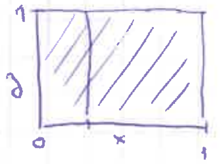
NMSA 332 1. cvičení – Podmíněné rozdělení

2 písemky 11-12 cvičení

1) $f(x,y) = (x+y) I_M \qquad M = \{0<x<1,\ 0<y<1\}$



a) $E[XY|X=x] = x\, E[Y|X=x] = \cancel{\frac{\ldots x^2}{6(x+1/2)}} = *$

$$f_{Y|X}(x,y) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{(x+y) I_M}{\int_0^1 (x+y)\,dy} = \frac{(x+y) I_M}{x+1/2} \qquad x\in(0,1)\ \ y\in(0,1)$$

$$E[Y|X] = \int_0^1 y\,\frac{(X+y) I_M}{X+1/2}\,dy = \frac{X/2}{X+1/2} + \frac{1/3}{X+1/2} = \cancel{\frac{X}{X+1/2}(5/6)}$$

$$* = x\cdot\left(\frac{x}{2x+1} + \frac{1}{3x+3/2}\right)$$

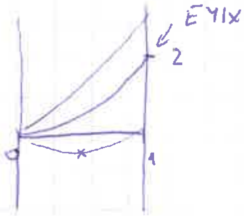
b) $E[XY|X] = X\left(\frac{X}{2X+1} + \frac{1}{3X+3/2}\right)$

c) $E[XY^2|X] = X\int_0^1 y^2\,\frac{(X+y)}{X+1/2}\,dy = \frac{X}{X+1/2}\left[\frac{X}{3} + \frac{1}{4}\right]$

Anděl (V 3.24). $EY^2<\infty$, $\binom{Y}{Z}$ má konkrétní rozdělení. Potom $E(Y-f(Z))^2 \geq E(Y-E(Y|Z))^2$ pro každou f měřitelnou. s Rovností $\iff$ $f(Z) = E(Y|Z)$ s.j.

podm. rozptyl: $var(Y|X) := E((Y-E(Y|X))^2|X)$

2) $Y|X \sim N(2X^3, 3X^2) \qquad X\sim R(0,1)$



$$E[Y|X] = \int_{\mathbb{R}} y\, f_{Y|X}(x,y)\,dy = \int_{\mathbb{R}} y\, \text{"}N(2x^3,3x^2)\text{"}\,dy = 2X^3$$

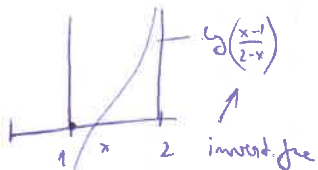
a) $E\left[\frac{Y}{X^2}\middle|X\right] = \frac{1}{X^2}E[Y|X] = 2X$

b) $E\frac{Y}{X^2} = E\,E\left[\frac{Y}{X^2}\middle|X\right] = E\,2X = 2\cdot 1/2 = 1$

c) $EY = E\,E[Y|X] = E\,2X^3 = 2\int_0^1 x^3\,dx = 2\cdot\frac{1}{4} = \frac{1}{2}$

d) $var\,Y = E\,var(Y|X) + var\,E(Y|X) = E\,3X^2 + var(2X^3) = 3\cdot 1/3 + 4\left(\frac{1}{7} - \left(\frac{1}{4}\right)^2\right) =$
$= 1 + 4/7 - 1/4$

3) $f(x,y) = \frac{1}{x}e^{-y/x} \qquad x\in(1,2),\ y>0$



$$f_X(x) = \int_0^\infty \frac{1}{x}e^{-y/x}\,dy = \left[\frac{1}{x}e^{-y/x}\cdot\left(-\frac{1}{x}\right)\right]_{y=0}^{\infty} = 1 \qquad x\in(1,2)$$

$$f_{Y|X}(x,y) = f_{Y|X}(y|x) = \frac{1}{x}e^{-y/x} \qquad Y|X\sim Exp\left(\frac{1}{X}\right) \quad X\sim R(1,2)$$

a) $E[Y|X=t] = t \quad t\in(1,2)$, $E[Y|X] = X$

b) $E\left[Y\middle|\lg\left(\frac{X-1}{2-X}\right) = t\right] = \frac{2e^t+1}{e^t+1}$

$$E\left[Y\middle|\underbrace{\lg\left(\frac{X-1}{2-X}\right)}_{T}\right] = \frac{2e^T+1}{e^T+1} = X$$

$\lg\frac{x-1}{2-x} = t$

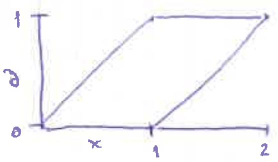
$\frac{x-1}{2-x} = e^t$

$x(1+e^t) = 2e^t+1$

$x = (2e^t+1)/(e^t+1)$

c) $E\left[\frac{Y}{X^6}\middle|\lg\left(\frac{X-1}{2-X}\right)\right] = \frac{1}{X^5}$

4) $f(x,y) = c\,y\,I_M(x,y) \qquad M = \{y\in(0,1),\ y\leq x\leq y+1\}$



$$\int_0^1\int_y^{y+1} c\,y\,dx\,dy = c\int_0^1 y\,dy = c\cdot 1/2 \quad\rightarrow c=2$$

a) $E\left[911X - \lg\frac{Y}{1-Y}\middle|Y\right] = 911\,E[X|Y] - \lg\frac{Y}{1-Y} = *$

~~$Y\sim R(0,1)$~~
~~$X|Y\sim R(Y,Y+1)$~~

$Y\sim 2y \quad y\in(0,1)$
$X|Y\sim R(Y,Y+1)$

$$f_{X|Y}(x|y) = \frac{2y\,I_M(x,y)}{\int_y^{y+1} 2y\,dx} = I_M(x,y)$$

$Y=y$ fixed

$$E[X|Y] = \int x\, I_M(x,y)\,dx = \int_y^{y+1} x\,dx = \left[\frac{x^2}{2}\right]_y^{y+1} = \frac{(y+1)^2-y^2}{2} = y+\frac{1}{2}$$

$$* = 911\cdot(Y+1/2) - \lg\left(\frac{Y}{1-Y}\right)$$

b) $E[\sin X|Y] = \int_y^{y+1} \sin x \cdot dx = [-\cos x]_y^{y+1} = \cos y - \cos(y+1)$

5) a) $E[X+Y|X] = X + E[Y|X] \underset{\text{nezáv.}}{=} X + EY$

b) $E[X+Y|X] = X + E[Y|X]$

projekcia maticou

$EY|X = f(x)$

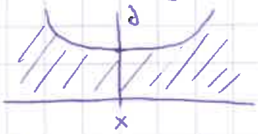
c) $Z = X+Y$

$E[X|Z] = E[Z-Y|Z] = Z - E[Y|Z] \overset{\text{zameniteľnosť}}{=} Z - E[X|Z]$

$E[X|Z] = Z/2 \Rightarrow E[X|X+Y] = \frac{X+Y}{2}$

6) $Y|X \sim R(0, X^2+1)$

$X \sim N(0,1)$



$EX^4 = 3!! = 3.1 = 3$

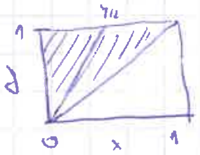
$EX^p = \sigma^p (p-1)!!$ p párne

a) $E[Y|e^X] \underset{\substack{Z = e^X \\ X = \ln Z}}{=} E[Y|Z] = \frac{(\ln Z)^2+1}{2} = \frac{X^2+1}{2}$

b) $EY = EE[Y|X] = E\frac{X^2+1}{2} = 1$ var $R(a,b) = (b-a)^2/12$

c) $\text{var}\, Y = \text{var}(E[Y|X]) + E(\text{var}(Y|X)) = \text{var}\, \frac{X^2+1}{2} + E\frac{(X^2+1)^2}{12} = \frac{3}{4} - \frac{1}{4} + \frac{1}{12}[3+2+1] = 1$

7) $(X,Y)^T \sim U(M)$ $M = \{0<x<1,\ x<y\}$



$f_{X,Y}(x,y) = 2I_M$

$f_{X|Y}(x|y) = \frac{2I_M}{\int_0^y 2\,dx} = \frac{I_M}{y}$

a) $E[\ln X|Y] = \int_0^y \frac{\ln x}{y}\, dx = \frac{1}{y}\left([x\ln x]_0^y - \int_0^y 1\,dx\right) = \ln y - 1$

b) $E[X|\ln Y] = E[X|Y] = \int_0^y \frac{x}{y}\, dx = \frac{y}{2}$

inverzná fce existuje

c) $E[\ln X|\ln Y] = \ln Y - 1$

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15) $X \sim Exp(\lambda)$ $g(\lambda) = 1/\lambda^2$ $[g'(\lambda)]^2 = 4/\lambda^6$ $EX = \frac{1}{\lambda}$ $\text{var}\, X = \frac{1}{\lambda^2}$ $EX^2 = \frac{2}{\lambda^2}$

i) $ET = cn\, EX_1^2 = cn\, 2/\lambda^2 \Rightarrow c = 1/2n$

ii) $\text{var}\, T = \frac{n}{4n^2}\text{var}\, X_1^2 = 5/4n\lambda^4$ $EX^k = \int_0^\infty \lambda e^{-\lambda x} x^k dx \overset{\lambda x = t}{=} \int_0^\infty e^{-t} t^k dt/\lambda^k = \frac{k!}{\lambda^k}$

$L(\lambda) = \lambda e^{-\lambda x} \rightsquigarrow l = \ln\lambda - \lambda x \rightsquigarrow l' = 1/\lambda - x \rightsquigarrow l'' = -1/\lambda^2 \rightsquigarrow J(\lambda) = 1/\lambda^2$

RC: $\frac{\lambda^2}{n}\cdot\frac{4}{\lambda^6} = \frac{4}{n\lambda^4} < \text{var}\, T = \frac{5}{\lambda^4 n}$

16) v normálnom modele sú S^2 a $\bar{X}$ nezávislé $\Rightarrow$ T_1 a T_2 sú nezávislé

ďalej $S^2(n-1) \sim \sigma^2\chi^2_{n-1}$

$\sqrt{S^2(n-1)} \sim \sigma\chi_{n-1}$ pokračovanie Mathematica skript

NMSA 332 2. cvičení – Rao–Cramér 9, navrhy 8, 10, 11, 12, 13

(8) 1, $\binom{X}{Y} \sim N_2\left(\binom{\theta}{\theta}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$ $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ $\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$ $|\Sigma| = 1-\rho^2$

a, $f_{\binom{X}{Y}}(x,y) = \frac{1}{2\pi |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}\left(\binom{x}{y}-\binom{\theta}{\theta}\right)' \Sigma^{-1} \left(\binom{x}{y}-\binom{\theta}{\theta}\right)\right\} = \frac{c'}{\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\left[\frac{(x-\theta)^2+(y-\theta)^2-2\rho(x-\theta)(y-\theta)}{1-\rho^2}\right]\right\}$

$\frac{1}{1-\rho^2}(x-\theta \quad y-\theta)\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} x-\theta \\ y-\theta \end{pmatrix} = \frac{1}{1-\rho^2}\left((x-\theta)^2+(y-\theta)^2-2\rho(x-\theta)(y-\theta)\right)$

$\ell_{\binom{X}{Y}}(\theta) = c - \frac{1}{2(1-\rho^2)}\left((x-\theta)^2+(y-\theta)^2-2\rho(x-\theta)(y-\theta)\right)$

$\frac{\partial \ell}{\partial \theta}(\theta) = -\frac{1}{\cancel{2}(1-\rho^2)}\left(-\cancel{2}(x-\theta)-\cancel{2}(y-\theta)+\cancel{2}\rho[(y-\theta)+(x-\theta)]\right)$

$\frac{\partial^2 \ell}{\partial \theta^2}(\theta) = -\frac{1}{1-\rho^2}(2-2\rho) = -\frac{2(1-\rho)}{1-\rho^2} = -2/(1+\rho) \Rightarrow J(\theta) = 2/(1+\rho)$

b, $\bar{X} \sim N(\theta, 1/n)$ $\operatorname{var} \bar{X} = 1/n$ R-C: $1/(2n(1+\rho)) = (1+\rho)/2n$

nestranný $\quad 1/\cancel{n} > (1+\rho)/(2\cancel{n})$ $\rightarrow$ nedosahuje R-C mezu $\quad$ až $\rho=1$ dosahuje

c, $Z = (X+Y)/2 \sim N\left(\theta, \frac{1}{4}+\frac{1}{4}+\frac{2\rho}{4}\right) = N\left(\theta, \frac{1+\rho}{2}\right)$

$\bar{Z} \sim N\left(\theta, \frac{1+\rho}{2n}\right)$ $\operatorname{var} \bar{Z} = \frac{1+\rho}{2n} =$ R-C: $\frac{1+\rho}{2n}$ $\rightarrow$ dosahuje R-C mezu

→(9) 2, $X \sim Po(\lambda)$

a, $f(x) = e^{-\lambda}\lambda^x/x!$ $x \in N_0$

$\ell(\lambda) = -\lambda + x \log \lambda + c$

$\partial \ell/\partial\lambda = -1 + x/\lambda$

$\partial^2 \ell/\partial^2\lambda = -x/\lambda^2$ $\quad J(\lambda) = E\, X/\lambda^2 = 1/\lambda$ $\quad$ b, $J_n(\lambda) = n/\lambda$

c, $g(\lambda) = 2\lambda$ $\quad Y = \sum X_i \sim Po(n\lambda)$ $\quad EY = n\lambda \Rightarrow T := \frac{2\sum X_i}{n}$

$\operatorname{var} T = \frac{4}{n^2} n \operatorname{var} X_1 = \frac{4\lambda}{n} =$ R-C: $\frac{4\lambda}{n}$ $\Rightarrow$ eficientní odhad

$g'(\lambda) = 2$

d, $T = \left(1-\frac{1}{n}\right)^{\sum X_i}$ $\quad ET^k = \sum_{j=0}^{\infty}\left(1-\frac{1}{n}\right)^{j\cdot k} e^{-n\lambda}\frac{(n\lambda)^j}{j!} = e^{-n\lambda} e^{n\lambda \cdot (1-\frac{1}{n})^k} = \cancel{\exp\{(1-\frac{1}{n})^k\}}$

$ET = e^{-\lambda} \Rightarrow$ nestranný $\quad g(\lambda) = e^{-\lambda}$

$\operatorname{var} T = e^{\cancel{-n\lambda} + \cancel{n\lambda} - 2\lambda + \lambda/n} - e^{-2\lambda} = e^{-2\lambda}(-1 + e^{\lambda/n})$ $\quad g'(\lambda) = -e^{-\lambda}$

R-C: $e^{-2\lambda}\frac{\lambda}{n}$

$\operatorname{var} T = e^{-2\lambda} \cdot \sum_{j=1}^{\infty}\left(\frac{\lambda}{n}\right)^j \frac{1}{j!} > e^{-2\lambda} \cdot \frac{\lambda}{n} \Rightarrow$ nedosahuje

e, jasné.

