcal{P}$ THEN $\mathrm{CSP}(\mathbb{A})$ IN P.

PROOF: $\mathcal{C} \longrightarrow \mathcal{C} \curvearrowright A^m/_{\mathrm{Aut}(\mathbb{B})} =: \mathcal{C}^m/_{\mathrm{Aut}(\mathbb{B})}$

$$\mathcal{C} \not\to \mathcal{P} \implies \mathcal{C}^m/_{\mathrm{Aut}(\mathbb{B})} \not\to \mathcal{P}$$

$\exists\, \mathbb{C} : \mathcal{C}^m/_{\mathrm{Aut}(\mathbb{B})} = \mathrm{Pol}(\mathbb{C})$ IDEMPOTENT
AND $\mathrm{CSP}(\mathbb{A}) \leq \mathrm{CSP}(\mathbb{C})$.

$\square$

CANONICITY EVERYWHERE (?)

THEOREM: BODIRSKY-PINSKER-TSANKOV

LET · $\mathcal{G} \curvearrowright A$ OLIGOMORPHIC
EXTREMELY AMENABLE
· $f: A^n \longrightarrow A$

THEN $\overline{\mathcal{G}f\mathcal{G}}$ CONTAINS A $\mathcal{G}$-CANONICAL FUNCTION.

QUESTION/CONJECTURE:

$\forall\, \mathcal{G} = \mathrm{Aut}(\mathbb{A})$ $\mathbb{A}$ FIN. HOMOGENEOUS

$\exists\, H \leqslant \mathcal{G}$ OLIGOMORPHIC EXTREMELY AMENABLE

$$\mathcal{G} = \mathrm{Inv}(\mathcal{D}) \subseteq \mathcal{C} \subseteq \mathcal{D}$$

$$\{ F \in \mathcal{D} \mid F \text{ CANONICAL WRT } \mathcal{G} \}$$

$$\mathcal{D} = \mathrm{Pol}(\mathbb{A}) \rightsquigarrow \mathcal{C} = \mathrm{Pol}(\mathbb{A}, \{\equiv^m_{\mathcal{G}}\}_{m \in \dots})$$

$\mathcal{G}$ EXT. AMENABLE $\Rightarrow$ $\chi : \mathcal{D} \longrightarrow 2^{\mathcal{C}} \setminus \{\emptyset\}$

$$F \mapsto \overline{\mathcal{G} F \mathcal{G}} \cap \mathcal{C}.$$

QUESTION: (~ 2014)

WHEN CAN $\xi : \mathcal{C} \to \mathcal{F}$ BE EXTENDED TO $\xi : \mathcal{D} \to \mathcal{F}$?

ALMOST NEVER:

$\mathcal{G}$ EXTREMELY AMENABLE $\Rightarrow$ INVARIANT LINEAR ORDER

"$\Rightarrow$" $\mathcal{E} \longrightarrow \mathcal{F}$ $\forall \mathcal{F}$

SECOND DILEMMA OF THE INFINITE SHEEP

EXTREME AMENABILITY: CAN'T LIVE WITH IT
CAN'T LIVE WITHOUT IT

APPROXIMATIONS

THEOREM "TOPOLOGICAL BIRKHOFF" (BODIRSKY-PINSKER)

TFAE:

- $\exists\, \xi : \mathfrak{D} \longrightarrow \mathcal{F}$
- $\exists\, n \geq 1,\ \underset{\subseteq A^n}{S},\ \underset{\subseteq S^2}{\sim} \in \mathrm{Inv}(\mathfrak{D})$:

$$\mathfrak{D} \curvearrowright S/_{\sim} \subseteq \mathcal{F}.$$

$\sim$ EQUIV. REL. ON $S =: \mathrm{supp}(\sim)$.

