The Containment Problem for Unambiguous Register Automata

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STACS 2019

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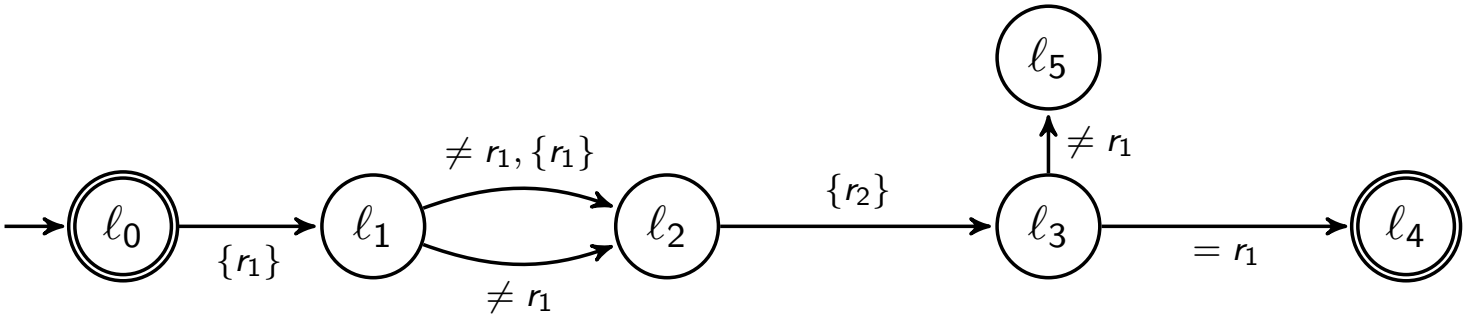

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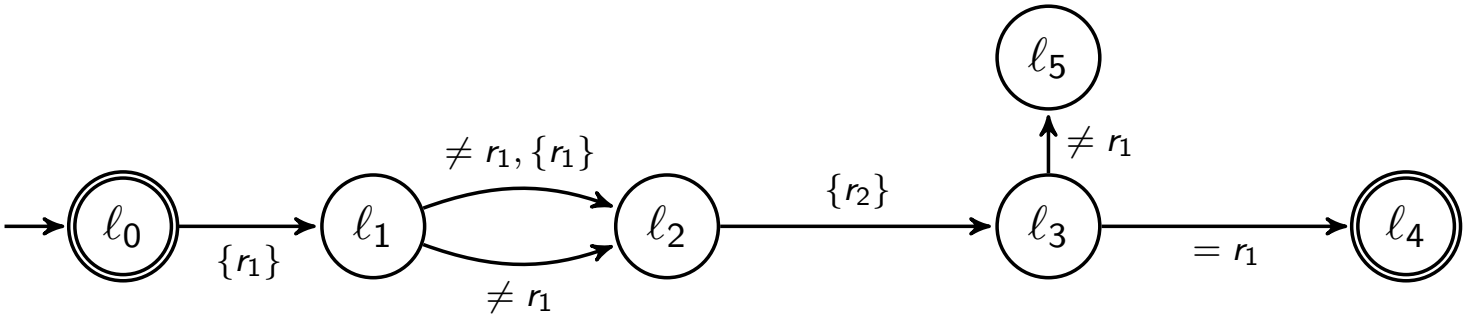

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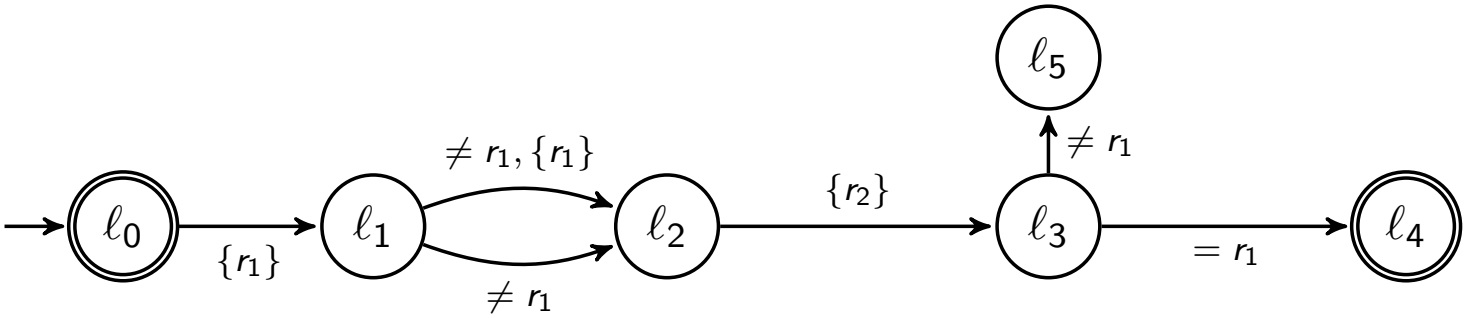