(10) 3, neregulární systém, support závisí na α

(11) 4, $X \sim N(\theta, \sigma^2)$

a, $L(\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\theta)^2\right\}$ $\quad \ell(\sigma) = c - \log\sigma - \frac{1}{2\sigma^2}(x-\theta)^2$

$\frac{\partial \ell}{\partial \sigma} = -\frac{1}{\sigma} + \frac{(x-\theta)^2}{\sigma^3}$ $\quad \frac{\partial^2 \ell}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{3(x-\theta)^2}{\sigma^4}$ $\quad J(\sigma) = \frac{3}{\sigma^2} - \frac{1}{\sigma^2} = \frac{2}{\sigma^2}$

b, $L(\sigma^2) = \dots$ $\quad \ell(\sigma^2) = c - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(x-\theta)^2$

$J(\sigma^2) = \frac{1}{(\sigma^2)^2} - \frac{1}{2(\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} = \frac{J(\sigma)}{(2\sigma)^2}$

$\frac{\partial \ell}{\partial \sigma^2} = -\frac{1}{2\sigma^2} + \frac{(x-\theta)^2}{2(\sigma^2)^2}$ $\quad \frac{\partial^2 \ell}{\partial (\sigma^2)^2} = \frac{1}{2(\sigma^2)^2} - \frac{(x-\theta)^2}{(\sigma^2)^3}$

c) jasné

12) $X \sim N(0,\sigma^2)$

i) mat. statistika I: nestrannosť S_m^2 (slide 2.2.3, veta 2.6)

na normality: (veta 2.8) $\frac{(m-1)S_m^2}{\sigma^2} \sim \chi^2_{m-1}$ $\quad \mathrm{var}\, \frac{m-1}{\sigma^2} S_m^2 = \mathrm{var}\, \chi^2_{m-1} = 2(m-1)$

$\Rightarrow \mathrm{var}\, S_m^2 = \frac{2\sigma^4}{m-1}$

$L(\sigma^2) =$ ako v 11) $\Rightarrow$ $J_m(\sigma^2) = \frac{1 \cdot m}{2(\sigma^2)^2}$ $\quad$ RC: $\frac{2\sigma^4}{m} < \mathrm{var}\, S_m^2 = \frac{2\sigma^4}{m-1}$

asymptoticky je to ale efektívne

ii) $T_m = \frac{1}{m}\sum X_i^2$ $\quad ET_m = EX^2 = \sigma^2$ $\quad \mathrm{var}\, T_m = \frac{1}{m^2}\sum \mathrm{var}\, X_i^2 = \frac{1}{m}(EX^4 - \sigma^4) = \frac{3\sigma^4 - \sigma^4}{m} =$

$= \frac{2\sigma^4}{m} =$ RC medza

iii) $\hat{\sigma}_m = \sqrt{\frac{\pi}{2}} \cdot \frac{1}{m}\sum |X_i|$ $\quad E\hat{\sigma}_m = \sqrt{\frac{\pi}{2}}\, E|X| = \sqrt{\frac{\pi}{2}} \int_{\mathbb{R}} \frac{|x|}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx = 2\sqrt{\frac{\pi}{2}} \int_0^\infty \frac{x e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} dx =$

$= \cancel{\sqrt{2\pi}} \int_0^\infty \frac{\sqrt{2\sigma^2}\, t}{\cancel{\sqrt{2\pi\sigma^2}}} e^{-t^2} \sqrt{2\sigma^2}\, dt = 2\sigma \int_0^\infty t e^{-t^2} dt = \sigma \int_0^\infty e^{-s} ds = \underline{\underline{\sigma}}$ $\quad \frac{x}{\sqrt{2\sigma^2}} = t$, $t^2 = s$

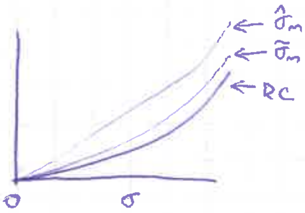
$\mathrm{var}\, \hat{\sigma}_m = \frac{\pi}{2m} \mathrm{var}\, |X_i| = \frac{\pi}{2m}\left(\sigma^2 - \left(\sigma\sqrt{\frac{2}{\pi}}\right)^2\right) = \sigma^2 \frac{\pi}{2m}\left(1 - \frac{2}{\pi}\right) = \sigma^2\left(\frac{\pi}{2m} - \frac{1}{m}\right) = \frac{\sigma^2}{2m}(\pi - 2)$

$RC = \frac{\sigma^2}{2m} < \mathrm{var}\, \hat{\sigma}_m = \frac{\sigma^2}{2m}(\pi - 2)$

iv) $\tilde{\sigma}_m = c\sqrt{\frac{1}{m}\sum X_i^2}$ $\quad E\tilde{\sigma}_m = c\int_0^\infty \sqrt{\frac{1}{m} y \sigma^2}\, f_{\chi^2_m}(y)\, dy \underset{\uparrow}{=} \frac{\sigma \cdot c}{\sqrt{m}} E\sqrt{Y} = \frac{\sigma c}{\sqrt{m}} \frac{\sqrt{2}\,\Gamma\left(\frac{1+m}{2}\right)}{\Gamma\left(\frac{m}{2}\right)}$

$\frac{\sum X_i^2}{\sigma^2} = \sum\left(\frac{X_i}{\sigma}\right)^2 \sim \sum N(0,1)^2 = \chi^2_m$ $\Rightarrow$ $\sum X_i^2 \sim \sigma^2 \chi^2_m$ $\quad Y \sim \chi^2_m$

$c = \sqrt{\frac{m}{2}} \frac{\Gamma(m/2)}{\Gamma\left(\frac{m+1}{2}\right)}$ $\quad \mathrm{var}\, \tilde{\sigma}_m = \frac{c^2}{m} E\sum X_i^2 - \sigma^2 = \frac{\cancel{m}}{2\cancel{m}}\left[\frac{\Gamma(m/2)}{\Gamma\left(\frac{m+1}{2}\right)}\right]^2 \sigma^2 m - \sigma^2$



$E\chi_m = \sqrt{2} \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma(m/2)}$

$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad (= (n-1)! \text{ pre } n \in \mathbb{N})$

13) $X \sim R(0,\theta)$ $\rightarrow$ neregulárny model

i) $2\bar{X} = \hat{\theta}_m$ $\quad E\hat{\theta}_m = 2EX_1 = \theta$ $\quad P(\max X_i \le t) = \left(\frac{t}{\theta}\right)^m \Rightarrow f_{\max}(t) = \frac{m t^{m-1}}{\theta^m}$

$\tilde{\theta}_m = \frac{m+1}{m}\max X_i$ $\quad E\tilde{\theta}_m = \frac{m+1}{m} E\max X_i = \frac{m+1}{m}\int_0^\theta t \frac{m t^{m-1}}{\theta^m} dt = \frac{(m+1)}{\theta^m}\left[\frac{t^{m+1}}{m+1}\right]_0^\theta = \theta$

ii) neregulárny model

14) $X \sim Alt(p)$

i) $\hat{p} = \bar{X}$ nestranný $\quad \mathrm{var}\, \hat{p} = \frac{p(1-p)}{m}$ $\quad L(p) = p^{\sum X_i}(1-p)^{m-\sum X_i}$

$\ell(p) = \sum x_i \log p + (m - \sum x_i)\log(1-p)$

$\frac{\partial \ell}{\partial p} = \frac{\sum x_i}{p} - \frac{m - \sum x_i}{1-p}$ $\quad \frac{\partial^2 \ell}{\partial p^2} = -\frac{\sum x_i}{p^2} - \frac{m - \sum x_i}{(1-p)^2}$

$J_m(p) = \frac{m}{p} + \frac{m}{1-p} = m\left(\frac{1}{p} + \frac{1}{1-p}\right) = \frac{m}{p(1-p)}$ $\quad$ RC: $\frac{p(1-p)}{m}$ dosahuje RC medzu

ii) jasné

NMSA 332   Suficientní štatistiky   vzorové 14), 21), 24), 26)

25) 14) $X_i \sim p(1-p)^k \quad k \in N_0$ $S = \sum X_i \sim$ neg. bin $= \binom{n+s-1}{s} p^n (1-p)^s$

a) $P(\mathbb{X} = x \mid S = s) = \frac{P(S=s \mid \mathbb{X}=x)\, P(\mathbb{X}=x)}{P(S=s)} = \frac{I[\sum x_i = s]\, p^n (1-p)^{\sum x_i}}{\binom{n+s-1}{s} p^n (1-p)^s}$

$= \begin{cases} 1/\binom{n+s-1}{s} & \text{ak } \sum x_i = s \\ 0 & \text{inak.} \end{cases}$ $\Rightarrow$ suf ic.

pre dané s ide o rovnomerné rozdelenie na všetkých n-ticiach v N_0 takých že $\sum_{i=1}^{n} x_i = s$.

b) $f(x; p) = p^n (1-p)^{\sum x_i}$ $\Rightarrow$ $S = \sum X_i$ je suf.

26) 18) a) $X_i \sim Po(\lambda)$ $S = \sum X_i \sim Po(n\lambda)$

$P(\mathbb{X} = x \mid S = s) = I[\sum x_i = s] \frac{e^{-n\lambda} \lambda^{\sum x_i} / (\prod x_i!)}{e^{-n\lambda} (n\lambda)^s / s!} = \begin{cases} \binom{s}{x_1 \dots x_n} \left(\frac{1}{n}\right)^{x_1} \cdots \left(\frac{1}{n}\right)^{x_n} & \text{ak } \sum x_i = s \\ 0 & \text{inak} \end{cases}$

$\Rightarrow$ suf ic a ide pre dané s o rozdelenie $M(s; \frac{1}{n}, \dots, \frac{1}{n})$

b) $P(\mathbb{X} = x) = \underbrace{e^{-n\lambda} \lambda^{\sum x_i}}_{g(\sum x_i, \lambda)} \underbrace{1/\prod (x_i!)}_{h(x)}$

27) 19) $X \sim R\{1 \dots M\}$ $M \in N$ $S = \max X_i$ $P(S \le s) = P(X_i \le s)^n = \left(\frac{s}{M}\right)^n$

$P(S = s) = P(S \le s) - P(S \le s-1) = \left(\frac{s}{M}\right)^n - \left(\frac{s-1}{M}\right)^n$ pre $s \in \{1 \dots M\}$

a) $P(\mathbb{X} = x \mid S = s) = I[\max x_i = s] \frac{(1/M)^n}{(s^n - (s-1)^n)/M^n} = \begin{cases} 1/(s^n - (s-1)^n) & \text{ak } \max x_i = s \\ 0 & \text{inak} \end{cases}$

$\Rightarrow$ S je suf ic

b) $P(X = x) = \left(\frac{1}{M}\right)^n \prod I[1 \le x_i \le M] = \left(\frac{1}{M}\right)^n I[\max x_i \le M]$

28) 20) $f(x; \sigma^2) = \frac{c}{\sigma^n} \exp\left\{-\frac{1}{2\sigma^2} \sum x_i^2\right\}$ z Lehman-Scheffé je $\sum X_i^2$, minimálna suf

$\Downarrow$

$\sum X_i^2$ je funkciou $\forall$ suf. štatistiky

i) $\mathbb{X}$ je suf (viem z nej dostať $\sum X_i^2$, prípadne $\mathbb{X} \mid \mathbb{X} \sim \delta_x$ nezávisí na σ^2)

ii) $(|X_1|, \dots, |X_n|)'$ je suf ($\sum X_i^2 = \sum |X_i|^2$)

iii) $\sum X_i$ nie je suf ($\sum X_i^2$ nie je fce $\sum X_i$, ak $\sum X_i = k$ neviem určiť $\sum X_i^2$)

iv) nie $\sum |X_i|$, ako v iii)

v) $\sum X_i^2$ je suf

vi) $\sum X_i^2 / n$ je suf

vii) $\left(\frac{1}{n} \sum_{i=1}^{n-1} X_i^2, X_n^2\right)$ je suf

ale $\sum X_i^2$ nie je funkcia $\sum X_i$

($X_1 = -1$, $X_2 = 1$ $\sum X_i = 0$ a $\sum X_i^2 = 2$

ale $X_1 = -2$, $X_2 = 2$ $\sum X_i = 0$ a $\sum X_i^2 = 4$

$\Rightarrow$ $\sum X_i^2$ nie je fce $\sum X_i$)

29) 21) $X \sim alt(p)$

i) $P(\mathbb{X} = x) = p^{\sum x_i} (1-p)^{n - \sum x_i}$ $\Rightarrow$ $\sum X_i$ je suf.

ii) $\frac{P(\mathbb{X} = x)}{P(\mathbb{X} = y)} = p^{\sum x_i - \sum y_i} (1-p)^{\sum y_i - \sum x_i}$ $\Rightarrow$ $\sum X_i$ je minimálna postač.

iii) nech $E_p w(X_1) \overset{!}{=} 0 = p(w(1)) + (1-p) w(0)$ $\forall p \in (0,1)$

$= p(w(1) - w(0)) + w(0)$ $\Rightarrow$ $w(0) = 0$ a $w(1) - w(0) = 0$.

$\Rightarrow$ úplná. Nie je ale postačujúca $\mathbb{X} \mid X_1 \sim (\delta_{x_1}, X_2 \dots X_n)'$ závisí na p

iv) nech $E_p w(\sum X_i) \overset{!}{=} 0 = \sum_{j=0}^{n} \binom{n}{j} \underbrace{p^j (1-p)^{n-j}}_{\text{LNZ fce v p pre } j = 0 \dots n} w(j)$ $\Rightarrow$ $w(j) = 0$ $\forall j$

$\sum X_i \sim Bi(n, p)$

a je úplná.

30) 22) $X_i \sim N(\mu,\sigma^2)$

$$\frac{f(\mathbf{x})}{f(\mathbf{y})} = \frac{1/\sigma^n}{1/\sigma^n}\,\frac{\exp\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\}}{\exp\{-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\}} = \exp\left\{-\frac{1}{2\sigma^2}\left[\sum x_i^2 - 2\mu\sum x_i + \cancel{n\mu^2} - \sum y_i^2 + 2\mu\sum y_i - \cancel{n\mu^2}\right]\right\}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[\sum(X_i^2-Y_i^2) - 2\mu\sum(X_i-Y_i)\right]\right\} \Rightarrow (\textstyle\sum X_i, \sum X_i^2)' \text{ je minim. suf.}$$

31) 23) $X\sim R(0,\theta)$ $\quad M=\max X_i \quad P(M\le m) = (m/\theta)^n \Rightarrow f_M(m) = \frac{n\, m^{n-1}}{\theta^n} \quad m\in[0,\theta]$

i) nech $\forall\theta>0$

$$0 \overset{!}{=} E_\theta\, w(M) = \int_0^\theta \frac{n\, m^{n-1}}{\theta^n}\, w(m)\,dm$$

potom aj $0=\int_0^\theta m^{n-1}w(m)\,dm \quad / \partial/\partial\theta$

$0 = \theta^{n-1}w(\theta) - 0^{n-1}w(0) \Rightarrow w(\theta)=0\ \forall\theta$ (s.v.) $\Rightarrow$ úplná

ii) $0 \overset{!}{=} E_\theta\, w(X_1) = \int_0^\theta \frac{w(x)}{\theta}\,dx \quad /\theta/\frac{\partial}{\partial\theta}$

$0=w(\theta)\quad \forall\theta$ (s.v.) $\Rightarrow$ úplná

32) 24) $X\sim R(\theta-1/2,\ \theta+1/2)$

i) $f(\mathbf{x}) = 1\cdot I[x_1\in(\theta-1/2,\theta+1/2)]\dots I[x_n\in(\theta-1/2,\theta+1/2)]$

$= I[\theta-1/2\le\min X_i\le\max X_i\le\theta+1/2] \Rightarrow (\min X_i, \max X_i)$ je suf

ii) $E(\max X_i - \min X_i)$ = nebude závisieť na θ kvôli invariancii voči posunutiu

$= (\theta+1/2-\frac{1}{n+1}) - (\theta-1/2+\frac{1}{n+1}) = 1-\frac{2}{n+1}$

33) 25) $X\sim$ Pareto(α,β)

$f(\mathbf{x}) = \frac{\beta^n\alpha^{n\beta}}{(\prod x_i)^{\beta+1}}\, I[\min x_i>\alpha] \Rightarrow$ suf je $(\min X_i, \prod X_i)$
alebo $(\min X_i, \sum\log X_i)$

34) 26) $X\sim N(\mu,\mu^2)$

i) $\frac{f(x)}{f(y)} = \exp\left\{-\frac{1}{2\mu^2}\left[\sum x_i^2 - 2\mu\sum x_i + \cancel{n\mu^2} - \sum y_i^2 + 2\mu\sum y_i - \cancel{n\mu^2}\right]\right\} =$

