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60) $P(Y=k|X) = \dfrac{\lambda(x)^k e^{-\lambda(x)}}{k!}$ $\quad \lambda(x) = e^{\beta x}$, X nezávisí na β

i) $L(\beta) = \prod P(Y_i = y_i | X = x_i) \cdot P(X = x_i) \left(= \prod P(Y_i = y_i, X_i = x_i)\right)$

$= \prod_{i=1}^{n} \dfrac{(e^{\beta x_i})^{y_i} e^{-e^{\beta x_i}}}{y_i!} P(X = x_i)$

$\ell(\beta) = \sum \left(y_i x_i \beta - e^{\beta x_i} - \log(y_i!) + \log P(X = x_i)\right)$

$= \sum y_i x_i \cdot \beta - \sum e^{\beta x_i} + c$

$\ell'(\beta) = \sum x_i y_i - \sum e^{\beta x_i} \cdot x_i \overset{!}{=} 0$

$\ell''(\beta) = -\sum e^{\beta x_i} x_i^2 \qquad J_n(\beta) = n\, E\, e^{\beta X} X^2 \qquad \sqrt{n}(\hat\beta - \beta) \xrightarrow{D} N\left(0, \dfrac{1}{E e^{\beta X} X^2}\right)$

61) $P(X = x) = \begin{cases} p & x \in \{-1, 1\} \\ 1-2p & x = 0 \end{cases}$ $\qquad$ dg: $y := \#\{X_i = 0\}$

$L(p) = (1-2p)^y p^{n-y} \qquad \ell(p) = y \log(1-2p) + (n-y)\log p$

$\ell'(p) = \dfrac{-2y}{1-2p} + \dfrac{n-y}{p} = 0$

$-\cancel{2py} + n - 2np - y + \cancel{2py} = 0$

$\hat p = (y-n)/(-2n) = (n-y)/2n = \sum I[X_i \neq 0]/2n$

$\ell''(p) = \dfrac{-2y \cdot 2}{(1-2p)^2} - \dfrac{n-y}{p^2} \qquad EY = E \sum I[X=0] = n P(X=0) = n(1-2p)$

$J_n(p) = \dfrac{4n\cancel{(1-2p)}}{(1-2p)^{\cancel{2}}} + \dfrac{n - n(1-2p)}{p^2} = \dfrac{4n}{1-2p} + \dfrac{2n}{p} =$

$= \dfrac{4np + 2n - 4np}{(1-2p)p} = n \cdot \dfrac{2}{p(1-2p)}$

$\sqrt{n}(\hat p - p) \xrightarrow{D} N(0, p(1-2p)/2)$

62) $L(\theta) = \prod \frac{1}{2} \exp(-|x_i - \theta|) = \frac{1}{2^n} \exp\left\{-\sum |x_i - \theta|\right\}$

$\ell(\theta) = c - \sum |x_i - \theta| \Rightarrow$ maximalizovať $-\sum |x_i - \theta|$ vo θ

riešením je medián $\{x_i\}$. (riešenie online) Lemma 3.4 MSI

MLE: Vektorový parameter

63) $X \sim N(\mu, \sigma^2)$

i) $L(\mu, \sigma^2) = c \cdot (\sigma^n)^{-1} \exp\left\{-\frac{1}{2\sigma^2} \sum (x_i - \mu)^2\right\}$

$\ell(\mu, \sigma^2) = c - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$

$D\ell = \left(\frac{1}{\sigma^2} \sum (x_i - \mu) \;;\; -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (x_i - \mu)^2\right)$

$\dfrac{\partial \ell}{\partial \mu} = 0 \Rightarrow \sum x_i = n\mu,\ \hat\mu = \overline{X} \qquad \dfrac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow n = \dfrac{\sum (x_i - \mu)^2}{\sigma^2},\ \hat\sigma^2 = \frac{1}{n} \sum (x_i - \mu)^2 \Big|_{\mu = \hat\mu}$

ii) $H_\ell(\mu, \sigma^2) = \begin{pmatrix} -\dfrac{n}{\sigma^2} & -\dfrac{\sum (x_i - \mu)}{\sigma^4} \\ -\dfrac{\sum (x_i - \mu)}{\sigma^4} & \dfrac{n}{2\sigma^4} - \dfrac{\sum (x_i - \mu)^2}{(\sigma^2)^3} \end{pmatrix}$