SPECIAL CASE: $\mathcal{F} = \mathcal{P}$

$$\mathfrak{D} \curvearrowright S/_{\sim} = \mathcal{P}$$

$(S, \sim)$ "NAKED SET" $\sim$-BLOCKS ARE $\mathrm{INV}(\mathfrak{D})$-INVARIANT

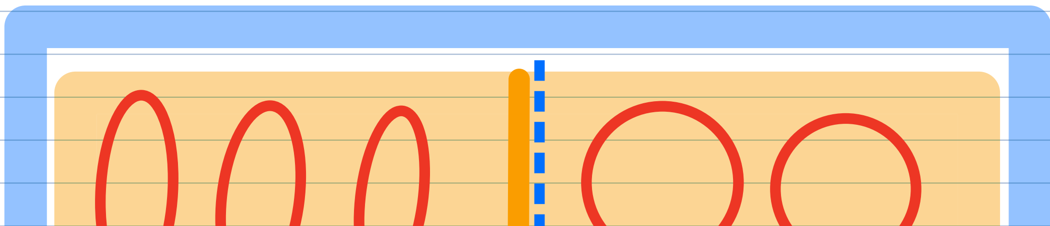
$$\begin{tikzcd} \mathcal{G} & \subseteq & \mathcal{C} \arrow[d, "{(S,\sim)}"] & \subseteq & \mathcal{D} \arrow[dll, dashed, "{(S',\eta)}"] \\ & & \mathcal{P} & & \end{tikzcd}$$

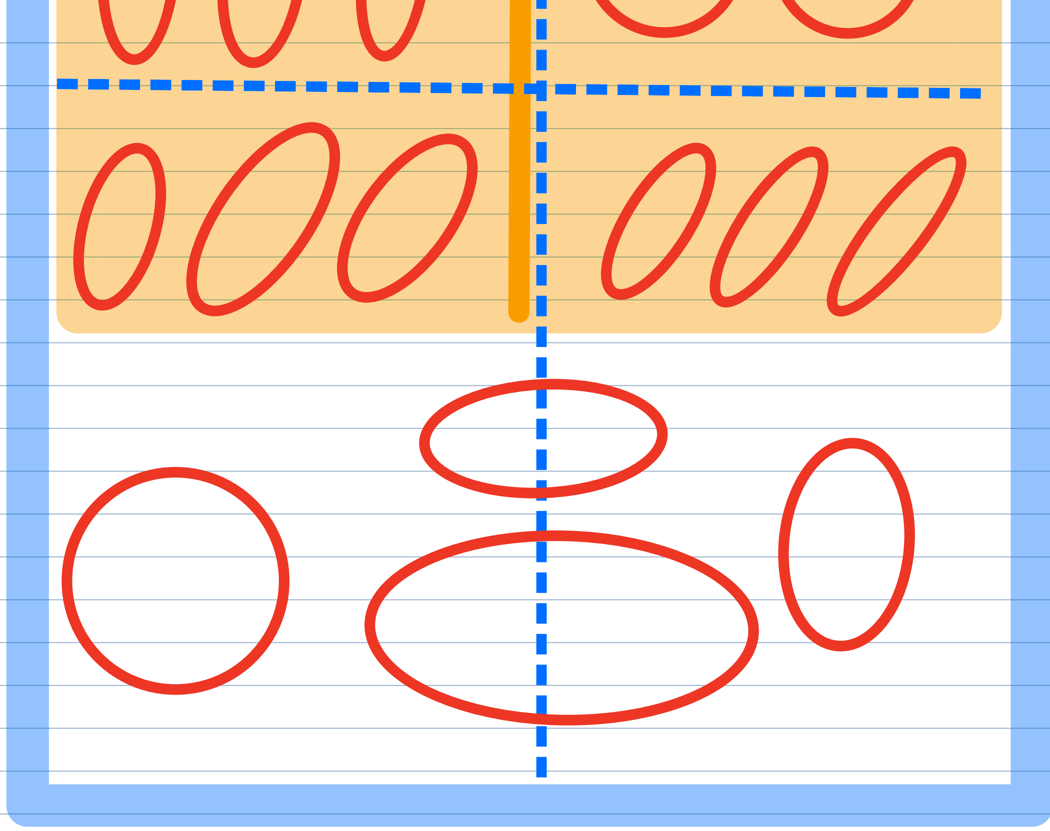
DEF: η APPROXIMATES $\sim$ IF

- $S \subseteq S'$
- $\eta|_S \subseteq \sim$.