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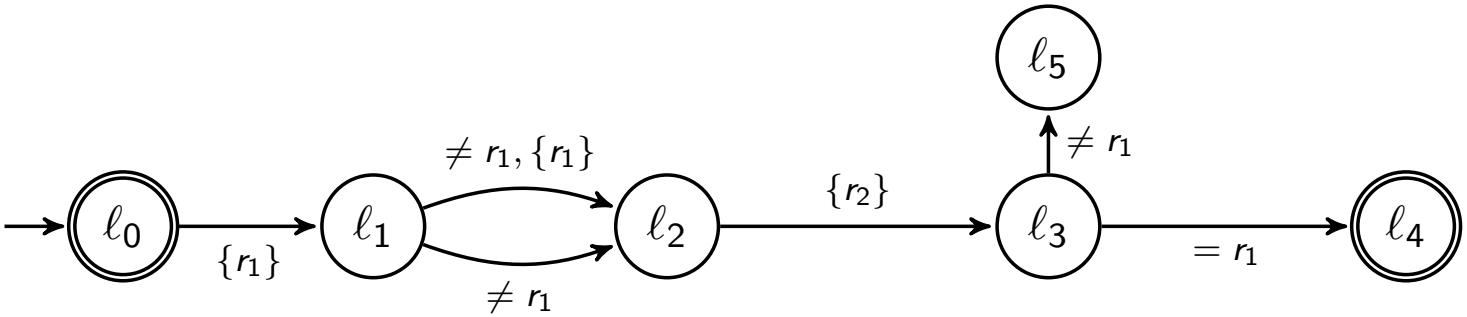

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- Recognizers of orbits:

$$\binom{a}{0}\binom{a}{1}\binom{a}{3}\binom{a}{0} \sim \binom{a}{4}\binom{a}{3}\binom{a}{1}\binom{a}{4}.$$

Projection and Emptiness

- Projection of $L \subseteq (\Sigma \times \mathbb{N})^*$ onto Σ^*: set of words $w \in \Sigma^*$ such that $(w_1, d_1) \dots (w_n, d_n) \in L$ for some $d_1, \dots, d_n \in \mathbb{N}$.

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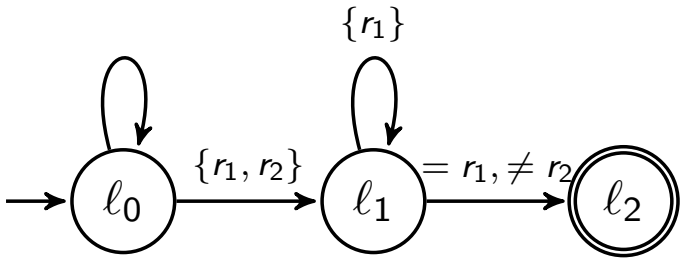

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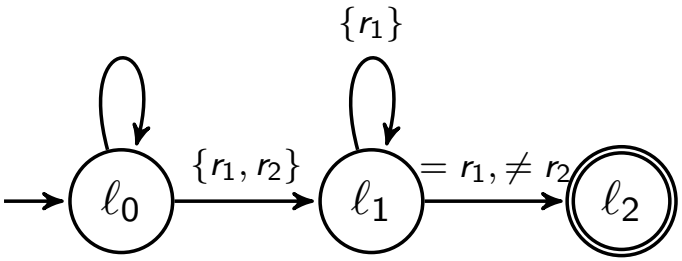