$= \exp\left\{-\frac{1}{2\mu^2}\left[\sum(x_i^2-y_i^2) - 2\mu\sum(x_i-y_i)\right]\right\} \quad (\sum X_i,\sum X_i^2)'$ je min. suf

ii) $E_\mu\left[\frac{(\sum X_i)^2}{n+1} - \sum X_i^2\right] = 0 \quad \forall\mu$

$\sum X_i\sim N(n\mu, n\mu^2) \Rightarrow E(\sum X_i)^2 = n\mu^2+\mu^2n^2 = \mu^2(n^2+n) = n\mu^2(n+1)$

$E\sum X_i^2 = n\,EX_1^2 = n\mu^2$

35) 27) $X\sim M(n; p_1\dots p_7)$

i) $\frac{P(\mathbf{X}=\mathbf{x})}{P(\mathbf{X}=\mathbf{y})} = \frac{\binom{n}{x_1\dots x_7}\prod p_j^{x_j}}{\binom{n}{y_1\dots y_7}\prod p_j^{y_j}} = c\cdot\prod p_j^{x_j-y_j} \Rightarrow \mathbf{X}$ je min postačujúca

n pevné $\Rightarrow X_7 = n-\sum_{i=1}^6 X_i$ a.s.

ii) $= c\cdot p_1^{\sum_1^5 x_j - \sum_1^5 y_j}\, p_6^{\sum_6^7 x_j - \sum_6^7 y_j} \Rightarrow (\sum_{i=1}^5 X_i,\ X_6+X_7)'$ je min. postačujúca

iii) $= c\, p^{\sum x_i - \sum y_i} \Rightarrow \sum_{i=1}^7 X_i$ je min. postačujúca

ale v tomto modele $p=1/4$

36) 28) $X\sim N(0,\sigma^2)$

i) $\sum X_i\sim N(0,n\sigma^2) \quad E_{\sigma^2}\sum X_i=0 \quad \forall\sigma^2 \Rightarrow$ nie je úplná

ii) $E_{\sigma^2}(\underbrace{\sin X_1 - 1}_{T}) = \int_R \underbrace{\sin x}_{\text{lichá fce}}\,\frac{e^{-x^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}}\,dx - 1 = -1$

$E_{\sigma^2}(T+1)=0 \quad \forall\sigma^2 \Rightarrow$ nie je úplná

NMSA 332

37, 29, $X \sim B(a,b)$ $\quad \frac{f(x)}{f(y)} = \frac{(\prod x_i)^{a-1} \prod (1-x_i)^{b-1}}{(\prod y_i)^{a-1} \prod (1-y_i)^{b-1}} = \left(\prod \frac{x_i}{y_i}\right)^{a-1} \left(\prod \frac{1-x_i}{1-y_i}\right)^{b-1}$

$\Rightarrow \left(\prod X_i, \prod(1-X_i)\right)^T$ alebo $\left(\sum \log X_i, \sum \log(1-X_i)\right)^T$ sú obe min. suf.

38, 30, $X \sim N(\mu_1, \sigma^2)$ $\quad Y \sim N(\mu_2, \sigma^2)$

i) $f(x,y) = \frac{c}{\sigma^{n+m}} \exp\left\{-\frac{1}{2\sigma^2}\left[\sum^{n} x_i^2 - 2\mu_1 \sum x_i + n\mu_1^2 - \sum^{m} y_i^2 + 2\mu_2 \sum y_i - m\mu_2^2\right]\right\}$

$\Rightarrow \left(\sum X_i, \sum X_i^2, \sum Y_i, \sum Y_i^2\right)^T$ je suf.

ii) $E_{\mu_1,\mu_2,\sigma^2} \; S^2_{n,X} - S^2_{m,Y} = 0 \quad \forall \mu_1, \mu_2, \sigma^2$

39, 31, $P(\mathbf{X} = \mathbf{x}) = \frac{e^{-n\lambda} \lambda^{\sum x_i}}{\prod x! \cdot C(k,\lambda)^n} \; I[0 \le x_1 \le k] \dots I[0 \le x_n \le k] \Rightarrow \left(\sum X_i, \max X_i\right)^T$ je suf.

Využitie suf. a úplných štatistík $\quad$ 39, 40 ukážka $\quad$ 45 ukážka

39 32, $X \sim Ge(p)$ $\quad P(X=x) = p(1-p)^x \quad x \in \mathbb{N}_0$

+1

a) $ET = E\, I[X=0] = p$

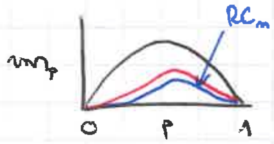
b) (pr 14) $\Rightarrow S = \sum X_i$ postačujúca $\quad E[T \mid S=s] = E[I[X=0] \mid S=s] = P(X=0 \mid S=s) =$

25

$= \frac{P(S=s \mid X_1=0)\, P(X_1=0)}{P(S=s)} = \frac{p \cdot P\left(\sum_{i=2}^{n} X_i = s\right)}{P\left(\sum_{i=1}^{n} X_i = s\right)} = \frac{p \binom{n+s-2}{s} p^{n-1}(1-p)^s}{\binom{n+s-1}{s} p^n (1-p)^s} =$

$P\left(\sum_{i=1}^{n} X_i = s\right) = \binom{n+s-1}{s} p^n (1-p)^s \quad s \in \mathbb{N}_0$

$= \frac{\frac{(n+s-2)!}{s!\,(n-2)!}}{\frac{(n+s-1)!}{s!\,(n-1)!}} = \frac{n-1}{n+s-1}$ $\qquad RC_n = \frac{p^2 q}{n}$



$m\,T = \frac{p(1-p)}{n}$

$m\,u(\bar{X})$ je menší chopitel'

$E[T \mid S] = \frac{n-1}{n+S-1} = \frac{1 - 1/n}{1 + \bar{X} - 1/n} = u(\bar{X})$

c) $P(X=x) = p(1-p)^x = \exp\{x \log(1-p)\} \cdot p \Rightarrow \sum X_i = S$ je úplná postač. pre $\theta = \log(1-p)$

$a(\theta) = 1 - e^\theta = 1 - (1-p) = p \qquad E_\theta T^2 = E_p T^2 < \infty \;\forall p \Rightarrow$ Lehmann-Scheffé $u(\bar{X})$ je najlepší nestr. odhad p

d) $E\, I[X_1 = 1] = P(X_1=1) = p(1-p) \qquad T = I[X_1=1]$ nestranný odhad.

$E[T \mid S=s] = p(1-p) \cdot \frac{P\left(\sum_{i=2}^{n} X_i = s-1\right)}{P\left(\sum X_i = s\right)} = p(1-p) \frac{\binom{n+s-3}{s-1} p^{n-1} (1-p)^{s-1}}{\binom{n+s-1}{s} p^n (1-p)^s} =$

$= \frac{\frac{(n+s-3)!}{(s-1)!\,(n-2)!}}{\frac{(n+s-1)!}{s!\,(n-1)!}} = \frac{(n-1) \cdot s}{(n+s-1)(n+s-2)}$

$E[T \mid S] = \frac{(n-1)\,S}{(n+S-1)(n+S-2)} = u(S)$

40 33, $\mathbf{X} \sim M(1; p, 1-2p, p)$ s hodnotami $\{-1, 0, 1\}$

a) $ET = P(X=1) = p$

b) $P(\mathbf{X} = (x_1, x_2, x_3)^T) = p^{x_1} (1-2p)^{x_2} p^{x_3}$ $\quad$ kde $(x_1, x_2, x_3)^T \in \{0,1\}^3$, $\sum x_i = 1$

$= p^{x_1+x_3} (1-2p)^{x_2}$ $\qquad$ počet -1 .. počet +1

$= p^{I[X \neq 0]} (1-2p)^{1 - I[X \neq 0]}$

c) $E\left[I[X=1] \,\middle|\, \underbrace{\sum I[X_i \neq 0]}_{S \sim Bi(n, 2p)} = s\right] = p \cdot \frac{P(S=s \mid X_1=1)}{P(S=s)} = p \frac{\binom{n-1}{s-1} (2p)^{s-1} (1-2p)^{n-s}}{\binom{n}{s} (2p)^s (1-2p)^{n-s}}$

$$= \frac{\frac{(m-1)!}{(s-1)!(m-s)!\cdot 2}}{\frac{m!}{s!(m-s)!}} = \frac{s}{2m} \qquad E[T|S] = u(S) = \frac{S}{2m} = \frac{1}{2}\frac{1}{m}\sum I[X_i=1]$$

d) $P(X=x) = \binom{m}{x_1 \dots x_3} \prod p_i^{x_i} = \exp\{\sum x_i \log p_i\} = \exp\{\log p \cdot (x_1+x_3) + \log(1-2p)\cdot x_2\}$

$= \exp\{(x_1+x_3)\log p + (1-(x_1+x_3))\log(1-2p)\} = \exp\{(x_1+x_3)\log\frac{p}{1-2p}\}\exp\{\log(1-2p)\}$

$\Rightarrow$ exponenciální rodina $\Rightarrow$ úplná suf. stat $\Rightarrow$ nejlepší nestranný odhad

41 34) $X \sim alt(p)$

a) $T = I[X_1 = 1]$ $\quad P(X=x) = p^x(1-p)^{1-x} = \exp\{x\log p + (1-x)\log(1-p)\} =$

$= \exp\{x \log\frac{p}{1-p}\}(1-p) \Rightarrow S = \sum X_i$ je úplná postačující $\quad$ z Pr S $\quad E[X_1|\sum X_i] \overset{X_i \text{ iid}}{=} \frac{\sum X_i}{m}$

$E[T|S=s] = E[\frac{S}{m}|S=s] = s/m \quad u(S) = \frac{S}{m} = \bar{X}$ je nejlepší nestranný odhad p

b) $T' = \bar{X}(1-\bar{X}) \quad \bar{X} \sim Bi(m,p)/m \quad Y \sim Bi(m,p) \quad \operatorname{var} Y = mp(1-p) = EY^2 - (mp)^2$

$EY^2 = mp(1-p) + m^2p^2$

$ET' = p - \frac{1}{m^2}EY^2 = p - \frac{p(1-p)}{m} - p^2 = p(1-p)(1-\frac{1}{m})$

$\Rightarrow T := \frac{m}{m-1}\bar{X}(1-\bar{X})$ je $ET = p(1-p)$

$E[T|S] = T$ (fce $\bar{X}$) je nejlepší nestr. odhad $p(1-p)$.

42 35) $X \sim Po(\lambda)$

a) $T_1 = \bar{X} \quad ET_1 = \lambda \quad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{1}{x!}e^{-\lambda}e^{x\log\lambda} \Rightarrow \sum X_i$ je úplná postačující pro $\log\lambda$

$E[T_1|\sum X_i] = T_1$ je nejlepší nestr. odhad $\lambda = \exp\{\log\lambda\}$

b) $T_2 := \frac{\sum I[X_i = 0]}{m} \quad ET_2 = e^{-\lambda} \quad \sum_{i=1}^m X_i \sim Po(m\lambda)$

$E[T_2|\sum X_i = s] = E[X_1 = 0|\sum X_i = s] = e^{-\lambda}\frac{e^{-(m-1)\lambda}[(m-1)\lambda]^s/s!}{e^{-m\lambda}(m\lambda)^s/s!} = \left(\frac{m-1}{m}\right)^s$

$\Rightarrow \left(1-\frac{1}{m}\right)^{\sum X_i}$ je nejlepší nestr. odhad $e^{-\lambda}$.

c) z Pr 9 $\quad T_1$ je eficientní, T_2 není. 10

43 36) $X \sim N(\mu,\sigma^2) \quad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\{-\frac{1}{2\sigma^2}(x^2 - 2\mu x + \mu^2)\} = \frac{1}{\sqrt{2\pi}}\sqrt{\xi}\exp\{-\frac{\xi}{2}(x^2) + \tau x\}e^{-\frac{\tau^2}{2\xi}}$

$(\sum X_i, \sum X_i^2)^T$ jsou úplné postač. pro $(\xi,\tau)^T \quad \frac{1}{\sigma^2} =: \xi \quad \frac{\mu}{\sigma^2} =: \tau$

$a(\xi,\tau) = 1/\sqrt{\xi} = \sigma \qquad$ a) $E[S_m^2 | \sum X_i, \sum X_i^2] = S_m^2$

a) $E\tilde{\sigma} = E a_m \frac{S\sqrt{m-1}}{\sigma}\cdot\sigma = \sigma a_m E\chi_{m-1} = \sigma$ nestranný $\quad$ Podobně jako v Pr. 13 $\left(\begin{smallmatrix}\text{kam}\\ \mu=0\end{smallmatrix}\right)$ ~~ale jako v Pr 12 není efektivní~~

$\frac{S^2(m-1)}{\sigma^2} \sim \chi^2_{m-1} \quad \frac{S\sqrt{m-1}}{\sigma} \sim \chi_{m-1}$

$E[\tilde{\sigma}|\sum X_i, \sum X_i^2] = \tilde{\sigma}$ nejlepší nestranný

b) medián není je z LS vzhledem k jednoznačnosti

b) $a(\xi,\tau) = \tau/\xi + u_\alpha \cdot 1/\sqrt{\xi} = \mu + u_\alpha\sigma \qquad E(\bar{X} + u_\alpha\tilde{\sigma}_m) = \mu + u_\alpha\sigma$ nestranný a je fce $(\sum X_i, \sum X_i^2)^T \Rightarrow$ nejlepší nestranný

c) $\mu^2 = -\sigma^2 + EX^2 \qquad T := \frac{1}{m}\sum X_i^2 - \cancel{\tilde{\sigma}}\, S^2 \qquad ET = EX^2 - \sigma^2 = \mu^2$ a je fce $(\sum X_i, \sum X_i^2)^T$

$\Rightarrow$ nejlepší nestranný $\qquad$ z Pr BiV) nedosahuje RC_m

e) $\sigma^2 = EX^2 - \mu^2$ $\quad$ odhadneme $\quad \frac{1}{m}\sum X_i^2 - S_m^2$ nestranný, fce úplný postač. stat, $\Rightarrow$ nejlepší nestranný

$\mu^2 = EX^2 - \sigma^2$

NMSA 332

44 37) $X \sim N(\mu, \mu^2)$ $f(\mathbf{x}) = \left(\frac{1}{\sqrt{2\pi\mu^2}}\right)^n \exp\left\{-\frac{1}{2\mu^2}\left[\sum x_i^2 - 2\mu \sum x_i + n\mu^2\right]\right\} = (2\pi\mu^2)^{-\frac{n}{2}} e^{-\frac{n}{2}}$.