QUALITY OF η:

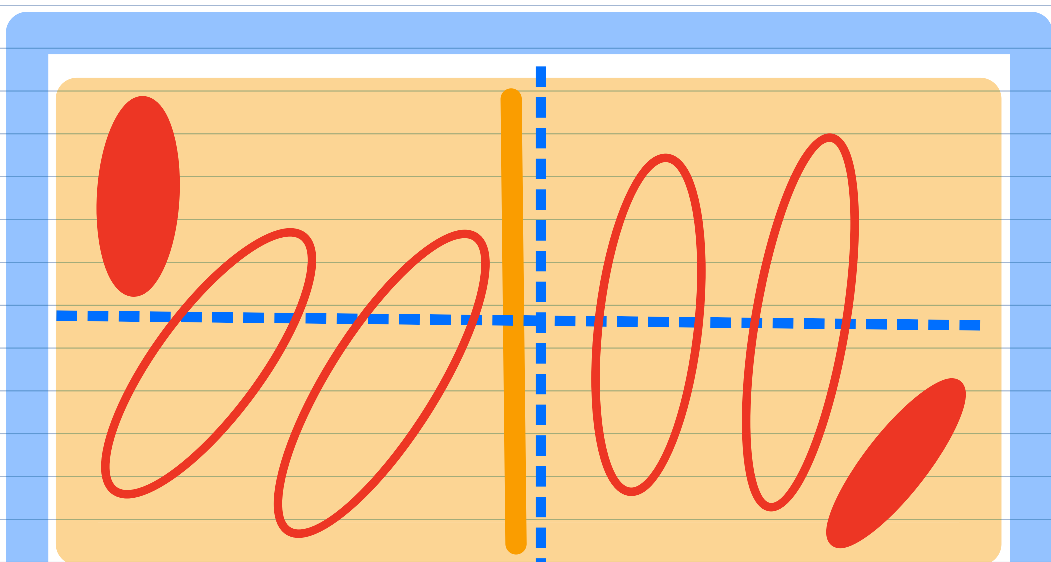
VERY SMOOTH: η-BLOCKS IN S ARE $\mathcal{G}$-INVARIANT

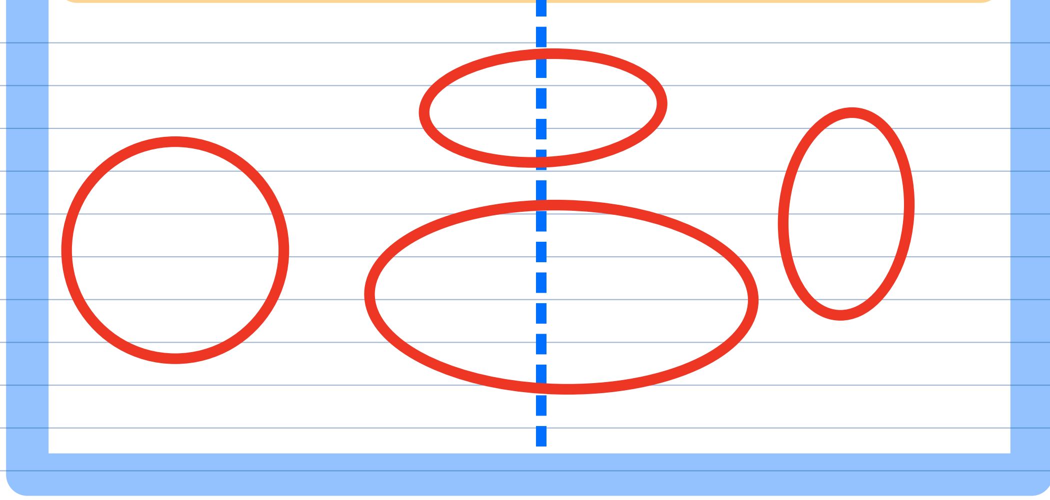




$\sim$ η $\mathcal{G}$-orbits

SMOOTH: $\forall \sim$-BLOCK C $\exists \eta$-BLOCK C':
$C \cap C'$ CONTAINS $\mathcal{G}$-ORBIT