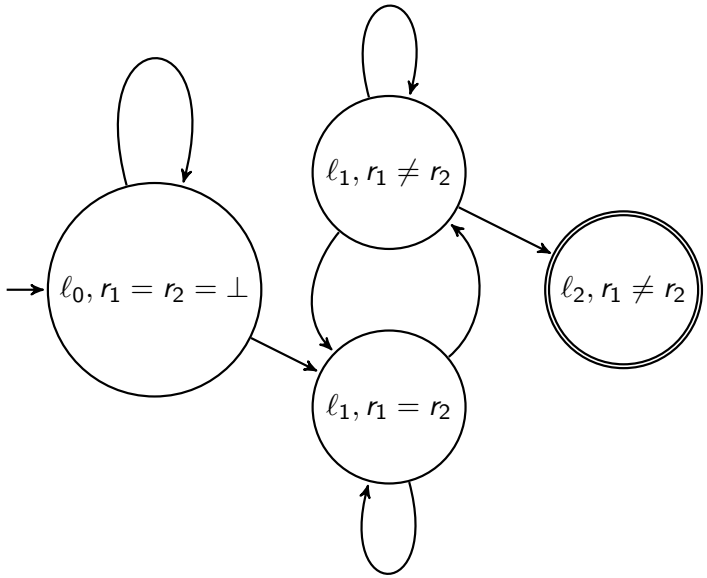

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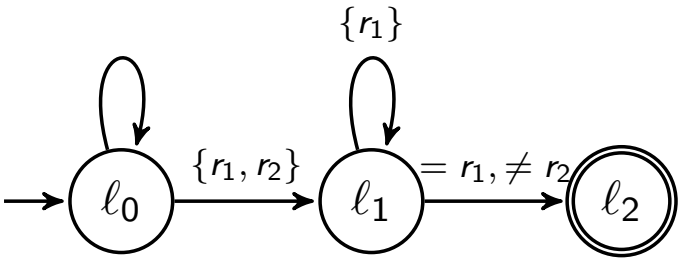
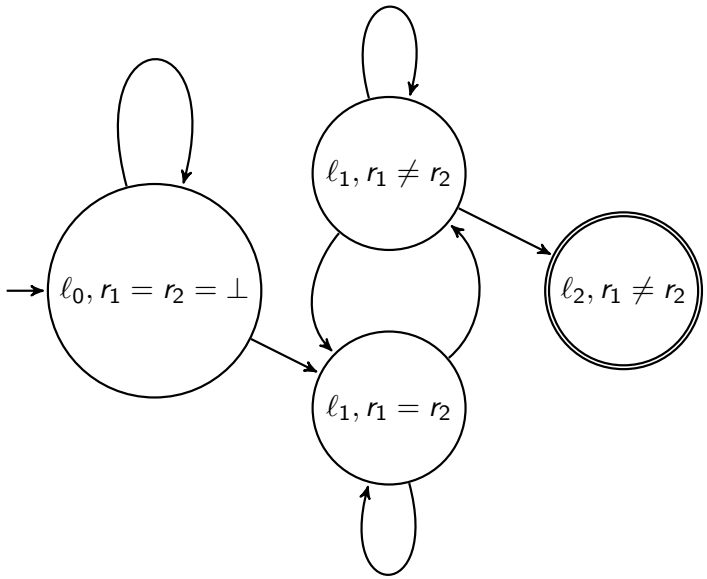
- Emptiness of L is decidable: $L = \emptyset \Leftrightarrow$ its projection is empty.





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- Important problems related to unambiguity (parity games in $UP \setminus P$?).

Deciding unambiguity

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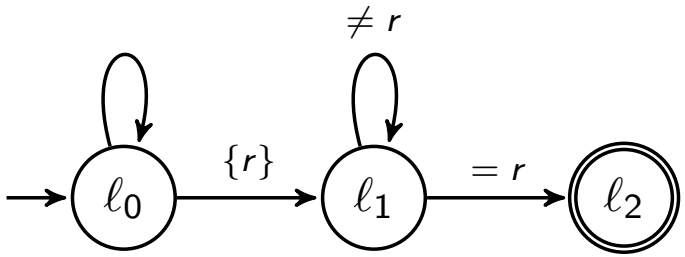