$\exp\left\{-\frac{1}{2\mu^2}\sum x_i^2 + \frac{1}{\mu}\sum x_i\right\}$ $\Rightarrow$ $(\sum X_i, \sum X_i^2)^T$ je sufficientná štatistika

$\frac{f(x)}{f(y)} = \exp\left\{-\frac{1}{2\mu^2}\left(\sum x_i^2 - \sum y_i^2\right) + \frac{1}{\mu}\left(\sum x_i - \sum y_i\right)\right\}$ je minimálna suf. štat.

i) $T_1 = \bar{X}$, $T_2 = a_n \sqrt{(n-1)S_n^2}$ sú obe fce $(\sum X_i, \sum X_i^2)'$

$E T_1 = \mu$ $E T_2 = a_n E \sqrt{\frac{(n-1)S_n^2}{\sigma^2}} \cdot \sigma = \sigma = \mu$ sú oba nestranné

ii) $\operatorname{var} T_1 = \frac{\mu^2}{n}$ $\operatorname{var} T_2 = a_n^2 \operatorname{var} \sqrt{\frac{(n-1)S_n^2}{\sigma^2}} \cdot \mu = \mu^2 a_n^2 \operatorname{var}\sqrt{\chi^2_{n-1}} =$

$= \mu^2 a_n^2 \left(n-1 - a_n^{-2}\right) = \mu^2 \left(\frac{n-1}{a_n^2} - 1\right)$

podľa př. 17 ani jeden odhad nedosahuje RC
aS_n je lepší

$(\sum X_i, \sum X_i^2)'$ nie je úplná $E_\mu\left[\frac{n}{n+1}(\bar{X})^2 - S^2\right] = 0$ $\forall \mu$

45 38) $X \sim f(x) = \lambda e^{-\lambda(x-\delta)} I[x > \delta]$ λ známe.

i) $f(\mathbf{x}) = \lambda^n e^{-\lambda \sum x_i} e^{n\lambda\delta} I[\min x_i > \delta]$ $\Rightarrow$ $\min X_i$ je sufficientná

$X_1' .. X_n' \sim Exp(\lambda)$ $\Rightarrow$ $\min X_i' \sim Exp(n\lambda)$ analogicky $\min X_i \sim (n\lambda) e^{-n\lambda(x-\delta)} I[x > \delta]$

nech $0 = E_\delta w(\min X_i) = \int_\delta^\infty (n\lambda) e^{-n\lambda(x-\delta)} w(x)\, dx = n\lambda e^{n\lambda\delta} \int_\delta^\infty w(x) e^{-n\lambda x} dx$

teda $0 = \int_\delta^\infty w(x) e^{-n\lambda x} dx$ $\forall \delta$ $/\frac{\partial}{\partial \delta}$

$0 = 0 - w(\delta) e^{-n\lambda\delta}$ $/ \cdot -e^{n\lambda\delta}$

$0 = w(\delta)$ $\forall \delta$ $\Rightarrow$ úplná postač. štatistika $\min X_i$

$E \min X_i = \delta + \frac{1}{n\lambda}$ $\Rightarrow$ $T = \min X_i - 1/n\lambda$ je najlepší nestr. odhad

ii) neregulárny systém hustôt $\operatorname{var} \min X_i = \operatorname{var} T = \frac{1}{(n\lambda)^2}$ a RC je iba rádu $\frac{c}{n}$
"prekonáva" väčšie čo by mohlo byť RC medzou

46 39) $X \sim f(x) = \lambda e^{-\lambda x} I[x > 0]$ $f(\mathbf{x}) = \lambda^n e^{-\lambda \sum x_i} I[\min x_i > 0]$

$\sum X_i$ je úplná postačujúca. $X_i \sim Exp(\lambda) \Rightarrow \sum X_i \sim \Gamma(n, 1/\lambda) \sim \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} I[x>0]$
$\overset{\shortparallel}{\Gamma(1, 1/\lambda)}$

i) $E \sum X_i = \frac{n}{\lambda}$ ale $E \frac{1}{\bar{X}} = n \int_0^\infty \frac{\lambda^n x^{n-2} e^{-\lambda x}}{\Gamma(n)} dx \underset{\lambda x = t}{=} \frac{n\lambda^{n-1}}{\Gamma(n)\lambda^{n-2}} \int_0^\infty t^{n-2} e^{-t} dt =$

$= \frac{n\lambda\Gamma(n-1)}{\Gamma(n)} = \frac{n}{n-1}\lambda \Rightarrow \frac{n-1}{\cancel{n}} \cdot \frac{\cancel{n}}{\sum X_i}$ je nestranný odhad λ

$\Rightarrow T = \frac{n-1}{\sum X_i}$ je najlepší nestranný odhad λ.

ii) n Pn 15: RC: $\frac{\lambda^2}{n}$ $\operatorname{var} \frac{n-1}{\sum X_i} = (n-1)^2 \cdot \operatorname{var} \underbrace{\frac{1}{\Gamma(n, 1/\lambda)}}_{\text{inverse gamma distribution}} = \frac{\lambda^2}{n-2}$

nedosahuje RC medzu.

iii) $E\left(\frac{1}{\bar{X}}\right)^k = (n\lambda)^k \frac{\Gamma(n-k)}{\Gamma(n)}$

(47) 40) $X \sim R(0,\theta)$ $\qquad f(\mathbf{x}) = \theta^{-n} I[0 < \min x_i]\, I[\max x_i < \theta]$

$\Rightarrow$ $\max X_i$ je suf. štat. podľa $\underline{\text{př } \cancel{23}}$ 31 je úplná

$$E \max X_i \overset{\text{př 13, př 30}}{=} \int_0^\theta \frac{x\, n\, x^{n-1}}{\theta^n}\,dx = \frac{n}{n+1}\,\theta \quad \Rightarrow \quad \frac{n+1}{n} \max X_i \text{ je najlepší nestranný odhad.}$$

RC medzná neexistuje – neregulárny systém

i) $E\, 2\bar{X} = \theta \qquad \operatorname{var} 2\bar{X} = 4\,\theta^2/12n = \theta^2/3n$

$$\operatorname{var}\left(\frac{n+1}{n}\right)\max X_i = \left(\frac{n+1}{n}\right)^2\left[\int_0^\theta \frac{x^2 n\, x^{n-1}}{\theta^n}\,dx\right] - \theta^2 = \left(\frac{n+1}{n}\right)^2 \frac{\theta^2 n}{n+2} - \theta^2$$

$$= \theta^2\left(\frac{(n+1)^2}{n(n+2)} - 1\right) = \theta^2/(n(n+2)) \quad \text{a} \quad \frac{1}{3n} \geq \frac{1}{n(n+2)} \text{ pre } n \geq 1$$

pre $n=1$ majú rovnaké rozptyly (rovnaké odhady)

ii) $\frac{n+1}{n} \max X_i$

iii) RC medzná neexistuje

(48) 41) $\mathbf{X} \sim M(1; p_1 \dots p_k) \qquad P(\mathbf{X} = \mathbf{x}) = \prod p_i^{x_i} = \prod e^{x_i \log p_i} = e^{\sum x_i \log p_i}$

i) $\left(\sum_{i=1}^n X_{i1}, \dots, \sum_{i=1}^n X_{ik}\right)'$ je úplná suficientná pre $\mathbf{p}$

ii) $$E \sum_{i=1}^n X_{i1} \sum_{i=1}^n X_{i2} \overset{*}{=} E\, Y_1 Y_2 = \operatorname{cov}(Y_1, Y_2) + EY_1\, EY_2 = -n p_1 p_2 + n^2 p_1 p_2 = n p_1 p_2 (n-1)$$

$* \;\; \mathbf{Y} = (Y_1 \dots Y_k) \sim M(n, p_1 \dots p_k)$

$\operatorname{cov}(Y_1, Y_2) = -n p_1 p_2 \qquad EY_i = n p_i$

$\Rightarrow \frac{1}{n(n-1)} \sum_{i=1}^n X_{i1} \cdot \sum_{i=1}^n X_{i2}$ je najlepší nestranný odhad $p_1 p_2$.

42 vzorný

NMSA 332

42) $X \sim N(\mu,\sigma^2)$ z Pr 36 je $(\sum X_i, \sum X_i^2)$ úplná postač. statistika

i) $f(x) = c\sqrt{\sigma^{-2}} \exp\left\{-\frac{1}{2\sigma^2}(x^2+\mu^2-2\mu x)\right\}$

$\ell(x) = c - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$

$\nabla\ell = \left(\frac{(x-\mu)}{\sigma^2},\ -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(x-\mu)^2\right)'$

$$H = \begin{pmatrix} -1/\sigma^2 & -\frac{x-\mu}{\sigma^4} \\ -\frac{x-\mu}{\sigma^4} & \frac{1}{2\sigma^4} - \frac{(x-\mu)^2}{\sigma^6} \end{pmatrix} \qquad J = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & \frac{1}{2\sigma^4} \end{pmatrix}$$

ii) z pr. 36 a Lehmann-Scheffé: najlepší nestr. odhad μ je fce $(\sum X, \sum X^2)'$
ale medián nie je taká funkcie $\Rightarrow$ nie je najlepší nestranný
je to možné overiť aj numericky na rozdelení výb. mediánu (veta 2.14, mat. stat I)

iii) $T = \bar{X}$ $\quad ET = \mu$ $\quad \mathrm{var}\,T = \sigma^2/m$

$g(\mu,\sigma^2) = \mu \qquad \nabla g = (1,0) \qquad \nabla g\, J_m^{-1} \nabla g' = \sigma^2/m \Rightarrow$ dosahuje RC medzu

keďže ide o fci úplnej postač. statistiky, nestranný $\Rightarrow$ najlepší nestranný odhad μ

iv) $E S_m^2 = \sigma^2 \qquad \mathrm{var}\, S_m^2 = \mathrm{var}\, \frac{\sigma^2 \chi^2_{m-1}}{m-1} = \frac{\sigma^4}{(m-1)^2}\mathrm{var}\,\chi^2_{m-1} = \frac{2\sigma^4}{m-1} > \frac{2\sigma^4}{m}$ RC

$\frac{(m-1)S_m^2}{\sigma^2} \sim \chi^2_{m-1}$

nenadobúda, ale je najlepší nestranný

$E X_{k} = \frac{\sqrt{2}\,\Gamma\left(\frac{k+1}{2}\right)}{\Gamma(k/2)}$

v) z pr 36: $T = \bar{X} + u_\alpha \tilde{\sigma}$ pre $\tilde{\sigma} = a_m\sqrt{(m-1)S_m^2}$ je najlepší nestranný odhad

$a_m = \frac{\Gamma\left(\frac{m-1}{2}\right)}{\sqrt{2}\,\Gamma\left(\frac{m}{2}\right)} = \frac{1}{E X_{m-1}}$

viď pr. 12 pre odvodenie $\mu = 0$

$g(\mu,\sigma^2) = \mu + u\sigma = \mu + u\sqrt{\sigma^2}$

$\nabla g = \left(1, \frac{u}{2\sqrt{\sigma^2}}\right) \qquad \nabla g\, J_m^{-1}\nabla g' = \left(1, \frac{u}{2\sqrt{\sigma^2}}\right)\begin{pmatrix} \frac{\sigma^2}{m} & 0 \\ 0 & \frac{2\sigma^4}{m}\end{pmatrix}\begin{pmatrix} 1 \\ \frac{u}{2\sqrt{\sigma^2}}\end{pmatrix} =$

$= \left(\frac{\sigma^2}{m}\ \ \frac{2u\sigma^4}{2m\sigma}\right)\begin{pmatrix} 1 \\ u/2\sigma \end{pmatrix} = \frac{\sigma^2}{m} + \frac{u^2\sigma^4}{2\sigma^2} = \sigma^2\left(\frac{1}{m} + \frac{u^2}{2m}\right)$ $\underline{\underline{RC}}$

$\mathrm{var}\,\tilde{\sigma} = b_m^2\left(E S_m^2 - (E S_m)^2\right) = b_m^2\sigma^2 - \sigma^2$

$b_m = \sqrt{m-1}\,a_m$

$\mathrm{var}\,T = \frac{\sigma^2}{m} + u^2\,\mathrm{var}\,\tilde{\sigma} = \frac{\sigma^2}{m} + u^2\,\mathrm{var}\left(a_m\sqrt{\frac{(m-1)S^2}{\sigma^2}}\cdot\sigma\right) = \frac{\sigma^2}{m} + u^2 a_m^2\sigma^2\,\mathrm{var}\,\chi_{m-1}$

$= \frac{\sigma^2}{m} + u^2 a_m^2\sigma^2\left(m-1-a_m^{-2}\right)$ nedosahuje R-C medzu $\quad \cancel{\mathrm{var}\,\chi_{m-1} = m-1-(EX_{m-1})^2}$

eficiencia $\frac{RC_m}{\mathrm{var}\,T_m} \xrightarrow{m\to\infty} \cancel{0.336}\ 1$

43) $X \sim LN(\mu,\sigma^2) \qquad f(x) = (\sigma x\sqrt{2\pi})^{-1}\exp\left\{-\frac{(\log x-\mu)^2}{2\sigma^2}\right\} I[x>0] \qquad (e^{N(0,1)}$, obecne $e^{N(\mu,\sigma^2)})$

i) $\ell = c - \frac{1}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(\log x-\mu)^2$

$\nabla\ell = \left(\frac{1\cdot 2}{2\sigma^2}(\log x-\mu),\ -\frac{1}{2\sigma^2} + \frac{1}{2\sigma^4}(\log x-\mu)^2\right)$

$$H = \begin{pmatrix} -1/\sigma^2 & -\frac{(\log x-\mu)}{\sigma^4} \\ -\frac{(\log x-\mu)}{\sigma^4} & \frac{1}{2\sigma^4} - \frac{1}{\sigma^6}(\log x-\mu)^2 \end{pmatrix} \qquad J = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4 \end{pmatrix} = J(\mu,\sigma^2)$$

$$E(\log X-\mu)^k = \int_0^\infty \frac{(\log x-\mu)^k}{\sigma x\sqrt{2\pi}}\exp\left\{-\frac{1}{2\sigma^2}(\log x-\mu)^2\right\}dx = \int_{\mathbb{R}} \frac{s^k}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{s^2}{2\sigma^2}\right\} = \begin{cases} 0 & k=1 \\ \sigma^2 & k=2 \end{cases}$$

$\log x - \mu = s$

ii) $g(\mu,\sigma^2) = \exp\{\mu + \sigma^2/2\}$ $\qquad \nabla g = \left(e^{\mu+\sigma^2/2},\ e^{\mu+\sigma^2/2}\cdot\frac{1}{2}\right)$

$= EX$

$$\nabla g\, J_m^{-1} \nabla g' = e^{(\mu+\sigma^2/2)\cdot 2}(1, 1/2)\begin{pmatrix}\sigma^2/m & 0\\ 0 & 2\sigma^4/m\end{pmatrix}\begin{pmatrix}1\\ 1/2\end{pmatrix} = e^{2\mu+\sigma^2}\cdot\left(\frac{\sigma^2}{m} + \frac{\sigma^4}{2m}\right)$$

iii) $E\bar{X} = EX = e^{\mu+\sigma^2/2}$

$$\operatorname{var}\bar{X} = \underbrace{(e^{\sigma^2}-1)\cdot e^{2\mu+\sigma^2}}_{\operatorname{var} X}/m = \frac{e^{2\mu+\sigma^2}}{m}\sum_{k=1}^{\infty}\frac{(\sigma^2)^k}{k!} > \frac{e^{2\mu+\sigma^2}}{m}\left(\sum_{k=1}^{2}\frac{(\sigma^2)^k}{k!}\right)$$

nedosahuje R-C medzu

$$f(x) = c\,\sigma^{-1}x^{-1}\exp\left\{-\frac{1}{2\sigma^2}\left[(\log x)^2 - 2\mu\log x + \mu^2\right]\right\}$$

$(\sum \log X_i,\ \sum(\log X_i)^2)'$ je úplná postačujúca $\Rightarrow$ $\bar{X}$ nie je najlepší nestr. odhad

44) i) neregulárna hustota

ii) $f(x) = \lambda e^{-\lambda(x-\theta)}\, I[x>\theta]$

$\ell = \log\lambda - \lambda(x-\theta)$, $\ell' = 1/\lambda - (x-\theta)$ $\qquad \ell'' = -1/\lambda^2$