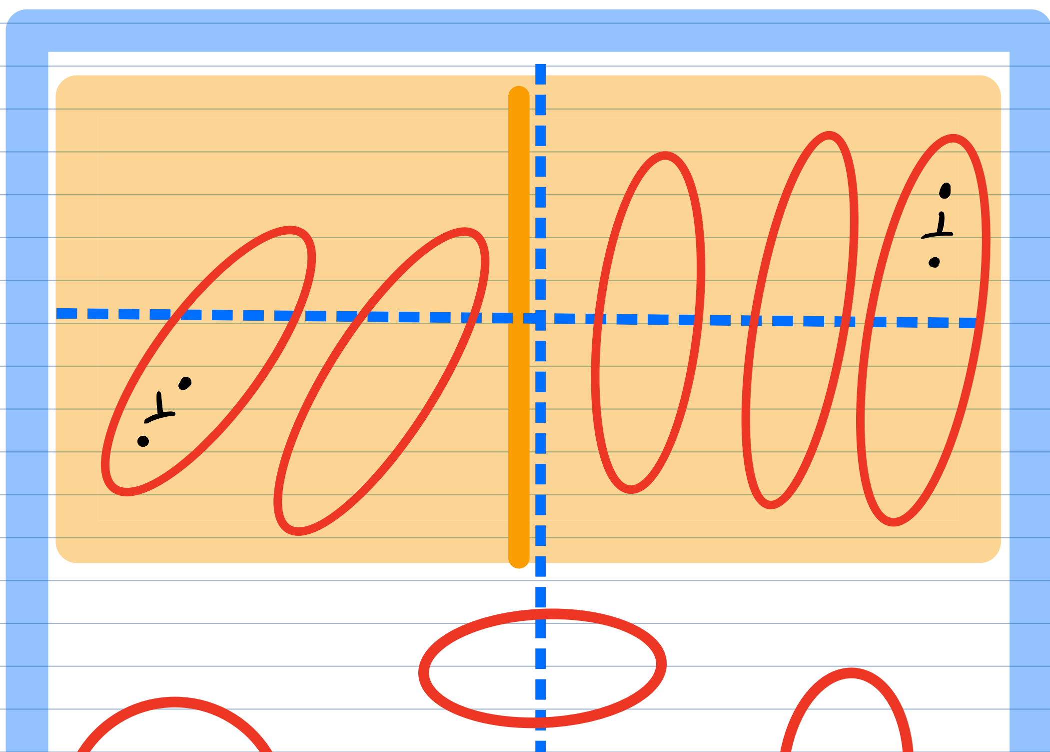


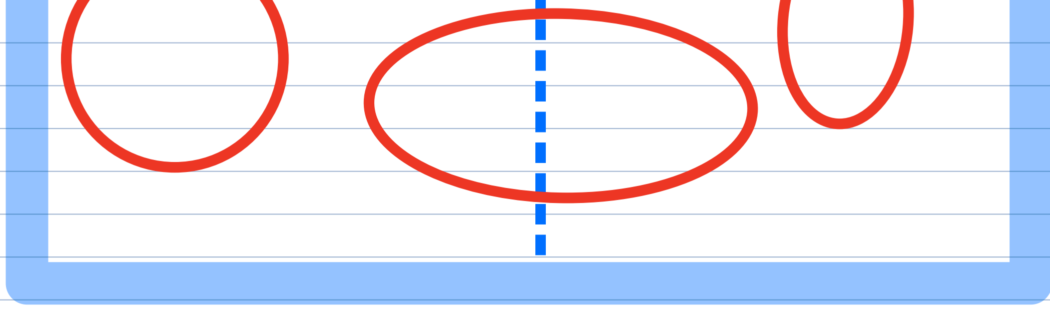


~ $\quad$ η $\quad$ G-orbits

TASTELESS: $\forall$ ~-BLOCK $\exists\,\eta$-BLOCK C':

$C \cap C'$ CONTAINS DISJOINT $\bar{a} \equiv_G \bar{b}$





$\sim$ η $\mathcal{G}$-orbits

__THEOREM__ (M.-PINSKER)

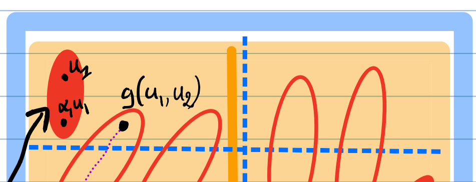
- $\mathcal{C} \subseteq \mathcal{D}$
- $\chi : \mathcal{D} \to 2^{\mathcal{C}} \setminus \{\emptyset\}$, $\quad f \mapsto \overline{\mathcal{G} f \mathcal{G}} \cap \mathcal{C}$
- $(S, \sim) \searrow \mathcal{F}$
- $\eta \in \mathrm{Inv}(\mathcal{D})$ SMOOTH APP. OF $\sim$.

