Příklad 1

X_i iid $\sim f(x) = \theta(1-x)^{\theta-1} I(x\in(0,1)) \qquad \theta>0$

a) $$\hat\theta_n = \frac{-n}{\sum_{i=1}^n \log(1-X_i)} = \frac{n}{\sum_{i=1}^n \underbrace{-\log(1-X_i)}_{=:Y_i}} = \frac{n}{\sum_1^n Y_i} = \frac{1}{\bar Y_n}$$

Vyšetřím konvergenci v P $\bar Y_n$:

$$EY_i = E -\log(1-X_i) = \int_0^1 -\log(1-x)\cdot\theta(1-x)^{\theta-1}\,dx =$$

$$= \left| \begin{array}{l} z = -\log(1-x) \\ e^{-z} = 1-x \\ -e^{-z}dz = -dx \end{array} \right| = \int_0^\infty z\cdot\theta\cdot\left[e^{-z}\right]^{\theta-1}\cdot e^{-z}\,dz$$

$$= \theta\int_0^\infty z\cdot e^{-z\theta}\,dz = \frac{1}{\theta}$$

↑ střední hodnota v $Exp(\theta)$

(nebo výpočet:

$$\theta\cdot\int_0^\infty z\cdot e^{-z\theta}dz = \left|\begin{array}{l} y = z\theta \\ dy = \theta dz\end{array}\right| = \theta\cdot\frac{1}{\theta}\int_0^\infty y\cdot e^{-y}\cdot\frac{1}{\theta}dy = \frac{1}{\theta}\cdot\Gamma(2) = \frac{1}{\theta}\cdot 1! = \frac{1}{\theta}$$

alternativně: per partes ....)

Ze SZVČ pro Y_i (předpoklady OK: iid, EY_i konečná)

$$\Rightarrow \bar Y_n \xrightarrow{sj} \frac{1}{\theta} \Rightarrow \bar Y_n \xrightarrow{P} \frac{1}{\theta}$$

↑ vztah mezi konvergencemi

Věta o spoj. transformaci pro $g(x) = \frac{1}{x}$

$$\Rightarrow \underbrace{\frac{1}{\bar Y_n}}_{\hat\theta_n} \xrightarrow{P} \theta$$

tj. $\hat\theta_n$ je konzistentní odhad θ

b) $\tilde{\theta}_n = \frac{1}{n}\sum_1^n X_i = \overline{X}_n$

$$EX_i = \int_0^1 \theta \cdot x(1-x)^{\theta-1}dx = \left| \begin{matrix} y = 1-x \\ x = 1-y \\ dy = -dy \end{matrix} \right| = \int_0^1 \theta(1-y)\cdot y^{\theta-1}dy$$

$$= \theta\int_0^1 y^{\theta-1}dy - \theta\cdot\int_0^1 y^{\theta}dy = \frac{\theta}{\theta} - \frac{\theta}{\theta+1} = 1 - \frac{\theta}{\theta+1} =$$

$$= \frac{\theta+1-\theta}{\theta+1} = \frac{1}{\theta+1}$$

dle SZVČ pro X_i (předpoklady OK: iid, EX_i konečná)

$$\overline{X}_n \xrightarrow{s.j.} \frac{1}{\theta+1} \Rightarrow \overline{X}_n \xrightarrow{P} \frac{1}{\theta+1} \neq \theta$$

$\Rightarrow \tilde{\theta}_n$ není konzistentní odhad θ

c) Asympt. rozdělení $\hat{\theta}_n = \frac{1}{\overline{Y}_n}$

$Y_i = -\log(1-X_i)$

$\operatorname{var} Y_i = EY_i^2 - (EY_i)^2$

$$EY_i^2 = \int_0^1 [-\log(1-x)]^2\,\theta\,(1-x)^{\theta-1}dx =$$

$$= \left| \begin{matrix} \text{stejná substituce} \\ z = -\log(1-x) \\ 1-x = e^{-z} \\ dx = e^{-z}dz \end{matrix} \right| = \int_0^\infty z^2\cdot\theta\cdot e^{-\theta z}dz = \left| \begin{matrix} y = \theta z \\ z = \frac{y}{\theta} \\ dz = \frac{dy}{\theta} \end{matrix} \right|$$