$J_m = m/\lambda^2$

45) $X \sim alt(p_1)\ m_1 \qquad Y \sim alt(p_2)\ m_2$

i) $\ell(p_1,p_2) = \log \prod p_1^{x_i}(1-p_1)^{1-x_i}\, p_2^{y_i}(1-p_2)^{1-y_i} = \sum\left(x_i\log p_1 + (1-x_i)\log(1-p_1)\right) + y_i\log p_2 + (1-y_i)\log(1-p_2)$

$$\nabla\ell = \left(\frac{\sum x_i}{p_1} - \frac{\sum(1-x_i)}{1-p_1},\ \frac{\sum y_i}{p_2} - \frac{\sum(1-y_i)}{1-p_2}\right)$$

$$H = \begin{pmatrix}-\frac{\sum x_i}{p_1^2} - \frac{\sum(1-x_i)}{(1-p_1)^2} & 0\\ 0 & -\frac{\sum y_i}{p_2^2} - \frac{\sum(1-y_i)}{(1-p_2)^2}\end{pmatrix} \qquad J_m = \begin{pmatrix}\frac{m_1}{p_1} + \frac{m_1}{1-p_1} & 0\\ 0 & \frac{m_2}{p_2} + \frac{m_2}{1-p_2}\end{pmatrix}$$

ii) $T = \bar{X}_{m_1} - \bar{Y}_{m_2} \qquad ET = p_1 - p_2 \qquad \operatorname{var} T = \frac{p_1(1-p_1)}{m_1} + \frac{p_2(1-p_2)}{m_2}$

$$g(p_1,p_2) = p_1 - p_2 \qquad \nabla g = (1\ \ -1) \qquad J_m^{-1} = \begin{pmatrix}\frac{p_1(1-p_1)}{m_1} & 0\\ 0 & \frac{p_2(1-p_2)}{m_2}\end{pmatrix}$$

$\nabla g\, J_m^{-1}\nabla g' = \operatorname{var} T \Rightarrow$ dosahuje RC medzu

iii) $\nabla g = \left(\frac{1-p_1}{p_1}\cdot\frac{1}{(1-p_1)^2},\ -\frac{1-p_2}{p_2}\frac{1}{(1-p_2)^2}\right) = \left(\frac{1}{p_1(1-p_1)},\ -\frac{1}{p_2(1-p_2)}\right)$

$$\nabla g\, J_m^{-1}\nabla g' = \frac{1}{m_1p_1(1-p_1)} + \frac{1}{m_2p_2(1-p_2)} \longrightarrow \infty \text{ pre } p_1\to 0,\ p_1\to 1,\ p_2\to 0,\ p_2\to 1$$

iv) $E\hat{\Theta} =$ "$\infty - \infty$" nie je definované

NMSA 332

46) $X \sim N(\mu_1,\sigma^2)$ n_1 $\qquad Y \sim N(\mu_2,\sigma^2)$ n_2

i) $$f(x,y) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{n_1} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu_1)^2\right\}\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{n_2}\exp\left\{-\frac{1}{2\sigma^2}\sum(y_i-\mu_2)^2\right\}$$

$$\ell(\mu_1,\mu_2,\sigma^2) = c - \frac{1}{2}(n_1+n_2)\log\sigma^2 - \frac{1}{2\sigma^2}\left[\sum(x_i-\mu_1)^2 + \sum(y_i-\mu_2)^2\right]$$

$$\nabla\ell = \left(\frac{\sum(x_i-\mu_1)}{\sigma^2}, \frac{\sum(y_i-\mu_2)}{\sigma^2}, -\frac{n_1+n_2}{2\sigma^2} + \frac{1}{2\sigma^4}\left[\sum(x_i-\mu_1)^2 + \sum(y_i-\mu_2)^2\right]\right)$$

$$H = \begin{pmatrix} -n_1/\sigma^2 & 0 & -\frac{\sum(x_i-\mu_1)}{\sigma^4} \\ 0 & -n_2/\sigma^2 & -\frac{\sum(y_i-\mu_2)}{\sigma^4} \\ -\frac{\sum(x_i-\mu_1)}{\sigma^4} & -\frac{\sum(y_i-\mu_2)}{\sigma^4} & \frac{n_1+n_2}{2\sigma^4} - \frac{1}{\sigma^6}\left[\sum(x_i-\mu_1)^2+\sum(y_i-\mu_2)^2\right] \end{pmatrix}$$

$$J_n = \begin{pmatrix} n_1/\sigma^2 & 0 & 0 \\ 0 & n_2/\sigma^2 & 0 \\ 0 & 0 & -\frac{n_1+n_2}{2\sigma^4} + \frac{(n_1+n_2)\sigma^2}{\sigma^6} \end{pmatrix} \Rightarrow \text{RC: } \left(\frac{n_1+n_2}{2\sigma^4}\right)^{-1}$$

ii) $$S^2 = \frac{1}{n_1+n_2-2}\left[(n_1-1)S_X^2 + (n_2-1)S_Y^2\right]$$

$$\operatorname{var} S^2 = \frac{1}{(n_1+n_2-2)^2}\left[\operatorname{var}\left(\chi^2_{n_1-1}\cdot\sigma^2\right) + \operatorname{var}\left(\chi^2_{n_2-1}\,\sigma^2\right)\right] = \frac{\sigma^4\cdot 2\,(n_1+n_2-2)}{(n_1+n_2-2)^2} = \frac{2\sigma^4}{n_1+n_2-2}$$

nedosahuje R-C meze (viz př 42)

47) i) $$f(y_i) = c\cdot\exp\left\{-\frac{\sum(y_i-\beta_0-\beta_1x_i)^2}{2}\right\}$$

$$\ell(\beta_0,\beta_1) = c - \frac{1}{2}\sum(y_i-\beta_0-\beta_1x_i)^2$$

$$\nabla\ell = \left(\sum(y_i-\beta_0-\beta_1x_i), \sum x_i(y_i-\beta_0-\beta_1x_i)\right)$$

$$H = \begin{pmatrix} -n & -\sum x_i \\ -\sum x_i & -\sum x_i^2 \end{pmatrix} \qquad J_n = \begin{pmatrix} n & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}$$

$$\frac{\sum(x_i-\bar x)}{\sum(x_i-\bar x)^2} = 0$$

$$\sum(x_i-\bar x)x_i = \sum x_i^2 - \sum x_i\cdot\bar x = n\left[\frac{1}{n}\sum x_i^2 - (\bar x)^2\right] = n\left[\frac{1}{n}\sum(x_i-\bar x)^2\right]$$

$=$

ii) $$E\hat\beta_1 = \frac{1}{\sum(x_i-\bar x)^2}\sum(x_i-\bar x)EY_i = \frac{\sum(x_i-\bar x)(\beta_0+\beta_1x_i)}{\sum(x_i-\bar x)^2} \neq \beta_1$$

ale pro $$\hat\beta_1 = \frac{\sum(x_i-\bar x)(Y_i-\bar Y)}{\sum(x_i-\bar x)^2} \Rightarrow E\hat\beta_1 = \frac{\sum(x_i-\bar x)E(Y_i-\bar Y)}{\sum(x_i-\bar x)^2} =$$

$$= \frac{\sum(x_i-\bar x)\left(\cancel{\beta_0}+\beta_1x_i - \frac{1}{n}\sum(\cancel{\beta_0}+\beta_1x_i)\right)}{\sum(x_i-\bar x)^2} = \beta_1$$

Y_i nezávislé

$$= \operatorname{var} \frac{\sum (x_i-\bar{x})\cdot Y_i}{\sum (x_i-\bar{x})^2} = \frac{1}{\left(\sum (x_i-\bar{x})^2\right)^2} \sum (x_i-\bar{x})^2 \overbrace{\operatorname{var} Y_i}^{1} = \frac{1}{\sum (x_i-\bar{x})^2}$$

$$\operatorname{var} \hat{\beta}_1 = [\text{věta 7.1 Hušková, Dupač}] = \frac{1}{\sum (x_i-\bar{x})^2}$$

$g(\beta_0, \beta_1) = (0 \; 1) \Rightarrow$ RC je prvek $(2,2)$ matice J_n^{-1}, tj

$$\frac{1}{n\sum x_i^2 - (\sum x_i)^2} \cdot n = \frac{1}{\sum (x_i-\bar{x})^2} \qquad \text{dosahuje RC-meze}$$

$$\sum (x_i-\bar{x})^2 = \sum x_i^2 - 2\bar{x} n \bar{x} + n(\bar{x})^2 = \sum x_i^2 - \frac{1}{n}\left(\sum x_i\right)^2$$

48) $\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2\left(\begin{pmatrix} \theta \\ \theta \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$

$$f(x,y) \overset{\text{def}}{=} \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \theta \\ \theta \end{pmatrix}\right)' \Sigma^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} \theta \\ \theta \end{pmatrix}\right)\right\}$$

$$l = c - \frac{1}{2}\log(1-\rho^2) - \frac{1}{2(1-\rho^2)}\left[(x-\theta)^2 + (y-\theta)^2 - 2\rho(x-\theta)(y-\theta)\right]$$

$$Dl = \left(\frac{1}{2(1-\rho^2)}\left[2(x-\theta) + 2(y-\theta) + 2\rho(x+y-2\theta)\right], \text{ funkce spočítaná mathematica skript} \right)$$

$EX^2 = EY^2 = 1+\theta^2 \qquad EXY = \rho+\theta^2$

$EX = EY = \theta$

$$J = \begin{pmatrix} \frac{2n}{1+\rho} & 0 \\ 0 & \frac{(1+\rho^2)n}{(\rho^2-1)^2} \end{pmatrix} \qquad J_n^{-1} = \begin{pmatrix} \frac{1+\rho}{2n} & 0 \\ 0 & \frac{(\rho^2-1)^2}{(1+\rho^2)n} \end{pmatrix}$$

RC pro θ je $(1+\rho)/2n$

i) $\operatorname{var} \bar{X} = 1/n < (1+\rho)/2n$ nedosahuje RC

ii) $\operatorname{var} \frac{1}{2}(\bar{X}+\bar{Y}) = \frac{1}{4}\left(\operatorname{var}\bar{X} + \operatorname{var}\bar{Y} + 2\operatorname{cov}(\bar{X},\bar{Y})\right) = \frac{1}{4}\left(\frac{1}{n} + \frac{1}{n} + \frac{2\rho}{n}\right)$

$= (1+\rho)/2n$ dosahuje RC

NMSA332

Metoda maximální věrohodnosti

49) $X \sim alt(p)$

i) $L(p) = \prod p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$

$\ell(p) = \sum x_i \log p + (n-\sum x_i)\log(1-p)$

$\ell'(p) = \sum x_i/p - (n-\sum x_i)/(1-p) \overset{!}{=} 0$

$(1-p)\sum x_i - (n-\sum x_i)p = 0$

$p(-\sum x_i - n + \sum x_i) = -\sum x_i$

$\hat p = \bar X$

$\ell''(p) = -\frac{\sum x_i}{p^2} - \frac{(n-\sum x_i)}{(1-p)^2}$

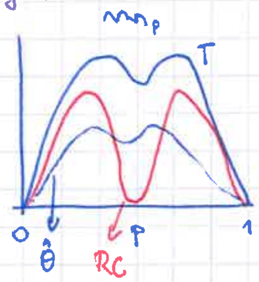
$J_n(p) = \frac{n}{p} + \frac{n}{1-p} = \frac{n}{p(1-p)}$

$\sqrt{n}(\bar X - p) \xrightarrow{D} N(0, \frac{p(1-p)}{\cancel{n}})$

ii) invariance: $\theta = p(1-p)$ $\hat\theta = \hat p(1-\hat p) = \bar X(1-\bar X)$ $g(t) = t(1-t)$ $g'(t) = (1-t) - t = 1-2t$

$\sqrt{n}(\bar X(1-\bar X) - \theta) \xrightarrow{D} N(0, (1-2p)^2 p(1-p))$ $p \neq 1/2$

iii) $\bar X$ je NNO, je eficientní; NNO θ je $T = \frac{n}{n-1}\bar x(1-\bar x)$, není je eficientní; $\hat\theta$ není je nestranný



50) $X \sim Po(\lambda)$ $L(\lambda) = e^{-n\lambda}\lambda^{\sum x_i}/\prod x_i!$ $\ell(\lambda) = -n\lambda + \sum x_i \log\lambda + c$

i) $\ell'(\lambda) = -n + \sum x_i/\lambda = 0$ $\hat\lambda = \bar X$

$\ell''(\lambda) = -\sum x_i/\lambda^2$ $J_n(\lambda) = n/\lambda$ $\sqrt{n}(\bar X - \lambda) \xrightarrow{D} N(0, \lambda)$

ii) $\theta = e^{-\lambda}$ $g(t) = e^{-t}$ $g'(t) = -e^{-t}$ $\sqrt{n}(e^{-\bar X} - e^{-\lambda}) \xrightarrow{D} N(0, e^{-2\lambda}\cdot\lambda)$

iii) $\bar X$ je NNO, je eficientní; NNO θ je $T = (1-\frac{1}{n})^{\sum x_i}$ není je eficientní

51) $X \sim Exp(\lambda)$ $L(\lambda) = \lambda^n e^{-\lambda\sum x_i}$ $\ell(\lambda) = n\log\lambda - \lambda\sum x_i$ $\ell'(\lambda) = n/\lambda - \sum x_i = 0$

i) $\hat\lambda = 1/\bar X$ $\ell''(\lambda) = -n/\lambda^2$ $J_n(\lambda) = n/\lambda^2$

ii) $\sqrt{n}(1/\bar X - \lambda) \xrightarrow{D} N(0, \lambda^2)$

iii) $\frac{n-1}{\sum x_i}$ je NNO λ ale není je eficientní

52) $X \sim Ge(p)$ $L(p) = p^n(1-p)^{\sum x_i}$ $\ell(p) = n\log p + \sum x_i \log(1-p)$

i) $\ell'(p) = n/p - \sum x_i/(1-p) = 0$

$n - np - p\sum x_i = 0$

$p(n + \sum x_i) = n$

$\hat p = \frac{n}{n+\sum x_i} = \frac{1}{1+\bar X}$

$\ell''(p) = -n/p^2 - \sum x_i/(1-p)^2$

$J_n(p) = n/p^2 - \frac{\cancel{(1-p)}}{p}\frac{n}{(1-p)^2} = n(\frac{1}{p^2} - \frac{1}{p(1-p)})$

$\sqrt{n}(\frac{1}{1+\bar X} - p) \xrightarrow{D} N(0, \frac{1-2p}{p^2(1-p)})$

ii) $\theta = p(1-p)$ $\hat\theta = \frac{1}{1+\bar X}(1 - \frac{1}{1+\bar X}) = \frac{1\cdot\bar X}{(1+\bar X)^2}$

$g(t) = t(1-t)$ $g'(t) = 1-2t$ $\sqrt{n}(\hat\theta - \theta) \xrightarrow{D} N(0, \frac{(1-2p)^2\cdot(1-2p)}{p(1-p)})$

53) $X \sim R(\theta - 1/2, \theta + 1/2)$ $L(\theta) = \prod I[\theta - 1/2 < x_i < \theta + 1/2] = I[\theta - 1/2 < \min x_i]\, I[\max x_i < \theta + 1/2]$

$= \begin{cases} 1 & \text{až } \theta < \min x_i + 1/2 \text{ a } \theta > \max x_i - 1/2 \\ 0 & \text{jinak} \end{cases}$ $\Rightarrow$ $\hat\theta$ je kterákoli hodnota $[\max x_i - 1/2, \min x_i + \frac{1}{2}]$

i)

ii) z MSI: $\min X_i \xrightarrow{P} \theta - 1/2$, $\max X_i \xrightarrow{P} \theta + 1/2$ $\Rightarrow$ $\max X_i - 1/2 \xrightarrow{P} \theta$, $\min X_i + 1/2 \xrightarrow{P} \theta$ $\Rightarrow$ každý odhad je tedy konzistentní

54) $X \sim R\{1..M\}$ $L(M) = \prod \frac{1}{M} I[x_i \in \{1..M\}] = \frac{1}{M^n} I[1 \le \min x_i \le \max x_i \le M] =$

i) $= \begin{cases} 1/M^n & \text{až } M \ge \max x_i \\ 0 & \text{jinak} \end{cases}$ $\Rightarrow$ $\hat M = \max x_i$

ii) $P(\hat M \le k) = P(X_i \le k)^n = (\frac{k}{M})^n$ $f_{\hat M}(k) = n k^{n-1}/M^n$ $= P(\hat M \le M - [\varepsilon])$