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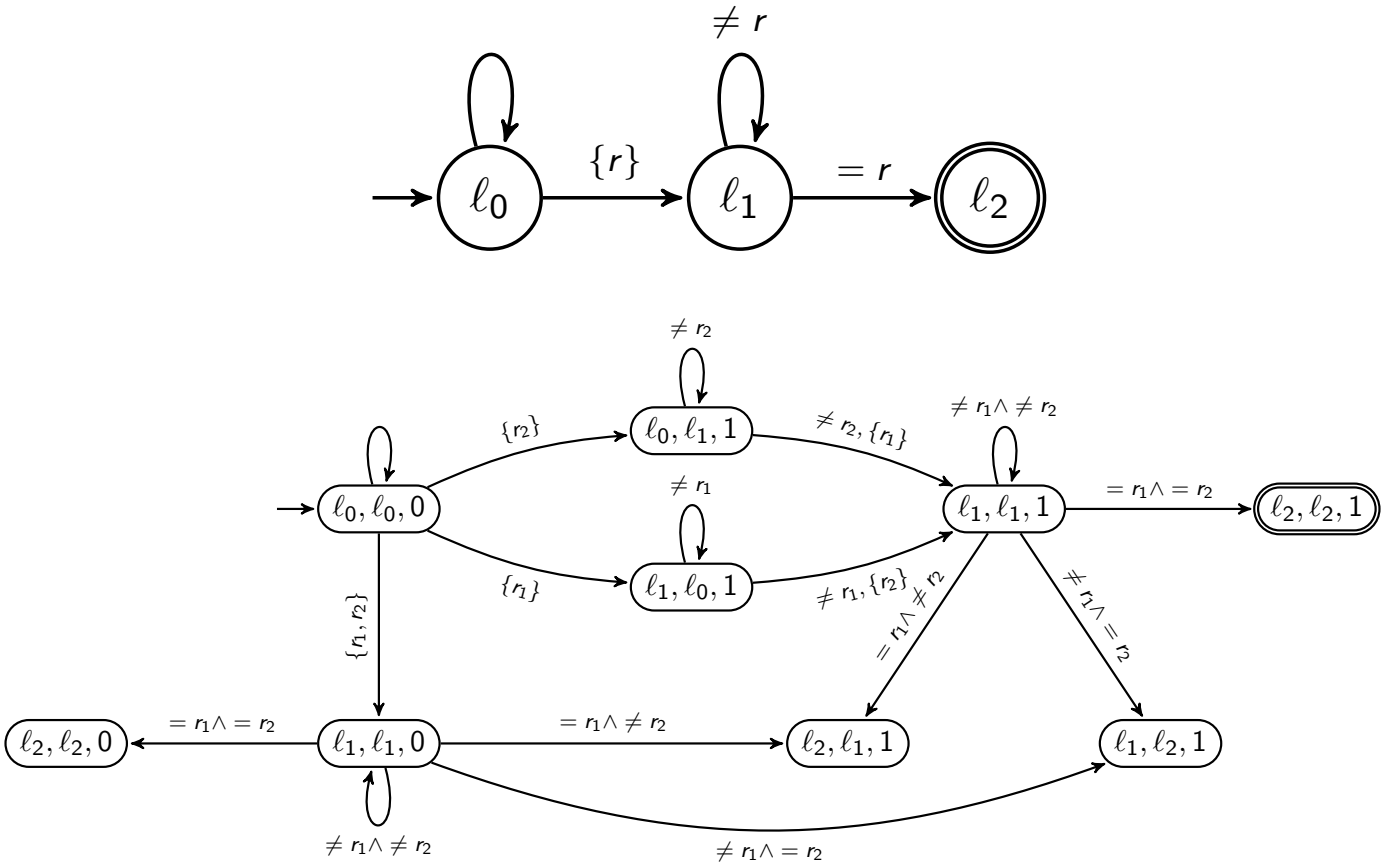

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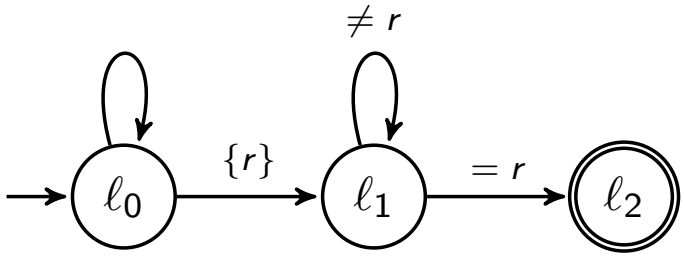

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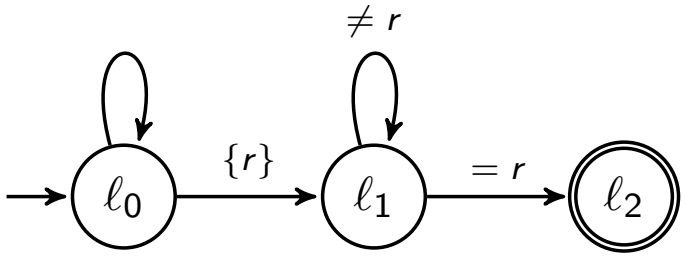