$$= \int_0^\infty \frac{y^2}{\theta^2}\cdot\theta\cdot e^{-y}\frac{dy}{\theta} = \frac{1}{\theta^2}\cdot\Gamma(3) = \frac{1}{\theta^2}\cdot 2! = \frac{2}{\theta^2}$$

$$\operatorname{var} Y_i = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

Tedy dle CLV (předp. OK: iid, $\operatorname{var} Y_i < \infty$)

$$\sqrt{n}(\overline{Y}_n - EY_1) \xrightarrow{D} N(0, \operatorname{var} Y_1)$$

$$\sqrt{n}\left(\overline{Y}_n - \frac{1}{\theta}\right) \xrightarrow{D} N\left(0, \frac{1}{\theta^2}\right)$$

$\hat{\theta}_n = g(\bar{Y}_n)$ $\quad g(x) = \frac{1}{x}$ $\quad g(EY_1) = \theta$

$g'(x) = -\frac{1}{x^2}$ $\quad [g'(x)]^2 = \frac{1}{x^4}$ $\quad [g'(EY_1)]^2 = \theta^4$

Δ věta $\Rightarrow$ $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} \underbrace{N(0, \theta^4 \cdot \frac{1}{\theta^2})}_{N(0,\theta^2)}$] Hledané as. rozdělení

d) Dolní int. odhad pro θ^2 $\rightarrow$ zkonstruujeme asympt. int. odhad

Více možných postupů.
Např. najdeme asympt. rozdělení $\hat{\theta}_n^2 = h(\hat{\theta}_n)$ $\quad h(x) = x^2$

$h'(x) = 2x \quad [h'(x)]^2 = 4x^2$

$[h'(\theta)]^2 = 4\theta^2$

Δ věta $\Rightarrow$

$$\sqrt{n}(\hat{\theta}_n^2 - \theta^2) \xrightarrow{\mathcal{D}} N(0, 4\theta^2 \cdot \theta^2) = N(0, 4\theta^4)$$

$$\sqrt{n}\,\frac{\hat{\theta}_n^2 - \theta^2}{2\theta^2} \xrightarrow{\mathcal{D}} N(0,1)$$

Víme: $\hat{\theta}_n \xrightarrow{P} \theta \Rightarrow$ V o spoj. transformaci $\left(\frac{\theta}{\hat{\theta}_n}\right)^2 \xrightarrow{P} 1$

$$\sqrt{n}\,\frac{\hat{\theta}_n^2 - \theta^2}{2\hat{\theta}_n^2} = \underbrace{\sqrt{n}\,\frac{\hat{\theta}_n^2 - \theta^2}{2\theta^2}}_{\downarrow \mathcal{D} \; N(0,1)} \cdot \underbrace{\frac{\theta^2}{\hat{\theta}_n^2}}_{\downarrow P \; 1} \xrightarrow{\mathcal{D}} N(0,1)$$

Podle Cramérovy Sluckého věty

$$\Rightarrow P\left(\sqrt{n}\,\frac{\hat{\theta}_n^2 - \theta^2}{2\hat{\theta}_n^2} < u_{1-\alpha}\right) \xrightarrow{n\to\infty} 1-\alpha$$

$u_{1-\alpha}$ kvantil $N(0,1)$ na hladině $1-\alpha$

$$P\left(\hat{\theta}_n^2 - \theta^2 < u_{1-\alpha} \cdot \frac{2\hat{\theta}_n^2}{\sqrt{n}}\right) \rightarrow 1-\alpha$$

$$P\left(\theta^2 > \hat{\theta}_n^2 - u_{1-\alpha} \cdot \frac{2\hat{\theta}_n^2}{\sqrt{n}}\right) \rightarrow 1-\alpha$$