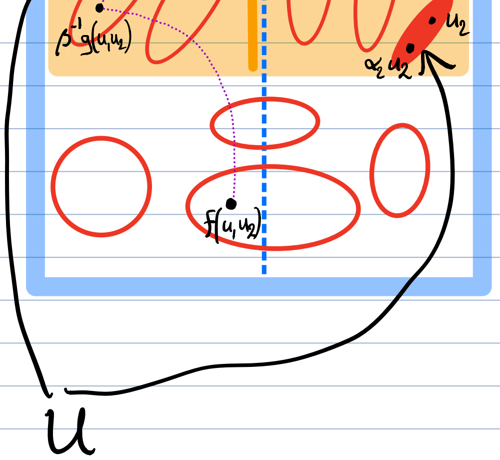
THEN $\exists\ \mathcal{D} \overset{H1}{\dashrightarrow} \mathcal{F}$

IF η VERY SMOOTH THEN $\mathcal{D} \to \mathcal{F}$.

PROOF: $\quad \forall g \in \chi(f) : g(u_1, \dots, u_n)\ (\sim \cdot \eta)\ f(u_1, \dots, u_n)$

$$= \beta f(\alpha_1 u_1, \dots, \alpha_n u_n)$$





$$\begin{array}{ccccc} u_1 & \cdots & u_n & \to & f(u_1,\dots,u_n) \\ \wr\eta & & \wr\eta & & \wr\eta \\ \alpha_1 u_1 & & \alpha u_n & \to & \beta^{-1} g(u_1,\dots,u_n) \\ & & & & \wr \\ & & & & g(u_1,\dots,u_n) \end{array}$$

$g, g' \in \mathcal{X}(f)$: $\quad g(u_1,\dots,u_n) \underset{(\sim\circ\eta)\cdot(\eta\circ\sim) \,=\, \sim\circ\eta\circ\sim}{\wr} g'(u_1,\dots,u_n)$

$$\Longrightarrow \; g(u_1,\dots,u_n) \sim g'(u_1,\dots,u_n).$$

$\square$

THEOREM: (M. - PINSKER) $\mathcal{G} \subseteq \mathcal{C} \subseteq \mathcal{D}$ $\quad$ $\mathcal{G}$ WITHOUT ALGEBRAICITY

$$\mathcal{C} \longrightarrow \mathcal{F}$$

$$\mathcal{C} \cap \mathcal{S}/_{\sim} \subseteq \mathcal{F}$$

∃ TASTELESS APPROXIMATION

OR

SOME SPECIFIC RELS $R \in \mathrm{Inv}(\mathcal{D})$ HAVE PSEUDOLOOPS $(\alpha_1 a, \alpha_2 a, \alpha_3 a, \ldots) \in R$

"UPGRADE" TO (VERY) SMOOTH

EXISTENCE OF $f \in \mathcal{D}$ SATISFYING DESIRABLE PROPERTIES

NO ALGEBRAICITY

TASTELESS

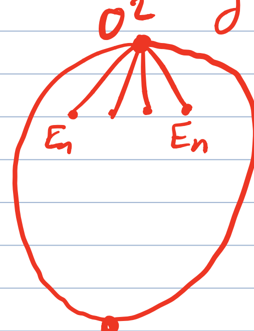
"PRIMITIVE"

SMOOTH $\Longleftarrow$ VERY SMOOTH

∃ TASTELESS $\Longrightarrow$ ∃ VERY SMOOTH

$\neq \in \mathrm{Inv}(\mathcal{C})$

$\forall O$ ORBIT OF $\mathcal{G} \curvearrowright A^n$



THEOREM: (M.-PINSKER)

LET $\mathcal{D}$ CONTAIN Aut(COUNTABLE UNIVERSAL HOMOGENEOUS TOURNAMENT)

S.T. $\overline{\mathrm{Inv}(\mathcal{D})} = \mathcal{D}^{(1)}$.