$P(|\hat M - M| > \varepsilon) = P(\hat M < M - \varepsilon) \le P(\hat M \le M - \varepsilon) \overset{\le}{=} (\frac{M-\varepsilon}{M})^n = (1 - \frac{\varepsilon}{M})^n \xrightarrow[n\to\infty]{} 0$

až $\varepsilon \in \mathbb{N}$

55) $Y_i \sim N(\theta x_i, 1)$ $L(\theta) = c\exp\{-\frac{1}{2}\sum(y_i - \theta x_i)^2\}$ $\ell(\theta) = c - \frac{1}{2}\sum(y_i - \theta x_i)^2$

i) $\ell'(\theta) = \sum(y_i - \theta x_i)\, x_i = 0$

$\sum y_i x_i = \theta\sum x_i^2$

$\frac{\sum y_i x_i}{\sum x_i^2} = \hat\theta$

ii) $E\hat\theta = \frac{1}{\sum x_i^2}\sum x_i EY_i = \theta$ nestranný

iii) $\ell'' = -\sum x_i^2$

$RC_n = 1/\sum x_i^2$

$mn\,\hat\theta = \frac{\sum x_i^2 \cdot 1}{(\sum x_i^2)^2} = \frac{1}{\sum x_i^2} = RC$

56) $P(X=k) = \frac{1}{1-(1-p)^m} \binom{m}{k} p^k (1-p)^{m-k} \qquad k=1,2 \dots m$

i) $L(p) = \frac{1}{(1-(1-p)^m)^n} \prod_{i=1}^{m} \binom{m}{x_i} \cdot p^{\sum x_i} (1-p)^{\sum (m-x_i)}$

$\ell(p) = -n \log(1-(1-p)^m) + c + \sum x_i \log p + \sum (m-x_i) \log(1-p)$

$\ell'(p) = \frac{-n}{1-(1-p)^m} \cdot m(1-p)^{m-1} + \frac{\sum x_i}{p} - \frac{\sum (m-x_i)}{1-p} = 0$

$\ell''(p) =$ mathematica script

57) $X \sim \theta x^{\theta-1} e^{-x^\theta} I[X>0] \qquad L(\theta) = \theta^n (\prod x_i)^{\theta-1} e^{-\sum x_i^\theta} I[\min X_i > 0]$

$x^\theta = e^{\theta \log x}$

i) $\ell(\theta) = n \log \theta + (\theta-1) \log(\prod x_i) - \sum x_i^\theta$

$\ell'(\theta) = \frac{n}{\theta} + \sum \log x_i - \sum x_i^\theta \log x_i = 0$

$\ell''(\theta) = -\frac{n}{\theta^2} - \sum x_i^\theta (\log x_i)^2$

} spojitá klesajúca fce, $\ell'(0) = \infty$, $\ell'(\infty) = -\infty$ jediné riešenie

ii) $EX = \int_0^\infty \theta x^\theta e^{-x^\theta} dx \underset{\substack{x^\theta = t \\ x = t^{1/\theta}}}{=} \int_0^\infty t^{1/\theta} e^{-t} dt = \Gamma\left(\frac{1}{\theta}+1\right)$

$E X^\theta (\log X)^2 =$ mathematica $= c'$

$\sqrt{n}(\hat\theta - \theta) \to N\left(0, \left(\frac{1}{\theta^2} + c'\right)^{-1}\right)$

58) $f(x) = e^{-(x-\theta)} / (1+e^{-(x-\theta)})^2 \qquad L(\theta) = e^{-\sum x_i + n\theta} / \prod (1+e^{-(x_i-\theta)})^2$

$\ell(\theta) = -\sum x_i + n\theta - \sum \log(1+e^{-(x_i-\theta)})^2$

i) $\ell'(\theta) = n - 2\sum \frac{1 \cdot e^{-(x_i-\theta)}}{1+e^{-(x_i-\theta)}} = n - 2\sum \frac{e^{-x_i}}{e^{-\theta}+e^{-x_i}} = 0$

$\ell''(\theta) = +2\sum \frac{e^{-x_i}(-e^{-\theta})}{(e^{-\theta}+e^{-x_i})^2}$

} spojitá kles. fce, $\ell'(-\infty) = n$, $\ell(\infty) = -n$ jediné riešenie

ii) $J_n(\theta) = \frac{n}{3}$

$E \frac{2e^{-X}e^{-\theta}}{(e^{-\theta}+e^{-X})^2} = \int_{\mathbb{R}} \frac{2e^{-x}\cancel{e^{\theta}}}{(e^{-\theta}+e^{-x})^2} \frac{e^{-x}\cancel{e^{\theta}}}{(1+e^{-(x-\theta)})^2} dx = \frac{2e^{2\theta}}{e^{-2\theta}} \int_1^\infty \frac{(t-1)}{t^4} = 2\left[\frac{-1}{2t^2} + \frac{1}{3t^3}\right]_1^\infty = \frac{1}{3}$

$1+e^{-(x-\theta)} = t \qquad t-1 = e^{-x}e^{\theta}$

$(-1) \cdot e^{-(x-\theta)} dx = dt \qquad (t-1)e^{-\theta} = e^{-x}$

$-e^{\theta}e^{-x} dx = dt$

$\sqrt{n}(\hat\theta - \theta) \xrightarrow{D} N(0, 3)$

59) $X \sim N(\theta, \theta^2) \qquad L(\theta) = c \cdot \theta^{-n} \exp\left\{-\frac{1}{2\theta^2}\sum(x_i-\theta)^2\right\} \qquad \ell(\theta) = -n\log\theta + c - \frac{1}{2\theta^2}\sum(x_i-\theta)^2$

i) $\ell'(\theta) = -\frac{n}{\theta} + \frac{1 \cdot \cancel{2}}{\cancel{2}\theta^2}\sum(x_i-\theta) + \frac{1 \cdot \cancel{2}}{\cancel{2}\theta^3}\sum(x_i-\theta)^2 = 0$

$-n + \frac{\sum x_i - n\theta}{\theta} + \frac{\sum x_i^2 - 2\theta\sum x_i + n\theta^2}{\theta^2} = 0$

$-n + \frac{\sum x_i}{\theta} - \cancel{n} + \frac{\sum x_i^2}{\theta^2} - \frac{2\sum x_i}{\theta} + \cancel{n} = 0$

$-n\theta^2 + \theta\sum x_i - 2\theta\sum x_i + \sum x_i^2 = 0$

$\theta^2 n + \theta\sum x_i - \sum x_i^2 = 0$

$\frac{-\sum x_i \pm \sqrt{(\sum x_i)^2 + 4\sum x_i^2 \cdot n}}{2n} = \hat\theta$

ii) $\hat\theta = \left[-\frac{1}{2}\bar X + \frac{1}{2}\sqrt{(\bar X)^2 + 4\frac{1}{n}\sum X_i^2}\right] \xrightarrow[n\to\infty]{s.j.} -\frac{\theta}{2} + \frac{1}{2}\sqrt{\theta^2 + 4 \cdot 2\theta^2} = -\frac{\theta}{2} + \frac{1}{2}\sqrt{9\theta^2} = \theta$

$EX^2 = 2\theta^2$

iii) $\ell''(\theta) = \frac{n}{\theta^2} + \left(-\frac{2}{\theta^3}\sum(x_i-\theta) + \frac{1(-n)}{\theta^2}\right) + \left(\frac{-3}{\theta^4}\sum(x_i-\theta)^2 - \frac{2\sum(x_i-\theta)}{\theta^3}\right)$

$J_n(\theta) = -\frac{n}{\theta^2} + \frac{n}{\theta^2} + \frac{3n}{\theta^2} = \frac{3n}{\theta^2} \qquad \sqrt{n}(\hat\theta - \theta) \xrightarrow{D} N\left(0, \frac{\theta^2}{3}\right)$

NMSA332

60) $P(Y=k|X) = \dfrac{\lambda(x)^k e^{-\lambda(x)}}{k!}$ $\quad \lambda(x) = e^{\beta x}$, X nezávisí na β

i) $L(\beta) = \prod P(Y_i = y_i | X = x_i) \cdot P(X = x_i) \left(= \prod P(Y_i = y_i, X_i = x_i) \right)$

$= \prod_{i=1}^{n} \dfrac{(e^{\beta x_i})^{y_i} e^{-e^{\beta x_i}}}{y_i!} P(X = x_i)$

$\ell(\beta) = \sum \left(y_i x_i \beta - e^{\beta x_i} - \log(y_i!) + \log P(X = x_i) \right)$

$= \sum y_i x_i \cdot \beta - \sum e^{\beta x_i} + c$

$\ell'(\beta) = \sum x_i y_i - \sum e^{\beta x_i} \cdot x_i \overset{!}{=} 0$

$\ell''(\beta) = -\sum e^{\beta x_i} x_i^2$ $\quad J_n(\beta) = n\, E\, e^{\beta X} X^2$ $\quad \sqrt{n}(\hat{\beta} - \beta) \xrightarrow{D} N(0, \frac{1}{E e^{\beta X} X^2})$

61) $P(X=x) = \begin{cases} p & x \in \{-1, 1\} \\ 1-2p & x = 0 \end{cases}$ $\quad$ def: $y := \#\{X_i = 0\}$

$L(p) = (1-2p)^y p^{n-y}$ $\quad \ell(p) = y \log(1-2p) + (n-y) \log p$

$\ell'(p) = \dfrac{-2y}{1-2p} + \dfrac{n-y}{p} = 0$

$-\cancel{2py} + n - 2np - y + \cancel{2py} = 0$

$\hat{p} = (y-n)/(-2n) = (n-y)/2n = \sum I[X_i \neq 0]/2n$

$\ell''(p) = \dfrac{-2y \cdot 2}{(1-2p)^2} - \dfrac{n-y}{p^2}$ $\quad EY = E \sum I[X=0] = n P(X=0) = n(1-2p)$

$J_n(p) = \dfrac{4n\cancel{(1-2p)}}{(1-2p)^{\cancel{2}}} + \dfrac{n - n(1-2p)}{p^2} = \dfrac{4n}{1-2p} + \dfrac{2n}{p} =$

$= \dfrac{4np + 2n - 4np}{(1-2p)p} = n \cdot \dfrac{2}{p(1-2p)}$

$\sqrt{n}(\hat{p} - p) \xrightarrow{D} N(0, p(1-2p)/2)$

62) $L(\theta) = \prod \frac{1}{2} \exp(-|x_i - \theta|) = \frac{1}{2^n} \exp\{-\sum |x_i - \theta|\}$

$\ell(\theta) = c - \sum |x_i - \theta| \Rightarrow$ maximalizovať $-\sum |x_i - \theta|$ na θ

riešením je medián $\{x_i\}$. (riešenie online) Lemma 3.4 MSI

MLE: Vektorový parameter

63) $X \sim N(\mu, \sigma^2)$

i) $L(\mu, \sigma^2) = c \cdot (\sigma^n)^{-1} \exp\left\{ -\frac{1}{2\sigma^2} \sum (x_i - \mu)^2 \right\}$

$\ell(\mu, \sigma^2) = c - \frac{n}{2} \log \sigma^2 - \frac{1}{2\sigma^2} \sum (x_i - \mu)^2$

$\nabla \ell = \left(\frac{1}{\sigma^2} \sum (x_i - \mu) \; ; \; -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum (x_i - \mu)^2 \right)$

$\frac{\partial \ell}{\partial \mu} = 0 \Rightarrow \sum x_i = n\mu$, $\hat{\mu} = \overline{X}$ $\quad \frac{\partial \ell}{\partial \sigma^2} = 0 \Rightarrow n = \frac{\sum (x_i - \mu)^2}{\sigma^2}$, $\hat{\sigma}^2 = \frac{1}{n} \sum (x_i - \mu)^2 \big|_{\mu = \hat{\mu}}$

ii) $H_\ell(\mu, \sigma^2) = \begin{pmatrix} -\frac{n}{\sigma^2} & -\frac{\sum(x_i - \mu)}{\sigma^4} \\ -\frac{\sum(x_i - \mu)}{\sigma^4} & \frac{n}{2\sigma^4} - \frac{\sum(x_i - \mu)^2}{(\sigma^2)^3} \end{pmatrix}$

$J_n(\mu, \sigma^2) = \begin{pmatrix} n/\sigma^2 & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$ $\quad \sqrt{n}\left(\begin{pmatrix} \overline{X} \\ \frac{n-1}{n} S^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \right) \xrightarrow{D} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix} \right)$

iii) IS pre μ: $\left[\bar{X} \mp u_{1-\alpha/2} \frac{\sqrt{\frac{n-1}{n} S^2}}{\sqrt{n}}\right]$ porovnaj $\left[\bar{X} \mp t_{n-1}(1-\alpha/2) \frac{S}{\sqrt{n}}\right]$

iv) $\hat{\theta} = \hat{\mu} + u_\alpha \hat{\sigma}$

$g(s,t) = s + u_\alpha \sqrt{t}$ $\quad \nabla g = \left(1, \frac{u_\alpha}{2\sqrt{t}}\right)$ $\quad \nabla g|_{\hat{\mu},\hat{\sigma}} = \left(1, \frac{u_\alpha}{2\sqrt{\sigma^2}}\right)$ $\quad \nabla g \, J_n^{-1} \nabla g' = \frac{1}{n}\left(\sigma^2 + u_\alpha^2 2\sigma^4\right)$

$\sqrt{n}\left((\hat{\theta}) - \theta\right) \xrightarrow{D} N\left(0, \sigma^2\left(1 + \frac{u_\alpha^2}{2}\right)\right)$ $\quad \frac{1}{n}\begin{pmatrix} 1 & \frac{u_\alpha}{2\sigma} \end{pmatrix} \begin{pmatrix} \sigma^2 & 0 \\ 0 & 2\sigma^4 \end{pmatrix} \begin{pmatrix} 1 \\ \frac{u_\alpha}{2\sigma} \end{pmatrix}$

AC median $\quad = \frac{1}{n}\left(\sigma^2 + \frac{u_\alpha^2 \sigma^2}{2}\right)$

64) $L(\mu,\sigma^2) = \sigma^{-n}\left(\prod x_i\right)^{-1} c \cdot \exp\left\{-\frac{1}{2\sigma^2} \sum (\log x_i - \mu)^2\right\}$

i) $\ell(\mu,\sigma^2) = -\frac{n}{2}\log\sigma^2 + c - \frac{1}{2\sigma^2}\sum(\log x_i - \mu)^2$

$\nabla \ell = \left(\frac{\sum(\log x_i - \mu)}{\sigma^2}, -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(\log x_i - \mu)^2 \right)$

$\nabla \ell = 0 \Rightarrow \hat{\mu} = \frac{1}{n}\sum \log x_i \quad \hat{\sigma}^2 = \frac{1}{n}\sum(\log x_i - \hat{\mu})^2$

ii) $H_\ell = \begin{pmatrix} -n/\sigma^2 & -\frac{\sum(\log x_i - \mu)}{\sigma^4} \\ -\frac{\sum(\log x_i - \mu)}{\sigma^4} & \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum(\log x_i - \mu)^2 \end{pmatrix}$ $\quad \log X \sim N(\mu,\sigma^2)$