$\Rightarrow$ hledaný interval $\left(\hat{\theta}_n - u_{1-\alpha} \cdot \frac{2\hat{\theta}_n^2}{\sqrt{n}}, \infty\right)$

dolní int. odhad s asympt. spolehlivostí $1-\alpha$

(Jiný postup např. najdeme interval pro $\theta \Rightarrow$ z něj transformací možné intervalový odhad θ^2 $\rightarrow$ jen pozor na znaménka)

e) Stabilizace rozptylu

$$\sqrt{n}\,(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{D}} N(0, \theta^2)$$

Chceme najít $g(x)$ tak, že $\sqrt{n}\,(g(\hat{\theta}_n) - g(\theta)) \xrightarrow{\mathcal{D}} N(0, \underbrace{g'(\theta)^2 \theta^2}_{=1})$

$$[g'(\theta)]^2 \cdot \theta^2 = 1$$

$$g'(\theta) = \frac{1}{\theta}$$

$$g(\theta) = \log \theta$$

$$\sqrt{n}\,(\log \hat{\theta}_n - \log \theta) \xrightarrow{\mathcal{D}} N(0, 1)$$

$$\Rightarrow P\left(-u_{1-\alpha/2} < \sqrt{n}\,(\log \hat{\theta}_n - \log \theta) < u_{1-\alpha/2}\right) \to 1 - \alpha$$

$$P\left(\log \hat{\theta}_n - \frac{u_{1-\alpha/2}}{\sqrt{n}} < \log \theta < \log \hat{\theta}_n + \frac{u_{1-\alpha/2}}{\sqrt{n}}\right) \to 1 - \alpha$$

oboustranný asympt. int. odhad $\log \theta$: $\left(\log \hat{\theta}_n \mp \frac{u_{1-\alpha/2}}{\sqrt{n}}\right)$

int. odhad $\theta = e^{\log \theta}$

rostoucí transformace

$$\left(e^{\log \hat{\theta}_n - \frac{u_{1-\alpha/2}}{\sqrt{n}}},\ e^{\log \hat{\theta}_n + \frac{u_{1-\alpha/2}}{\sqrt{n}}}\right)$$

$$\left(\hat{\theta}_n \cdot e^{-\frac{u_{1-\alpha/2}}{\sqrt{n}}},\ \hat{\theta}_n \cdot e^{\frac{u_{1-\alpha/2}}{\sqrt{n}}}\right)$$

Příklad 2: $X_i \sim Bi(2,p)$

$P(X_i=0) = (1-p)^2 \qquad P(X_i=1) = 2\cdot p(1-p) \qquad P(X_i=2) = p^2$

a) $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n} I[X_i=2] \qquad \theta_X = p^2$

$$E\hat{\theta}_n = \frac{1}{n}\cdot\sum_{i=1}^{n} \underbrace{E\, I[X_i=2]}_{P(X_i=2)=p^2} = \frac{np^2}{n} = \theta_X$$

$\Rightarrow \hat{\theta}_n$ je nestranný odhad θ_X

b) $\tilde{\theta}_n = \frac{\bar{X}_n^2}{4} = \frac{1}{4}\cdot g(\bar{X}_n)$ pro $g(x) = x^2$ konvexní funkce

Jensenova nerovnost: $E g(\bar{X}_n) > g(EX)$

$$\Rightarrow E\tilde{\theta}_n = \frac{1}{4} g(\bar{X}_n) > \frac{1}{4}\cdot g(EX) = \frac{1}{4}\cdot(2p)^2 = p^2 = \theta_X$$

$EX = 2\cdot p$ $\qquad \Rightarrow \tilde{\theta}_n$ není nestranný odhad θ_X

↑ $Bi(2,p)$ rozdělení

(Lze i přímý výpočet (v tomto případě):

$$E\tilde{\theta}_n = \frac{E\bar{X}_n^2}{4} = \frac{var\,\bar{X}_n + (E\bar{X}_n)^2}{4} = \frac{\frac{2p(1-p)}{n} + 4p^2}{4}$$

$$= p^2 + \underbrace{\frac{p(1-p)}{2n}}_{>0} > p^2 = \theta_X$$

$$\neq \theta_X \text{)}$$

c) $\begin{pmatrix}\hat{\theta}_n \\ \tilde{\theta}_n\end{pmatrix} = \begin{pmatrix}\bar{Y}_n \\ \frac{(\bar{X}_n)^2}{4}\end{pmatrix} \qquad \bar{Y}_n = \frac{1}{n}\sum \underbrace{I[X_i=2]}_{Y_i}$