$J_n(\mu, \sigma^2) = \begin{pmatrix} n/\sigma^2 & 0 \\ 0 & \dfrac{n}{2\sigma^4} \end{pmatrix} \qquad \sqrt{n}\left(\begin{pmatrix} \overline{X} \\ \frac{n-1}{n} S^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}\right) \xrightarrow{D} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix}\right)$

iii) IS pre μ: $\left[\bar{X} \mp \frac{u_{1-\alpha/2}\sqrt{\frac{m-1}{m}S^2}}{\sqrt{m}}\right]$ porovnaj $\left[\bar{X} \mp t_{m-1}(1-\alpha/2)\frac{S}{\sqrt{m}}\right]$

iv) $\hat{\theta} = \hat{\mu} + u_\alpha \hat{\sigma}$

$g(s,t) = s + u_\alpha\sqrt{t}$ $\quad \nabla g = \left(1, \frac{u_\alpha}{2\sqrt{t}}\right)$ $\quad \nabla g|_{\hat\mu,\hat\sigma} = \left(1, \frac{u_\alpha}{2\sqrt{\sigma^2}}\right)$ $\quad \nabla g\, J_m^{-1} \nabla g' = \frac{1}{m}\left(\sigma^2 + u_\alpha^2 2\sigma^4\right)$

$\sqrt{m}\left(\left(\hat{\theta}\right) - \theta\right) \xrightarrow{D} N\left(0, \sigma^2\left(1 + \frac{u_\alpha^2}{2}\right)\right)$ $\quad \frac{1}{m}\left(1 \;\; \frac{u_\alpha}{2\sigma}\right)\begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}\begin{pmatrix}1\\ \frac{u_\alpha}{2\sigma}\end{pmatrix}$

RC medián $\quad = \frac{1}{m}\left(\sigma^2 + \frac{u_\alpha^2\sigma^2}{2}\right)$

64) $L(\mu,\sigma^2) = \sigma^{-m}(\prod x_i)^{-1}c \cdot \exp\left\{-\frac{1}{2\sigma^2}\sum(\log x_i - \mu)^2\right\}$

i) $\ell(\mu,\sigma^2) = -\frac{m}{2}\log\sigma^2 + c - \frac{1}{2\sigma^2}\sum(\log x_i - \mu)^2$

$\nabla\ell = \left(\frac{\sum(\log x_i - \mu)}{\sigma^2}, -\frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(\log x_i - \mu)^2\right)$

$\nabla\ell = 0 \Rightarrow \hat{\mu} = \frac{1}{m}\sum\log x_i \qquad \hat{\sigma}^2 = \frac{1}{m}\sum(\log x_i - \hat{\mu})^2$

ii) $H_\ell = \begin{pmatrix} -m/\sigma^2 & -\frac{\sum(\log x_i - \mu)}{\sigma^4} \\ -\frac{\sum(\log x_i-\mu)}{\sigma^4} & \frac{m}{2\sigma^4} - \frac{1}{\sigma^6}\sum(\log x_i - \mu)^2\end{pmatrix}$ $\qquad \log X \sim N(\mu,\sigma^2)$

$J_m = \begin{pmatrix} +m/\sigma^2 & 0 \\ 0 & +\frac{m}{2\sigma^4}\end{pmatrix}$ $\qquad \sqrt{m}\left(\begin{pmatrix}\hat\mu\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}\right)$

iii) $\left\{\begin{pmatrix}\mu\\ \sigma^2\end{pmatrix} : m\left(\begin{pmatrix}\hat\mu\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right)^T \begin{pmatrix}1/\hat\sigma^2 & 0\\ 0 & 1/2\hat\sigma^4\end{pmatrix}\left(\begin{pmatrix}\hat\mu\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \le \chi^2_2(1-\alpha)\right\}$

tj $\quad m\left[\frac{(\hat\mu-\mu)^2}{\hat\sigma^2} + \frac{(\hat\sigma^2 - \sigma^2)^2}{2\hat\sigma^4}\right] \le \chi^2_2(1-\alpha)$

iv) $\left[\hat\mu - \frac{u_{1-\alpha}\hat\sigma}{\sqrt{m}}, \infty\right)$ $\qquad$ dolný odhad