$J_n = \begin{pmatrix} +n/\sigma^2 & 0 \\ 0 & +\frac{n}{2\sigma^4} \end{pmatrix}$ $\quad \sqrt{n}\left(\begin{pmatrix}\hat{\mu} \\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu \\ \sigma^2\end{pmatrix}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix}\sigma^2 & 0 \\ 0 & 2\sigma^4\end{pmatrix}\right)$

iii) $\left\{ \begin{pmatrix}\mu \\ \sigma^2\end{pmatrix} : n\left(\begin{pmatrix}\hat{\mu} \\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu \\ \sigma^2\end{pmatrix}\right)^T \begin{pmatrix} 1/\hat{\sigma}^2 & 0 \\ 0 & 1/2\hat{\sigma}^4 \end{pmatrix} \left(\begin{pmatrix}\hat{\mu} \\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu \\ \sigma^2\end{pmatrix}\right) \le \chi^2_2(1-\alpha) \right\}$

tj. $n\left[\frac{(\hat{\mu}-\mu)^2}{\hat{\sigma}^2} + \frac{(\hat{\sigma}^2 - \sigma^2)^2}{2\hat{\sigma}^4}\right] \le \chi^2_2(1-\alpha)$

iv) $\left[\hat{\mu} - u_{1-\alpha}\frac{\hat{\sigma}}{\sqrt{n}}, \infty\right)$ dolný odhad

65) $X \sim \lambda e^{-\lambda(x-\delta)}$; $x > \delta$

i) $L(\lambda,\delta) = \lambda^n \exp\left\{-\lambda\sum(x_i - \delta)\right\} I[\min x_i > \delta]$

$\ell(\lambda,\delta) = n\log\lambda - \lambda\sum(x_i-\delta) + \log I[\min x_i > \delta]$

$= n\log\lambda - \lambda\sum x_i + n\lambda\delta$ $\quad$ ak $\min x_i > \delta$

pre pevné $\lambda > 0$ maximalizuje $\hat{\delta} = \min X_i$

pre daný δ $\quad \frac{\partial \ell(\lambda,\delta)}{\partial\lambda} = \frac{n}{\lambda} - \sum x_i + n\delta = 0 \Rightarrow \hat{\lambda} = \frac{1}{\bar{X} - \hat{\delta}}$

ii) $\hat{\lambda} \xrightarrow{P} \lambda$ z pr 51 52 (pre porovnanie a Slutského vete) $\quad \Big\} \begin{pmatrix}\hat{\lambda} \\ \hat{\delta}\end{pmatrix} \xrightarrow{P} \begin{pmatrix}\lambda \\ \delta\end{pmatrix}$

$\hat{\delta}$ sa správa ako $Exp(n\lambda)$ posunuté o $\delta \Rightarrow \hat{\delta} \xrightarrow{P} \delta$

iii) $n \cdot Exp(n\lambda) \sim Exp(\lambda)$, tj. $n\hat{\delta} \sim Exp(\lambda) + \delta n$

$P(n(\hat{\delta} - \delta) \le x) \longrightarrow (=) 1 - e^{-\lambda x}$

tj. $(\sqrt{n})^2(\hat{\delta} - \delta) \xrightarrow{D} Exp(\lambda)$

66) $X \sim R(a,b)$ $\quad L(a,b) = \left[\frac{1}{b-a}\right]^n I[a < \min x_i \le \max x_i < b]$

i) L je maximalizované ak $b-a$ je minimalizované tak, že $a < \min X_i < \max X_i < b$

$\begin{pmatrix}\hat{a} \\ \hat{b}\end{pmatrix} = \begin{pmatrix}\min X_i \\ \max X_i\end{pmatrix}$. $\quad$ ii) z MSI: $\left.\begin{matrix}\hat{a} \xrightarrow{P} a \\ \hat{b} \xrightarrow{P} b\end{matrix}\right\} \Rightarrow \begin{pmatrix}\hat{a} \\ \hat{b}\end{pmatrix} \xrightarrow{P} \begin{pmatrix}a \\ b\end{pmatrix}$

iii) $P(\hat{b} \le x) = \left(\frac{x-a}{b-a}\right)^n$ $\quad x \in [a,b]$

1 ak $x \ge 0$

$P(n(\hat{b} - b) \le x) = P(\hat{b} \le b + x/n) = \left(\frac{b + x/n - a}{b-a}\right)^n = \left(1 + \frac{x/(b-a)}{n}\right)^n \xrightarrow[n\to\infty]{} e^{x/(b-a)}$ pre $x < 0$

$n(b - \hat{b}) \xrightarrow{D} Exp\left(\frac{1}{b-a}\right)$

NMSA 332

67) $X \sim M(1, p_1 \dots p_k)$ $\quad L(p) = \prod p_j^{y_j} I[\sum p_j = 1]$ $\Big|$ $y_j = \#\{X_i : X_i = (0..0,\overset{j\text{-ty el.}}{1},0..0)\}$

i) Lagrangeove multiplikátory:

maximalizujeme $\ell(p) = \sum y_j \log p_j$ za podmienky $\sum p_i = 1$

tj. $\sum y_j \log p_j + \lambda(1 - \sum p_j) = f(p, \lambda)$

$$\frac{\partial}{\partial p_i} f(p,\lambda) = \frac{y_i}{p_i} - \lambda = 0 \Rightarrow p_i = y_i/\lambda$$

$$\frac{\partial}{\partial \lambda} f(p,\lambda) = 1 - \sum p_j = 0 \Rightarrow \sum \frac{y_i}{\lambda} = 1 \Rightarrow \hat\lambda = \sum y_i \Rightarrow \hat p_i = \frac{y_i}{\sum y_j} = \frac{\sum_{i=1}^m I[X_{ij}=1]}{m}$$

ii) definujeme $\hat p = \frac{1}{m}\sum_{i=1}^m \left(I[X_{i1}=1], I[X_{i2}=1], \dots, I[X_{ik}=1]\right)' = \frac{1}{m}\sum X_i$

ide o priemer iid náhodných vektorov $\Rightarrow$ $E\, I[X_{ij}=1] = P(X_i = (0,..0,\overset{j\text{-ty el.}}{1},0..0)) = p_j$

$E\hat p = p$ $\quad$ var $I[X_{ij}=1] = p_j - p_j^2$

cov $I[X_{ij}=1]\, I[X_{ij'}=1] = 0 - p_j p_{j'}$

$$\text{var}\,\hat p = \frac{1}{m}\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \dots & -p_1p_k \\ -p_1p_2 & p_2(1-p_2) & \dots & -p_2p_k \\ & & \ddots & \\ -p_1p_k & -p_2p_k & \dots & p_k(1-p_k)\end{pmatrix} =: \frac{1}{m}\Sigma(p)$$

CLV: $\sqrt{m}(\hat p - p) \xrightarrow{D} N_k(0, \Sigma(p))$

68) ako v pr 67: $X \sim M(1, p_1 - p_4)$ $\quad X \sim (CACA, DD, CAD, DCA)^T$

$= M(1, p, q, \frac{1-p-q}{2}, \frac{1-p-q}{2})$

i) $\hat p = \frac{\#\{CACA\}}{m} = 604/1987$ $\quad \hat q = \frac{\#\{D,D\}}{m} = 609/1987$

$\theta(p,q) = \frac{2p}{1+p-q}$ $\quad \hat\theta = \frac{2\hat p}{1+\hat p - \hat q}$ $\quad g(s,t) = \frac{2s}{1+s-t}$ $\quad \nabla g = \left(\frac{2(1-t)}{(1+s-t)^2}, \frac{2s}{(1+s-t)^2}\right)'$

$\nabla g((p,q)) = \left(\frac{2(1-q)}{1+p-q}, \frac{2p}{(1+p-q)^2}\right)'$ $\quad \sqrt{m}\left(\binom{\hat p}{\hat q} - \binom{p}{q}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix} p(1-p) & -pq \\ -pq & q(1-q)\end{pmatrix}\right)$

Δ-veta: $\sqrt{m}(\hat\theta - \theta) \xrightarrow[\text{mult.}]{D} N\left(0, \frac{4p(1-q)(1-p-q)}{(1+p-q)^4}\right)$

ii) IS: $\left[\frac{2\hat p}{1+\hat p - \hat q} \mp \frac{u_{1-\alpha/2}}{\sqrt{m}}\sqrt{\frac{4\hat p(1-\hat q)(1-\hat p-\hat q)}{(1+\hat p+\hat q)^4}}\right] = [0{,}58 ; 0{,}63]$ $\quad \hat\theta = 0{,}609$

69) $P(Y=i, N=j) = P(Y=i \mid N=j)P(N=j) \Rightarrow Y|N \sim Bi(N,p)$ $\quad N \sim Po(\lambda)$

i) $L(p,\lambda) = \prod P(Y_i=y_i, N_i=m_i) = \prod P(Y_i=y_i \mid N_i=m_i) \cdot \prod P(N_i=m_i)$

$\ell(p,\lambda) = \sum \log P_p(Y_i=y_i \mid N_i=m_i) + \sum \log P_\lambda(N_i=m_i)$

$\frac{\partial \ell}{\partial p}$ = "vierohodnosť z $Bi(N_i,p)$", tj. $\frac{\sum y_i}{p} - \frac{\sum(m_i-y_i)}{1-p} = 0 \Rightarrow \hat p = \left(\frac{\sum m_i}{\sum y_i}\right)^{-1}$

$\frac{\partial \ell}{\partial \lambda}$ = "vierohodnosť z $P(\lambda)$", tj. $\hat\lambda = \frac{1}{m}\sum m_i$

ii) H = ako vždy, derivujeme na zvlášť p a λ

$$H = \begin{pmatrix} -\frac{\sum y_i}{p^2} - \frac{\sum(m_i-y_i)}{(1-p)^2} & 0 \\ 0 & -\frac{\sum m_i}{\lambda^2}\end{pmatrix} \qquad J_m = \begin{pmatrix} \frac{m\lambda}{p} + \frac{m\lambda}{1-p} & 0 \\ 0 & m/\lambda \end{pmatrix}$$

$EY = E\,E[Y|N] = E\,Bi(N,p) = EN p = \lambda p$ $\quad \sqrt{m}\left(\binom{\hat p}{\hat\lambda} - \binom{p}{\lambda}\right) \xrightarrow{D} N_2\left(0, \begin{pmatrix} \frac{p(1-p)}{\lambda} & 0 \\ 0 & \lambda \end{pmatrix}\right)$

$E(N-Y) = E\,E[N-Y|N] = E[N - Bi(N,p)] = \lambda(1-p)$

$EN = \lambda$

10) $Y|X \sim N(\beta' X, \sigma^2)$ $\quad X$ nezávisí na β, σ^2.

i) $L(\beta,\sigma^2) = \prod f_{Y|X}(y_i|x_i) f_X(x_i) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^m \exp\left\{-\frac{1}{2\sigma^2}\sum(y_i - \beta' x_i)^2\right\} \prod f_X(x_i)$

$$\ell(\beta,\sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(y_i - \beta' x_i)^2 + d = c' - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(y_i - \beta' x_i)'(y_i - \beta' x_i)$$

$$= c' - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta)$$

$$D\ell = \left(\frac{X'(Y-X\beta)}{\sigma^2}, -\frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}(Y-X\beta)'(Y-X\beta)\right)$$

$X'Y - X'X\beta = 0$ $\qquad -m\sigma^2 + (Y-X\beta)'(Y-X\beta) = 0$

$\hat\beta = (X'X)^{-1}X'Y$ $\qquad \hat\sigma^2 = \frac{1}{m}(Y - X\hat\beta)'(Y - X\hat\beta)$

ii) $$H_\ell = \begin{pmatrix} -\frac{X'X}{\sigma^2} & -\frac{X'(Y-X\beta)}{\sigma^4} \\ -\frac{X'(Y-X\beta)}{\sigma^4} & \frac{m}{2\sigma^4} - \frac{(Y-X\beta)'(Y-X\beta)}{\sigma^6} \end{pmatrix} \qquad J_m = \begin{pmatrix} \frac{E(X'X)}{\sigma^2} & 0 \\ 0^T & \frac{m}{2\sigma^4} \end{pmatrix}$$

$$E(Y-X\beta)'(Y-X\beta) = E\sum\varepsilon_i^2 = m\sigma^2$$

$$\sqrt{m}\left(\begin{pmatrix}\hat\beta\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\beta \\ \sigma^2\end{pmatrix}\right) \xrightarrow{D} N_{p+1}\left(0, \begin{pmatrix}\sigma^2 (EX'X)^{-1} m & 0 \\ 0^T & 2\sigma^4\end{pmatrix}\right)$$

iii) $\sqrt{m}(\hat\beta - \beta) \xrightarrow{D} N_p(0, \sigma^2 [EX'X]^{-1} m)$

iv) $P(\beta \in \{x \in \mathbb{R}^p : (\hat\beta - x)'[EX'X](\hat\beta - x) \le \hat\sigma^2 \chi^2_p(1-\alpha)\}) \to 1-\alpha$

11) $P(Y=1|X) = \frac{e^{\beta' x}}{1+e^{\beta' x}}$ $\qquad P(Y=0|X) = 1 - P(Y=1|X)$

$S(x,\beta) := e^{\beta' x}/(1+e^{\beta' x})$

$$L(\beta) = \prod S(x_i\beta)^{y_i}(1 - S(x_i,\beta))^{1-y_i} = e^{\sum y_i \beta' x_i} / \prod(1+e^{\beta' x_i})$$

$$\ell(\beta) = \sum y_i \beta' x_i - \sum \log(1+e^{\beta' x_i})$$

$$D\ell(\beta) = \sum y_i x_i - \sum \frac{x_i e^{\beta' x_i}}{1+e^{\beta' x_i}} = \sum x_i (y_i - S(x_i,\beta))$$

$$H_\ell(\beta) = -\sum x_i \left(\frac{x_i' e^{\beta' x_i}(1+e^{\beta' x_i}) - e^{\beta' x_i} e^{\beta' x_i} x_i'}{(1+e^{\beta' x_i})^2}\right) = -\sum x_i x_i' \frac{e^{\beta' x_i}}{1+e^{\beta' x_i}} \frac{1}{1+e^{\beta' x_i}} =$$

$$= -\sum x_i x_i' S(x_i,\beta)(1 - S(x_i,\beta)) = -X'WX \quad \text{pro } X = \begin{pmatrix} x_1' \\ \vdots \\ x_m'\end{pmatrix}$$

$W := \operatorname{diag}(S(x_1,\beta)(1-S(x_1,\beta)), \dots, S(x_m,\beta)(1-S(x_m,\beta)))$

a) $\sqrt{m}(\hat\beta - \beta) \xrightarrow{d} N_p(0, [E(X'WX)]^{-1} m)$

b) $\beta_1 \in [\hat\beta_1 \mp u_{1-\alpha/2} (E X'WX)^{-1}_{11}]$.

12) $X \sim Exp(1/\eta)$ $\qquad Y|X \sim Exp(1/x\theta)$

a) $L(\theta,\eta) = (\prod x_i)^{-1}\theta^{-m}\eta^{-m}\exp\left\{-\sum\frac{y_i}{x_i\theta} - \sum\frac{x_i}{\eta}\right\}$

$$\ell(\theta,\eta) = c - m\log\theta - m\log\eta - \sum\frac{y_i}{x_i\theta} - \sum\frac{x_i}{\eta}$$

$$D\ell(\theta,\eta) = \left(-\frac{m}{\theta} + \sum\frac{y_i}{x_i\theta^2};\ -\frac{m}{\eta} + \sum\frac{x_i}{\eta^2}\right) \stackrel{!}{=} 0 \Rightarrow \hat\eta = \bar X \quad \hat\theta = \frac{1}{m}\sum\frac{Y_i}{X_i}$$

$$H_\ell(\theta,\eta) = \begin{pmatrix} \frac{m}{\theta^2} - \sum\frac{2y_i}{x_i\theta^3}, & 0 \\ 0, & \frac{m}{\eta^2} - \sum\frac{2x_i}{\eta^3}\end{pmatrix}$$

$$EX = \eta \quad E\left[\frac{Y}{X}\right] = E\left[E\left[\frac{Y}{X}\Big| X\right]\right] = E\left[\frac{1}{X}E[Y|X]\right] = E\left[\frac{X\theta}{X}\right] = \theta$$

72) cont. $J_m(\theta,\eta) = \begin{pmatrix} m/\theta^2 & 0 \\ 0 & m/\eta^2 \end{pmatrix}$

$$\sqrt{m}\left(\begin{pmatrix}\hat\theta\\ \hat\eta\end{pmatrix} - \begin{pmatrix}\theta\\ \eta\end{pmatrix}\right) \xrightarrow{\mathcal{D}} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\theta^2 & 0\\ 0 & \eta^2\end{pmatrix}\right)$$

b) $\sqrt{m}(\hat\theta - \theta) \xrightarrow{\mathcal{D}} N(0, \theta^2)$

c) $\theta \in [\hat\theta \mp u_{1-\alpha/2}\, \hat\theta/\sqrt{m}]$

d) ak $\theta_0 \in CI$ pre θ nezamietam H_0, inak zamietam.

mór d) exponenciálny systém, úplná postač. pre $\frac{1}{\theta}$ je $\sum \frac{y_i}{x_i}$, $E\,\frac{Y_i}{X_i} = \theta$ nestr. odhad, fce úplnej postač., najlepší nestr.