TFAE:

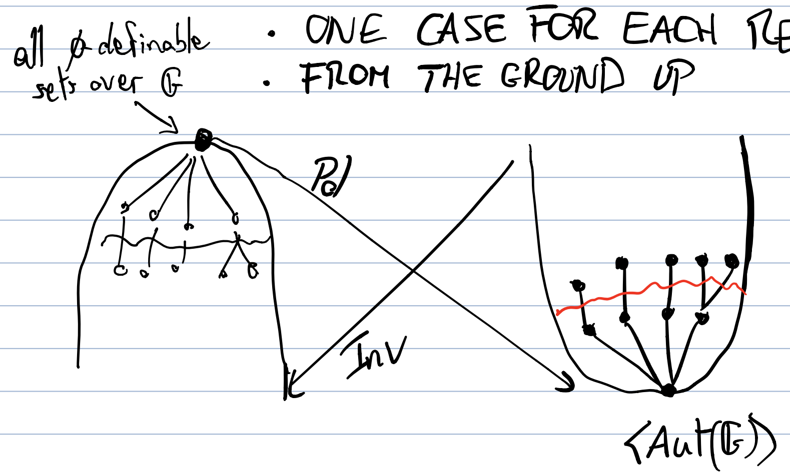
- $\mathcal{D} \not\to \mathcal{P}$
- $\mathcal{D} \not\to \mathcal{P}$
- $\exists f \in \mathcal{D}^{(n)} \quad u, v \in \mathcal{D}^{(1)} : \quad u \circ f(x_1, \dots, x_n) \approx v \circ f(x_2, \dots, x_n, x_1)$
- ... f CANONICAL W.R.T. $\mathrm{Aut}(\mathbb{T})$.

SAME STATEMENTS HOLD OVER

HOMOGENEOUS GRAPHS, RANDOM POSET, UNARY STRUCTURES, ...

1st GEN. PROOF:

- ONE CASE FOR EACH REDUCT (NEED TO KNOW THEM FIRST...)
- FROM THE GROUND UP



2nd GEN. PROOF: $\quad \mathcal{D} = \mathrm{Pol}(\mathbb{A})$

$$\mathcal{C}_{\mathbb{T}} \longrightarrow \mathcal{P} \iff \nexists f \text{ CYCLIC CAN. WRT. } \mathrm{Aut}(\mathbb{T})$$

$$\Downarrow$$

$$\mathcal{C}_{(\mathbb{T}<)} \subseteq \mathcal{C}_{\mathbb{T}} \subseteq \mathcal{C}_{\mathbb{A}}$$

$$\Downarrow$$

$$\mathcal{C}_{\mathbb{A}} \xrightarrow[(\Sigma,\sim)]{} \mathcal{P}$$

or

(VERY SMOOTH) APPROXIMATION

$$\Downarrow$$

$$\mathcal{D} \longrightarrow \mathcal{P}$$

LOOP LEMMA

$$\downarrow$$

$$\exists f \in \mathcal{C}_{\pi}^{(3)} : e_1 f(xxy) \approx e_2 f(xyx) \approx e_3 f(yxx) \approx e(x)$$

$$\lightning$$

THEOREM: (M.-PINSKER)

LET $\mathcal{D}$ CONTAIN Aut(COUNTABLE UNIVERSAL HOMOGENEOUS TOURNAMENT)

S.T. $\overline{\mathrm{Inv}(\mathcal{D})} = \mathcal{D}^{(1)}$.

TFAE:

- $\mathcal{D} \not\to \mathscr{A}$
- $\mathcal{D} \not\to \mathscr{A}$
- $\forall n \geq 3\ \exists f \in \mathcal{D}^{(n)}\ u_1 \cdots u_n \in \mathcal{D}^{(1)} : u_1 f(yx\cdots x) \approx \cdots \approx u_n f(x\cdots xy)$

$\mathscr{A}$ = ANY CLONE OF AFFINE MAPS OVER

A FINITE MODULE

- SAME STATEMENT HOLDS OVER STRUCTURES AS ABOVE.
- NO "1ST GEN. PROOF".
- ESSENTIALLY NOTHING TO ADD TO PROVE IT (REPLACE $\mathcal{P}$ BY $\mathcal{A}$)