65) $X \sim \lambda e^{-\lambda(x-\delta)}; \; x > \delta$

i) $L(\lambda,\delta) = \lambda^n \exp\{-\lambda\sum(x_i - \delta)\}\, I[\min x_i > \delta]$

$\ell(\lambda,\delta) = m\log\lambda - \lambda\sum(x_i-\delta) + \log I[\min x_i > \delta]$

$\quad = m\log\lambda - \lambda\sum x_i + m\lambda\delta \qquad$ ak $\min x_i > \delta$

pre pevné $\lambda > 0$ maximalizuje $\hat\delta = \min X_i$

pre dané δ $\quad \frac{\partial\ell(\lambda,\delta)}{\partial\lambda} = \frac{m}{\lambda} - \sum x_i + m\delta = 0 \Rightarrow \hat\lambda = \frac{1}{\bar X - \hat\delta}$

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ii) $\hat\lambda \xrightarrow{P} \lambda$ z pr 51 (po posunutí a Sluckého vete) $\quad \left.\right\} \begin{pmatrix}\hat\lambda\\ \hat\delta\end{pmatrix} \xrightarrow{P} \begin{pmatrix}\lambda\\ \delta\end{pmatrix}$

$\hat\delta$ sa správa ako $Exp(m\lambda)$ posunuté o $\delta \Rightarrow \hat\delta \xrightarrow{P} \delta$

iii) $m \cdot Exp(m\lambda) \sim Exp(\lambda)$, tj $m\hat\delta \sim Exp(\lambda) + \delta m$

$P(m(\hat\delta - \delta) \le x) \longrightarrow (=) 1 - e^{-\lambda x}$

tj $(\sqrt{m})^2(\hat\delta - \delta) \xrightarrow{D} Exp(\lambda)$

66) $X \sim R(a,b) \qquad L(a,b) = \left[\frac{1}{b-a}\right]^m I[a < \min x_i \le \max x_i < b]$

i) L je maximalizované ak $b-a$ je minimalizované tak, že $a < \min X_i < \max X_i < b$

$\begin{pmatrix}\hat a\\ \hat b\end{pmatrix} = \begin{pmatrix}\min X_i\\ \max X_i\end{pmatrix}$. $\qquad$ ii) z MSI: $\left.\begin{array}{l}\hat a \xrightarrow{P} a\\ \hat b \xrightarrow{P} b\end{array}\right\} \Rightarrow \begin{pmatrix}\hat a\\ \hat b\end{pmatrix} \xrightarrow{P} \begin{pmatrix}a\\ b\end{pmatrix}$

iii) $P(\hat b \le x) = \left(\frac{x-a}{b-a}\right)^m \quad x \in [a,b]$

$\qquad$ 1 ak $x \ge 0$

$P(m(\hat b - b) \le x) = P(\hat b \le b + x/m) = \left(\frac{b + x/m - a}{b-a}\right)^m = \left(1 + \frac{x/(b-a)}{m}\right)^m \xrightarrow[m\to\infty]{} e^{x/(b-a)}$ pre $x < 0$

$m(b - \hat b) \xrightarrow{D} Exp\left(\frac{1}{b-a}\right)$

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67) $X \sim M(1, p_1 \dots p_k)$ $\quad L(p) = \prod p_j^{y_j} I[\sum p_j = 1]$ $\Big|$ $y_j = \#\{X_i : X_i = (0..0,\overset{j\text{-tý el.}}{1},0..0)\}$

i) Lagrangeovy multiplikátory:

maximalizujeme $\ell(p) = \sum y_j \log p_j$ za podmínky $\sum p_i = 1$

tj. $\sum y_j \log p_j + \lambda(1-\sum p_j) = f(p,\lambda)$

$\frac{\partial}{\partial p_i} f(p,\lambda) = \frac{y_i}{p_i} - \lambda = 0 \Rightarrow p_i = y_i/\lambda$