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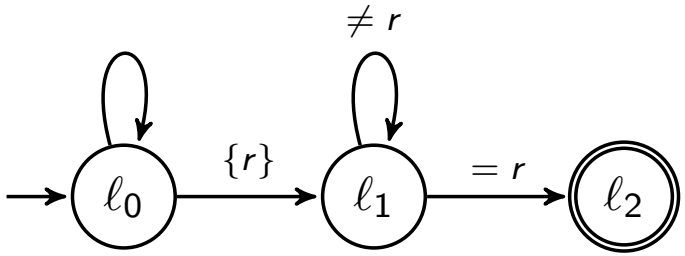

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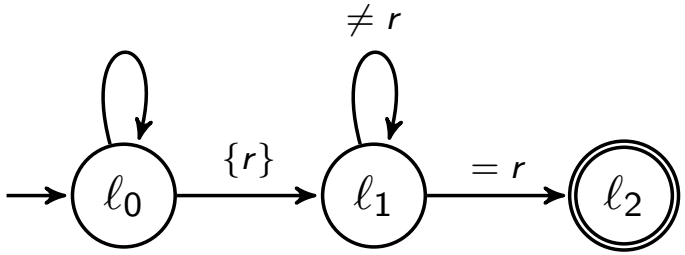

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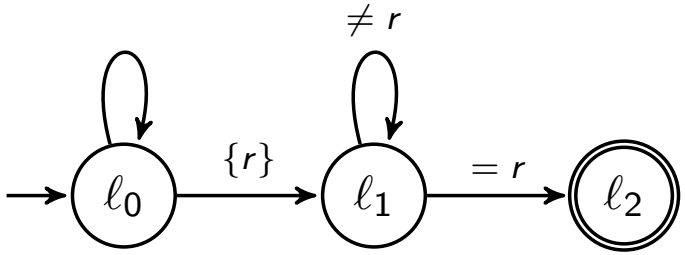

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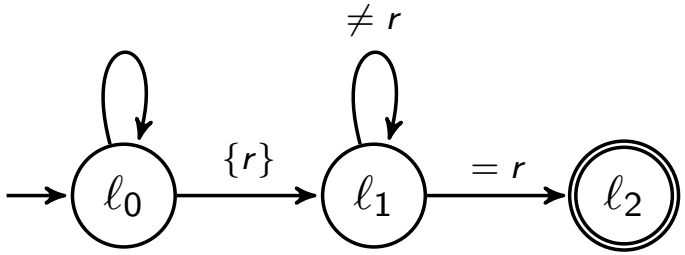

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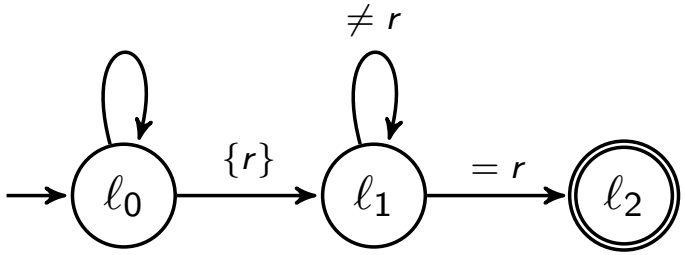

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- In particular L not recognizable by deterministic RA.

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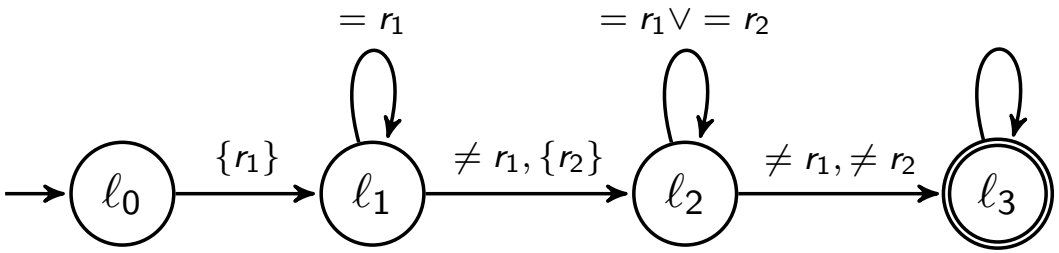