Nejprve najdeme asympt. rozdělení $\begin{pmatrix}\bar{Y}_n \\ \bar{X}_n\end{pmatrix}$

$$Z_i = \begin{pmatrix}Y_i \\ X_i\end{pmatrix} = \begin{pmatrix}I[X_i=2] \\ X_i\end{pmatrix} \qquad EZ_i = \begin{pmatrix}EY_1 \\ EX_1\end{pmatrix} = \begin{pmatrix}p^2 \\ 2p\end{pmatrix}$$

$$\operatorname{var} Z_i = \begin{pmatrix} \operatorname{var} Y_1 & \operatorname{cov}(Y_1, X_1) \\ \operatorname{cov}(Y_1, X_1) & \operatorname{var} X_1 \end{pmatrix} =: \Sigma$$

$$Y_1 = \begin{cases} 1 \dots p^2 \\ 0 \dots 1-p^2 \end{cases} \quad \operatorname{Alt}(p^2) \Rightarrow \operatorname{var} Y_1 = p^2(1-p^2)$$

$\operatorname{var} X_1 = 2 \cdot p(1-p)$ (jde o $Bi(2,p)$)

$$\operatorname{cov}(X_1, Y_1) = EX_1Y_1 - EX_1 \cdot EY_1$$

$$EX_1Y_1 = EX_1 \cdot I[X_1 = 2] = \sum_{i=0}^{2} i\, I[i=2] \cdot P(X=i) =$$

$$= 2 \cdot p^2$$

$$EX_1 = 2p \quad EY_1 = p^2 \Rightarrow \operatorname{cov}(X_1, Y_1) = 2p^2 - 2p^3 =$$
$$= 2p^2(1-p)$$

$$\Sigma = \begin{pmatrix} p^2(1-p^2) & 2p^2(1-p) \\ 2p^2(1-p) & 2p(1-p) \end{pmatrix} = p(1-p) \begin{pmatrix} p(1+p) & 2p \\ 2p & 2 \end{pmatrix}$$

dle CLV pro $Z_1 \dots Z_n$ (předp. OK: iid rozptyl konečný)

$$\sqrt{n}\,(\bar{Z}_n - EZ_1) \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

$$\sqrt{n}\left(\begin{pmatrix} \bar{Y}_n \\ \bar{X}_n \end{pmatrix} - \begin{pmatrix} p^2 \\ 2p \end{pmatrix}\right) \xrightarrow{\mathcal{D}} N(0, \Sigma)$$

$$\begin{pmatrix} \hat{\theta}_n \\ \tilde{\theta}_n \end{pmatrix} = g\begin{pmatrix} \bar{Y}_n \\ \bar{X}_n \end{pmatrix} \qquad g(u,v) = \begin{pmatrix} u \\ \frac{v^2}{4} \end{pmatrix}$$

$$Dg(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{2v}{4} \end{pmatrix} \qquad Dg(p^2, 2p) = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$$

Delta věta:

$$\sqrt{n}\left(\begin{pmatrix} \hat{\theta}_n \\ \tilde{\theta}_n \end{pmatrix} - \begin{pmatrix} p^2 \\ p^2 \end{pmatrix}\right) \xrightarrow{\mathcal{D}} N\Bigg(0, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \cdot \Sigma \cdot \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}}_{W}\Bigg)$$

$$W = p(1-p) \cdot \begin{pmatrix} p(1+p) & 2p \\ 2p^2 & 2p \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} =$$

$$= p(1-p) \cdot \begin{pmatrix} p(1+p) & 2p^2 \\ 2p^2 & 2p^2 \end{pmatrix}$$

as. rozptyl $\hat{\theta}_n$: $p^2(1-p^2)$

as. rozptyl $\tilde{\theta}_n$: $2p^3(1-p)$

porovnání $\quad p^2(1-p^2) \;?\; 2p^3(1-p) \qquad /:(1-p)$ $\quad :p^2$

$$1+p \;?\; 2p$$

$$1 > p$$

↑ platí toto, protože $p \in (0,1)$

$\Rightarrow$ $\tilde{\theta}_n$ má menší asympt. rozptyl

$\Rightarrow$ doporučíme $\tilde{\theta}_n$

Příklad 3.