$\frac{\partial}{\partial \lambda} f(p,\lambda) = 1 - \sum p_j = 0 \Rightarrow \sum \frac{y_i}{\lambda} = 1 \Rightarrow \hat\lambda = \sum y_i \Rightarrow \hat p_i = \frac{y_i}{\sum y_j} = \frac{\sum_{j=1}^{m} [X_{ij}=1]}{m}$

ii) definujeme $\hat p = \frac{1}{m}\sum_{i=1}^{m}\left(I[X_{i1}=1], I[X_{i2}=1], \dots, I[X_{ik}=1]\right)' = \frac{1}{m}\sum X_i$

jde o průměr iid náhodných vektorů $\Rightarrow$ $E\, I[X_{ij}=1] = P(X_i = (0,..0,\overset{j\text{-tý el.}}{1},0..0)) = p_j$

$E\hat p = p$ $\quad var\, I[X_{ij}=1] = p_j - p_j^2$

$cov\, I[X_{ij}=1] I[X_{ij'}=1] = 0 - p_j p_{j'}$

$$var\, \hat p = \frac{1}{m}\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_k \\ -p_1p_2 & p_2(1-p_2) & \dots & -p_2p_k \\ & & \ddots & \\ -p_1p_k & -p_2p_k & \dots & p_k(1-p_k)\end{pmatrix} =: \frac{1}{m}\Sigma(p)$$

CLV: $\sqrt{m}(\hat p - p) \xrightarrow{D} N_k(0, \Sigma(p))$

68) dle př 67: $X \sim M(1, p_1 - p_4)$ $\quad X \sim (CHCH, DD, CHD, DCH)^T$
$= M(1, p, q, \frac{1-p-q}{2}, \frac{1-p-q}{2})$

i) $\hat p = \frac{\#\{CHCH\}}{m} = 604/1987$ $\quad \hat q = \frac{\#\{D,D\}}{m} = 609/1987$

$\theta(p,q) = \frac{2p}{1+p-q}$ $\quad \hat\theta = \frac{2\hat p}{1+\hat p-\hat q}$ $\quad g(s,t) = \frac{2s}{1+s-t}$ $\quad \nabla g = \left(\frac{2(1-t)}{(1+s-t)^2}, \frac{2s}{(1+s-t)^2}\right)'$

$\nabla g((p,q)) = \left(\frac{2(1-q)}{1+p-q}, \frac{2p}{(1+p-q)^2}\right)'$ $\quad \sqrt{m}\left(\binom{\hat p}{\hat q} - \binom{p}{q}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix} p(1-p) & -pq \\ -pq & q(1-q)\end{pmatrix}\right)$

Δ-věta: $\sqrt{m}(\hat\theta - \theta) \xrightarrow[\text{meth.}]{D} N\left(0, \frac{4p(1-q)(1-p-q)}{(1+p-q)^4}\right)$

ii) IS: $\left[\frac{2\hat p}{1+\hat p-\hat q} \mp \frac{u_{1-\alpha/2}}{\sqrt{m}}\sqrt{\frac{4\hat p(1-\hat q)(1-\hat p-\hat q)}{(1+\hat p+\hat q)^4}}\right] = [0{,}58 ; 0{,}63]$ $\quad \hat\theta = 0{,}609$

69) $P(Y=i, N=j) = P(Y=i \mid N=j)P(N=j) \Rightarrow$ $\quad Y|N \sim Bi(N,p)$ $\quad N \sim Po(\lambda)$

i) $L(p,\lambda) = \prod P(Y_i = y_i, N_i = m_i) = \prod P(Y_i = y_i \mid N_i = m_i) \cdot \prod P(N_i = m_i)$