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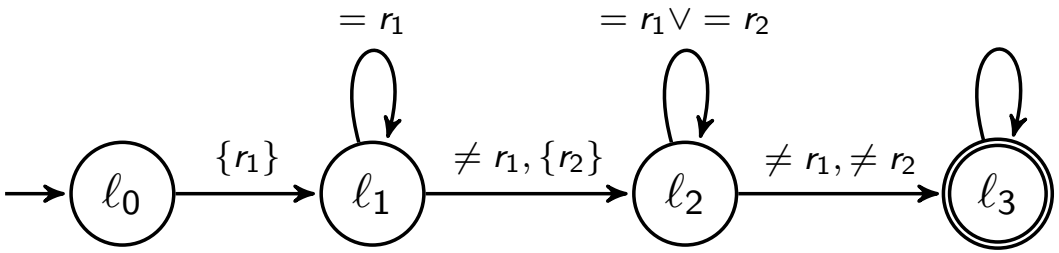

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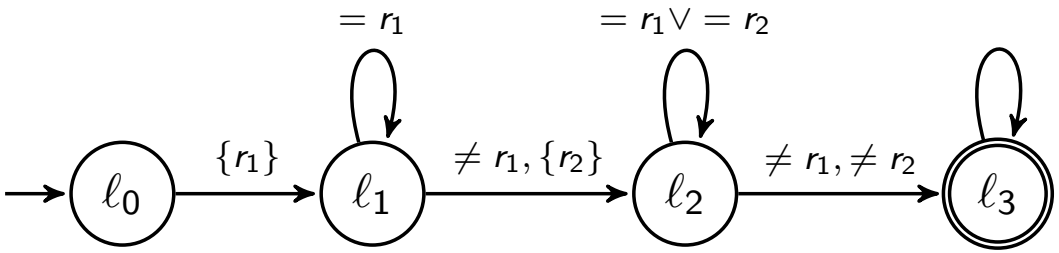

- Needs 2 registers.
- Exists a 1-register unambiguous RA.

The Universality and Containment Problems

- Universality: Given $\mathcal{B}$, determine if $L(\mathcal{B}) = (\Sigma \times \mathbb{N})^*$.

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  $\rightsquigarrow$ "on-the-fly" complementation.

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- $((\ell^{\mathcal{A}}, d_1, \ldots, d_m), C)$ bad if $\ell^{\mathcal{A}}$ accepting and C not accepting.
- $L(\mathcal{A}) \not\subseteq L(\mathcal{B}) \Leftrightarrow \exists$bad reachable configuration in $(\mathbb{S}, \to)$.

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The unambiguous case: try to bound size of configurations.

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- $x_1 = x_2 \wedge x_2 \neq x_3$ and $x_1 \neq x_2 \wedge x_1 = x_3$ are 3-types (there are 5 in total),

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An *n*-type is a satisfiable conjunction $\varphi(x_1, \ldots, x_n)$ of $=$ and $\neq$ that is maximal (any formula containing φ is equivalent to φ or insatisfiable).

- $x_1 = x_2$ and $x_1 \neq x_2$ are the only 2-types,
- $x_1 = x_2 \wedge x_2 \neq x_3$ and $x_1 \neq x_2 \wedge x_1 = x_3$ are 3-types (there are 5 in total),
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- $\mathrm{tp}(d_1, \ldots, d_n) = \mathrm{tp}(e_1, \ldots, e_n) \Leftrightarrow$ $\exists$ permutation α of $\mathbb{N}$ s.t. $\alpha(d_i) = e_i$.

Types... and how to use them

Consider

$$C = \{(\ell, 1, 2), (\ell'', 1, 2), (\ell', 3, 4), (\ell', 2, 5), (\ell'', 4, 5), (\ell, 1, 3), (\ell'', 1, 3)\}$$

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$$L_\varphi(d_1, d_2) := \{\ell \mid \exists e_1, e_2 : (\ell, e_1, e_2) \in C \text{ and } \mathbb{N} \models \varphi(d_1, d_2, e_1, e_2)\}.$$

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- Generalize $\equiv_C$ to synchronized configurations.

Proposition (M-Quaas '18)

C reachable. $\overline{a}, \overline{b}$ such that $\overline{a} \equiv_C \overline{b}$.
Let $C' = C \setminus \{(\ell, \overline{b}) \in C\}$.
C reaches a bad configuration in k steps iff
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$\mathcal{B}$	DRA	URA	NRA
1 register	PSPACE-comp.	EXPSPACE	Ackermann-comp.
*	PSPACE-comp.	2-EXPSPACE	Undecidable

Open problems

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- Timed automata: decidability for ≥ 2 clocks?