$X_i \sim R[\theta, \theta+1] \qquad f(x) = \begin{cases} 1 & x \in [\theta, \theta+1] \\ 0 & \text{jinak} \end{cases}$

$$T_n = \min X_i + \max X_i - 1$$

? konz. odhad θ

Ukážeme, že
- $\min X_i \xrightarrow{P} \theta$
- $\max X_i - 1 \xrightarrow{P} \theta$

$\Rightarrow \begin{pmatrix} \min X_i \\ \max X_i - 1 \end{pmatrix} \xrightarrow{P} \begin{pmatrix} \theta \\ \theta \end{pmatrix}$

$\Rightarrow T_n = \min X_i + \max X_i - 1 \xrightarrow{P} 2\theta$

V. o spoj. transf.

$\Rightarrow$ T_n není konzistentní

a $\frac{T_n}{2}$ by byl konzistentní

ad min: $\varepsilon > 0$ chceme ukázat, že $P(|\min X_i - \theta| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

$$P(|\min X_i - \theta| > \varepsilon) = P(\min X_i > \theta+\varepsilon) + \underbrace{P(\min X_i < \theta - \varepsilon)}_{0}$$

protože $X_i \in [\theta, \theta+1]$ s.j.

$$= P(\min X_i > \theta+\varepsilon) = P\Big(\bigcap_{i=1}^{n} [X_i > \theta+\varepsilon]\Big) \underset{\text{nez.}}{=}$$

$\min_{1\le i\le n} X_i > a \iff X_i > a \ \forall i = 1..n$

$$= \prod_{i=1}^{n} \underbrace{P(X_i > \theta+\varepsilon)}_{1-F(\theta+\varepsilon)} = [1-F(\theta+\varepsilon)]^n = \circledast$$

F dist. funkce X_i

$$F(x) = \begin{cases} 0 & x<\theta \\ x-\theta & x\in[\theta,\theta+1] \\ 1 & x>\theta+1 \end{cases}$$

$$\circledast = \begin{cases} [1-1]^n = 0 & \text{pro } \theta+\varepsilon > \theta+1 \text{ tj. pro } \varepsilon > 1 \\ [1-(\theta+\varepsilon-\theta)]^n = [1-\varepsilon]^n \xrightarrow[n\to\infty]{} 0 & \text{pro } 0<\varepsilon<1 \end{cases}$$

tedy $\min X_i \xrightarrow{P} \theta$

- ad max: $\max X_i - 1 \xrightarrow{P} \theta \iff \underbrace{\max X_i \xrightarrow{P} \theta+1}_{\text{toto dokážeme}}$

$\underline{\varepsilon > 0}$ $P(|\max X_i - 1 - \theta| > \varepsilon) =$

$$= \underbrace{P(\max X_i > 1+\theta+\varepsilon)}_{0 \text{ protože } X_i < 1+\theta \text{ s.j.}} + P(\max X_i < 1+\theta-\varepsilon) =$$

$$= P\Big(\bigcap_{i=1}^{n} [X_i < 1+\theta-\varepsilon]\Big) = \prod_{i=1}^{n} P(X_i < 1+\theta-\varepsilon) =$$

$$= F(1+\theta-\varepsilon)^n = \begin{cases} (1-\varepsilon)^n \longrightarrow 0 \quad n\to\infty & \text{pro } 1+\theta-\varepsilon < \theta+1 \text{ zároveň } 1+\theta-\varepsilon > \theta,\ \varepsilon<1 \\ 0 & \text{pro } 1+\theta-\varepsilon < \theta,\ \varepsilon > 1 \end{cases}$$

tj. $\max X_i - 1 \xrightarrow{P} \theta$