$\ell(p,\lambda) = \sum \log P_p(Y_i = y_i \mid N_i = m_i) + \sum \log P_\lambda(N_i = m_i)$

$\frac{\partial \ell}{\partial p}$ = "věrohodnost z $Bi(N_i,p)$", tj. $\frac{\sum y_i}{p} - \frac{\sum(m_i - y_i)}{1-p} = 0 \Rightarrow \hat p = \left(\frac{\sum m_i}{\sum y_i}\right)^{-1}$

$\frac{\partial \ell}{\partial \lambda}$ = "věrohodnost z $Po(\lambda)$", tj. $\hat\lambda = \frac{1}{m}\sum m_i$

ii) H = dle vzorce, dekompozice na části s p a λ

$$H = \begin{pmatrix} -\frac{\sum y_i}{p^2} - \frac{\sum(m_i - y_i)}{(1-p)^2} & 0 \\ 0 & -\frac{\sum m_i}{\lambda^2}\end{pmatrix} \qquad J_m = \begin{pmatrix} \frac{m\lambda}{p} + \frac{m\lambda}{1-p} & 0 \\ 0 & m/\lambda \end{pmatrix}$$

$EY = E\,E[Y|N] = E\,Bi(N,p) = EN p = \lambda p$

$E(N-Y) = E\,E[N-Y|N] = E[N - Bi(N,p)] = \lambda(1-p)$

$EN = \lambda$

$\sqrt{m}\left(\binom{\hat p}{\hat\lambda} - \binom{p}{\lambda}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix} \frac{p(1-p)}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}\right)$

10) $Y|X \sim N(\beta' X, \sigma^2)$ $\quad X$ nezávisí na β, σ^2.

i) $L(\beta, \sigma^2) = \prod f_{Y|X}(y_i|x_i) f_X(x_i) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^m \exp\left\{-\frac{1}{2\sigma^2} \sum (y_i - \beta' x_i)^2\right\} \prod f_X(x_i)$

$$\ell(\beta, \sigma^2) = c - \frac{m}{2}\log \sigma^2 - \frac{1}{2\sigma^2} \sum (y_i - \beta' x_i)^2 + d = c' - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y_i - \beta' x_i)'(y_i - \beta' x_i)$$

$$D\ell = \left(\frac{X'(Y - X\beta)}{\sigma^2}, -\frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}(Y-X\beta)'(Y-X\beta) \right) \qquad = c' - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)$$

$X'Y - X'X\beta = 0$ $\qquad -m\sigma^2 + (Y - X\beta)'(Y - X\beta) = 0$

$\underline{\underline{\hat\beta = (X'X)^{-1}X'Y}}$ $\qquad \hat\sigma^2 = \underline{\underline{\frac{1}{m}(Y - X\hat\beta)'(Y - X\hat\beta)}}$

ii) $$H_\ell = \begin{pmatrix} -\frac{X'X}{\sigma^2} & -\frac{X'(Y-X\beta)}{\sigma^4} \\ -\frac{X'(Y-X\beta)}{\sigma^4} & \frac{m}{2\sigma^4} - \frac{(Y-X\beta)'(Y-X\beta)}{\sigma^6} \end{pmatrix} \qquad J_m = \begin{pmatrix} \frac{E(X'X)}{\sigma^2} & 0 \\ 0^T & \frac{m}{2\sigma^4} \end{pmatrix}$$

$$E(Y - X\beta)'(Y - X\beta) = E\sum \varepsilon_i^2 = m\sigma^2$$

$$\sqrt{m}\left(\begin{pmatrix}\hat\beta \\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\beta \\ \sigma^2\end{pmatrix}\right) \xrightarrow{D} N_{p+1}\left(0, \begin{pmatrix} \sigma^2 (EX'X)^{-1} m & 0 \\ 0^T & 2\sigma^4 \end{pmatrix}\right)$$

iii) $\sqrt{m}(\hat\beta - \beta) \xrightarrow{D} N_p(0, \sigma^2 [EX'X]^{-1} m)$

iv) $P(\beta \in \{x \in \mathbb{R}^p : (\hat\beta - x)'[EX'X](\hat\beta - x) \le \hat\sigma^2 \chi^2_p(1-\alpha)\}) \to 1-\alpha$

11) $P(Y=1|X) = \frac{e^{\beta' x}}{1 + e^{\beta' x}}$ $\qquad P(Y=0|X) = 1 - P(Y=1|X)$

$S(x,\beta) := e^{\beta' x}/(1 + e^{\beta' x})$

$$L(\beta) = \prod S(x_i \beta)^{y_i}(1 - S(x_i, \beta))^{1-y_i} = e^{\sum y_i \beta' x_i} / \prod(1 + e^{\beta' x_i})$$

$$\ell(\beta) = \sum y_i \beta' x_i - \sum \log(1 + e^{\beta' x_i})$$

$$D\ell(\beta) = \sum y_i x_i - \sum \frac{x_i e^{\beta' x_i}}{1 + e^{\beta' x_i}} = \sum x_i (y_i - S(x_i, \beta))$$

$$H_\ell(\beta) = -\sum x_i \left(\frac{x_i' e^{\beta' x_i}(1 + e^{\beta' x_i}) - e^{\beta' x_i} e^{\beta' x_i} x_i'}{(1 + e^{\beta' x_i})^2} \right) = -\sum x_i x_i' \frac{e^{\beta' x_i}}{1 + e^{\beta' x_i}} \frac{1}{1 + e^{\beta' x_i}} =$$

$$= -\sum x_i x_i' S(x_i, \beta)(1 - S(x_i, \beta)) = -X'WX \quad \text{pro } X = \begin{pmatrix} x_1' \\ \vdots \\ x_m' \end{pmatrix}$$

$$W := \text{diag}(S(x_1, \beta)(1 - S(x_1, \beta)), \dots, S(x_m, \beta)(1 - S(x_m, \beta)))$$

a) $\sqrt{m}(\hat\beta - \beta) \xrightarrow{d} N_p(0, [E(X'WX)]^{-1} m)$

b) $\beta_1 \in [\hat\beta_1 \mp u_{1-\alpha/2} (E X'WX)^{-1}_{11}]$.

12) $X \sim Exp(1/\eta)$ $\qquad Y|X \sim Exp(1/x\theta)$

a) $L(\theta, \eta) = (\prod x_i)^{-1} \theta^{-m} \eta^{-m} \exp\left\{-\sum \frac{y_i}{x_i \theta} - \sum \frac{x_i}{\eta}\right\}$

$$\ell(\theta, \eta) = c - m \log\theta - m \log \eta - \sum \frac{y_i}{x_i\theta} - \sum \frac{x_i}{\eta}$$

$$D\ell(\theta, \eta) = \left(-\frac{m}{\theta} + \sum \frac{y_i}{x_i \theta^2}; -\frac{m}{\eta} + \sum \frac{x_i}{\eta^2}\right) \stackrel{!}{=} 0 \Rightarrow \hat\eta = \bar X \qquad \hat\theta = \frac{1}{m}\sum \frac{y_i}{x_i}$$

$$H_\ell(\theta, \eta) = \begin{pmatrix} \frac{m}{\theta^2} - \sum \frac{2 y_i}{x_i \theta^3}, & 0 \\ 0, & \frac{m}{\eta^2} - \sum \frac{2 x_i}{\eta^3} \end{pmatrix}$$

$$EX = \eta \qquad E\left[\frac{Y}{X}\right] = E\left[E\left[\frac{Y}{X} \middle| X\right]\right] = E\left[\frac{1}{X} E[Y|X]\right] = E\left[\frac{X\theta}{X}\right] = \theta$$

72) cont. $\mathcal{J}_n(\theta,\eta) = \begin{pmatrix} n/\theta^2 & 0 \\ 0 & n/\eta^2 \end{pmatrix}$

$$\sqrt{n}\left(\begin{pmatrix}\hat\theta\\ \hat\eta\end{pmatrix} - \begin{pmatrix}\theta\\ \eta\end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\theta^2 & 0\\ 0 & \eta^2\end{pmatrix}\right)$$

b) $\sqrt{n}(\hat\theta - \theta) \xrightarrow{\mathcal{D}} N(0, \theta^2)$

c) $\theta \in [\hat\theta \mp u_{1-\alpha/2}\, \hat\theta/\sqrt{n}]$

d) až $\theta_0 \in CI$ pro θ nezamítám H_0, jinak zamítám.

nové d) exponenciální rodina, úplná postač. pro $\frac{1}{\theta}$ je $\sum \frac{y_i}{x_i}$, $E\frac{y_i}{x_i} = \theta$ nestr. odhad, fce úplné postač., nejlepší nestranný.

8. asymptotické testy bez rušivých parametrů

73) $X \sim alt(p)$ z Př. 49 +1 rovnice

80 (82) $U_n(p) = \frac{\sum X_i}{p} - \frac{n - \sum X_i}{1-p}$ $\qquad \hat p = \bar X \qquad \mathcal{J}_n(p) = \frac{n}{p(1-p)}$

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- $W_n = n(\bar X - p_0)^2 \cdot \frac{1}{\hat p(1-\hat p)} = \left[\sqrt{n}\, \frac{\bar X - p_0}{(\bar X(1-\bar X))^{1/2}}\right]^2$ ⟺ klas. asympt. test

- $R_n = \dfrac{\left(\frac{\sum X_i}{p_0} - \frac{n - \sum X_i}{1-p_0}\right)^2}{n/(p_0(1-p_0))} = \left[\sqrt{n}\sqrt{p_0(1-p_0)}\left(\frac{\bar X}{p_0} - \frac{1-\bar X}{1-p_0}\right)\right]^2 = \left[\frac{\sqrt{n}}{\sqrt{p_0(1-p_0)}}(\bar X - p_0)\right]^2$

⟺ Wilsonova metoda

- $LR_n = 2\left[\sum X_i \log \bar X + (n - \sum X_i)\log(1-\bar X) - \sum X_i \log p_0 - (n - \sum X_i)\log(1-p_0)\right]$
$\qquad = 2n\left[\bar X \log\frac{\bar X}{p_0} + (1-\bar X)\log\frac{1-\bar X}{1-p_0}\right]$

každý test zamítám až $T_n > \chi^2_1(1-\alpha)$

74) $X \sim Po(\lambda)$ z Př 50 +1 rovnice

81 $U_n(\lambda) = -n + \sum X_i/\lambda \qquad \hat\lambda = \bar X \qquad \mathcal{J}(\lambda) = 1/\lambda$

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- $W_n = n(\bar X - \lambda_0)^2 \frac{1}{\bar X} = \left(\sqrt{n}\, \frac{\bar X - \lambda_0}{\sqrt{\bar X}}\right)^2$
- $R_n = \left(-n + \sum X_i/\lambda_0\right)^2 / (n/\lambda_0) = \left(\sqrt{n}\, \frac{\bar X - \lambda_0}{\sqrt{\lambda_0}}\right)^2$
- $LR_n = 2\left(-n\bar X + \sum X_i \log \bar X + n\lambda_0 - \sum X_i \log \lambda_0\right) = 2n\left[\log\left(\frac{\bar X}{\lambda_0}\right)\cdot \bar X - (\bar X - \lambda)\right]$

každý test zamítáme až $T_n > \chi^2_1(1-\alpha)$

75) $\tilde X \sim Po(\lambda)$ ale $X = \tilde X \mid \tilde X > 0$

82 $P(X = k) = P(\tilde X = k \mid \tilde X > 0) = P(\tilde X = k, \tilde X > 0)/P(\tilde X > 0) = \frac{e^{-\lambda}\lambda^k/k!}{1 - e^{-\lambda}} \qquad k \geq 1$

85 $P(\tilde X > 0) = 1 - e^{-\lambda}\lambda^0/0! = 1 - e^{-\lambda}$