8. asymptotické testy bez rušivých parametrov

73) $X \sim alt(p)$ $\quad$ R Pr. 49+1 riešenie

80 (82) $U_m(p) = \frac{\sum X_i}{p} - \frac{m - \sum X_i}{1-p}$ $\qquad \hat p = \bar X \qquad J_m(p) = \frac{m}{p(1-p)}$

83

- $W_m = m(\bar X - p_0)^2 \cdot \frac{1}{\hat p(1-\hat p)} = \left[\sqrt{m}\, \frac{\bar X - p_0}{(\bar X(1-\bar X))^{1/2}}\right]^2$ $\iff$ klas. asympt. test

- $R_m = \frac{\left(\frac{\sum X_i}{p_0} - \frac{m - \sum X_i}{1-p_0}\right)^2}{m/(p_0(1-p_0))} = \left[\sqrt{m}\sqrt{p_0(1-p_0)}\left(\frac{\bar X}{p_0} - \frac{1-\bar X}{1-p_0}\right)\right]^2 = \left[\frac{\sqrt{m}}{\sqrt{p_0(1-p_0)}}(\bar X - p_0)\right]^2$

$\iff$ Wilsonova metóda

- $LR_m = 2\left[\sum X_i \log \bar X + (m - \sum X_i)\log(1-\bar X) - \sum X_i \log p_0 - (m - \sum X_i)\log(1-p_0)\right]$
$\quad = 2m\left[\bar X \log\frac{\bar X}{p_0} + (1-\bar X)\log\frac{1-\bar X}{1-p_0}\right]$

každý test zamietam ak $T_m > \chi^2_1(1-\alpha)$

74) $X \sim Po(\lambda)$ $\quad$ R Pr 50+1 riešenie

81 $U_m(\lambda) = -m + \sum X_i/\lambda$ $\qquad \hat\lambda = \bar X \qquad J(\lambda) = 1/\lambda$

84

- $W_m = m(\bar X - \lambda_0)^2\, \frac{1}{\bar X} = \left(\sqrt{m}\, \frac{\bar X - \lambda_0}{\sqrt{\bar X}}\right)^2$
- $R_m = \left(-m + \sum X_i/\lambda_0\right)^2 / (m/\lambda_0) = \left(\sqrt{m}\, \frac{\bar X - \lambda_0}{\sqrt{\lambda_0}}\right)^2$
- $LR_m = 2\left(-m\bar X + \sum X_i \log \bar X + m\lambda_0 - \sum X_i \log\lambda_0\right) = 2m\left[\log\left(\frac{\bar X}{\lambda_0}\right)\cdot \bar X - (\bar X - \lambda)\right]$

každý test zamietame ak $T_m > \chi^2_1(1-\alpha)$

75) $\tilde X \sim Po(\lambda)|$ $\quad$ ale $\quad X = \tilde X \mid \tilde X > 0$

82 $P(X = k) = P(\tilde X = k \mid \tilde X > 0) = P(\tilde X = k, \tilde X > 0)/P(\tilde X > 0) = \frac{e^{-\lambda}\lambda^k/k!}{1 - e^{-\lambda}}$ $\qquad k \geq 1$

85 $\quad P(\tilde X > 0) = 1 - e^{-\lambda}\lambda^0/0! = 1 - e^{-\lambda}$

$$L(\lambda) = \prod \frac{e^{-\lambda}\lambda^{x_i}/x_i!}{(1-e^{-\lambda})} = \frac{e^{-m\lambda}\lambda^{\sum x_i}/\prod x_i!}{(1-e^{-\lambda})^m}$$

$$\ell(\lambda) = -m\lambda + \sum x_i \log\lambda - \sum \log x_i! - m\log(1-e^{-\lambda})$$

$$U(\lambda) = -m + \sum x_i/\lambda + -\frac{me^{-\lambda}}{1-e^{-\lambda}} \overset{!}{=} 0$$

$$\hookrightarrow \lambda \frac{e^{\lambda}}{e^{\lambda}-1} = \bar{X}$$

$$H_\ell(\lambda) = -\sum x_i/\lambda^2 - \frac{me^{\lambda}}{(e^{\lambda}-1)^2} \qquad EX = \frac{1}{1-e^{-\lambda}}\sum_{x=1}^{\infty} x\, e^{-\lambda}\frac{\lambda^x}{x!} =$$

$$J_m(\lambda) = \frac{m}{\lambda(1-e^{-\lambda})} + \frac{me^{\lambda}}{(1-e^{\lambda})^2} \qquad = \frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = \frac{\lambda}{1-e^{-\lambda}}$$

a) $\sqrt{m}(\hat{\lambda}-\lambda) \xrightarrow{\mathcal{D}} N\left(0, \left[\frac{1}{\lambda(1-e^{-\lambda})} + \frac{e^{\lambda}}{(1-e^{\lambda})^2}\right]^{-1}\right)$

b) numericky $\hat{\lambda} \approx 0{,}31$ $\quad \sqrt{m}(\hat{\lambda}-\lambda) \overset{as}{\approx} N(0, 0{,}033)$ Mathematica

c) $\theta = P(\bar{X}=0) = e^{-\lambda}$ $\quad \hat{\theta} = e^{-\hat{\lambda}} \doteq 0{,}73$

interval Δ-metodou $g(t) = e^{-t}$ $\quad g'(\lambda) = -e^{-\lambda}$

$$\sqrt{m}(\hat{\theta}-\theta) \xrightarrow{d} N\left(0, e^{-2\lambda}\left[\frac{1}{\lambda(1-e^{-\lambda})} + \frac{e^{\lambda}}{(1-e^{-\lambda})^2}\right]^{-1}\right)$$

CI: $\theta \in [0{,}725 ; 0{,}735]$ $\quad$ namisleno $\theta = 0{,}75$.

d) $\hat{\theta} = 0{,}73$. , resp θ v intervalu hore v c)

86 83 76) $X \sim Exp(\lambda)$ z prikladu Cn 51.

$U_m(\lambda) = m/\lambda - \sum x_i$ $\quad \hat{\lambda} = 1/\bar{X}$ $\quad J(\lambda) = 1/\lambda^2$

- $W_m = m\left(\frac{1}{\bar{X}} - \lambda_0\right)^2 \bar{X}^2$
- $R_m = (m/\lambda_0 - \sum x_i)^2/(m/\lambda_0^2)$
- $LR_m = 2m(-\log\bar{X} - (\bar{X})^{-1}\bar{X} - \log\lambda_0 + \bar{X}\lambda_0)$

87 84 77) $X \sim Ge(p)$ Cn 52

- $W_m = m\left(\frac{1}{1+\bar{X}} - p_0\right)^2\left(\frac{1}{\hat{p}^2} - \frac{1}{\hat{p}(1-\hat{p})}\right)$
- $R_m = \left(\frac{m}{p_0} - \frac{\sum x_i}{1-p_0}\right)^2 \Big/ \left(m\left(\frac{1}{p_0^2} - \frac{1}{p_0(1-p_0)}\right)\right)$
- $LR_m = 2m\left(\log\hat{p} + \bar{X}\log(1-\hat{p}) - \log p_0 - \bar{X}\log(1-p_0)\right)$

88 85 78) $\binom{X}{Y} \overset{iid}{\sim} f(x,y) = \beta x e^{-\beta x y}$ $\quad x\in(0,1), y>0$

a) $L(\beta) = \beta^m \prod x_i \cdot e^{-\beta\sum x_i y_i}$

$$\ell(\beta) = m\log\beta + c - \beta\sum x_i y_i$$

$$U(\beta) = \frac{m}{\beta} - \sum x_i y_i = 0$$

$$\hat{\beta} = \underline{\underline{\frac{m}{\sum x_i y_i}}}$$

88 78) cont. b) $\frac{\partial U}{\partial \beta}(\beta) = -m/\beta^2$ $\qquad J(\beta) = 1/\beta^2$

- $W_m = (\hat\beta - \beta)^2 m / \hat\beta^2$
- $R_m = \left(\frac{m}{\beta_0} - \sum x_i y_i\right)^2 / (m/\beta_0^2)$
- $LR_m = m2\left(\log\hat\beta - \hat\beta \frac{1}{m}\sum x_i y_i - \log\beta_0 + \beta_0 \frac{1}{m}\sum x_i y_i\right)$

79) $Y|X \sim Po(e^{\beta X})$ r Pr 60+1 $\theta = e^{\beta}$

86

89 $L(\theta) = \prod \frac{\theta^{y_i x_i} e^{-\theta^{x_i}}}{\cancel{x_i}\, y_i!} \cdot c = \theta^{\sum y_i x_i} e^{-\sum \theta^{x_i}} \cdot c'$

$\ell(\theta) = \sum y_i x_i \log\theta - \sum \theta^{x_i}$ [illegible] nedokážeme riešiť

a) $U(\theta) = \sum x_i y_i/\theta - \sum x_i \theta^{x_i-1} = 0$

$U'(\theta) = -\sum x_i y_i/\theta^2 - \sum x_i(x_i-1)\,\theta^{x_i-2}$

$J_m(\theta) = E\left[\frac{mXY}{\theta^2} + m X(X-1)\theta^{X-2}\right]$ $\qquad EXY = E[E(XY|X)] = E X E(Y|X) = E X e^{\beta X}$

b) explicitne vyjadrím ich

- $R_m = \left[\sum x_i y_i/\theta_0 - \sum x_i \theta_0^{x_i-1}\right]^2 / J_m(\theta_0)$

c) $\{\theta \in R : R_m(\theta_0) \le \chi^2_1(1-\alpha)\}$ je interval pre $\theta = e^{\beta}$

d) pre β urobíme ich numericky, prípadne $\log\theta = \beta$ teda log-transformáciou množiny z c).

80) $X \sim$ Logist (θ) r Pr 58+1

87

90 explicitne ich

- $R_m = \left(m - \sum_{i=1}^{m} \frac{2e^{-x_i}}{e^{-\theta} - e^{-x_i}}\right)^2 / \left(\frac{m}{3}\right)$

81) $X \sim N(\mu, \sigma^2)$ r Or 63+1

88

91

- $W_m = \left((\bar X; \frac{1}{m}\sum(X_i-\bar X)^2) - (\mu,\sigma^2)\right)\begin{pmatrix} m/\hat\sigma^2 & 0 \\ 0 & m/2\hat\sigma^4 \end{pmatrix}\begin{pmatrix} \bar X - \mu \\ \frac{1}{m}\sum(X_i-\bar X)^2 - \sigma^2 \end{pmatrix}$

$= \frac{(\bar X-\mu)^2 m}{\hat\sigma^2} + \frac{(\hat\sigma^2-\sigma^2)m}{2\hat\sigma^4}$

- $R_m = \left(\frac{1}{\sigma_0^2}\sum(X_i-\mu_0),\ -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(X_i-\mu_0)^2\right)\begin{pmatrix} \sigma_0^2/m & 0 \\ 0 & 2\sigma_0^4/m \end{pmatrix}\begin{pmatrix} \frac{1}{\sigma_0^2}\sum(X_i-\mu_0) \\ -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(X_i-\mu_0)^2 \end{pmatrix}$

$= \frac{\sigma_0^2}{m}\left(\sum(X_i-\mu_0)\right)^2 + \frac{2}{m}\left(-\frac{m}{2} + \frac{1}{2\sigma_0^2}\sum(X_i-\mu_0)^2\right)^2$

- $LR_m = 2m\left(-\frac{\log\hat\sigma^2}{2} - \frac{1}{2\hat\sigma^2}\frac{1}{m}\sum(X_i-\bar X)^2 + \frac{\log\sigma_0^2}{2} + \frac{1}{2\sigma_0^2}\frac{1}{m}\sum(X_i-\mu_0)^2\right)$

porovnávam s $\chi^2_2(1-\alpha)$

82) $X \sim \log N(\mu, \sigma^2)$ Or 64

12.89) a) $\hat{\mu} = \frac{1}{m}\sum \log X_i \qquad \hat{\sigma}^2 = \frac{1}{m}\sum(\log X_i - \hat{\mu})^2$

b) $\sqrt{m}\left(\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}\right)$

c) $\left\{\mu,\sigma^2 : m(\hat{\mu}-\mu, \hat{\sigma}^2-\sigma^2)\begin{pmatrix}1/\hat{\sigma}^2 & 0\\ 0 & 1/2\hat{\sigma}^4\end{pmatrix}\begin{pmatrix}\hat{\mu}-\mu\\ \hat{\sigma}^2-\sigma^2\end{pmatrix} \leq \chi^2_2(1-\alpha)\right\}$

d) $(\mu,\sigma) = (0,1) : H_0$

- $W_m = m(\hat{\mu}-\mu_0, \hat{\sigma}^2-\sigma^2)\begin{pmatrix}1/\hat{\sigma}^2 & 0\\ 0 & 1/2\hat{\sigma}^4\end{pmatrix}\begin{pmatrix}\hat{\mu}-\mu_0\\ \hat{\sigma}^2-\sigma^2\end{pmatrix}$

- $R_m = \left(\sum \frac{\log X_i - \mu_0}{\sigma_0^2}, -\frac{m}{2\sigma_0^2} + \frac{1}{2\sigma_0^4}\sum(\log X_i-\mu_0)^2\right)\begin{pmatrix}\sigma_0^2/m & 0\\ 0 & 2\sigma_0^4/m\end{pmatrix}\begin{pmatrix}\sum \log X_i\\ -\frac{m}{2}+\frac{1}{2}\sum(\log X_i)^2\end{pmatrix}$

$$= \frac{(\sum \log X_i)^2}{m} + \frac{2}{m}\left(-\frac{m}{2} + \frac{1}{2}\sum(\log X_i)^2\right)^2$$

- $LR_m = 2m\left(-\frac{\log \hat{\sigma}^2}{2} - \frac{1}{2\hat{\sigma}^2}\frac{1}{m}\sum(\log X_i - \hat{\mu})^2 + 0 + \frac{1}{2m}\sum(\log X_i)^2\right)$

porovnáme s $\chi^2_2(1-\alpha)$

e) $\sqrt{m}(\hat{\mu}-\mu) \xrightarrow{d} N(0, 2\sigma^4)$

zamítáme $H_0 : \mu = \mu_0$ až $\mu_0 \notin \left[\hat{\mu} \mp u_{1-\alpha/2}\frac{\sqrt{2\hat{\sigma}^4}}{\sqrt{m}}\right]$

90) 83) $\equiv$ Pr 78

91) 84) $Y|X \sim N(\beta_1 X + \beta_2 X^2, 1)$

93

$L(\beta) = \prod c \cdot \exp\left\{-\frac{1}{2}(y_i - \beta_1 x_i - \beta_2 x_i^2)^2\right\} = c \cdot \exp\left\{-\frac{1}{2}\sum(y_i - \beta_1 x_i - \beta_2 x_i^2)^2\right\}$

$\ell(\beta) = -\frac{1}{2}\sum(y_i - \beta_1 x_i - \beta_2 x_i^2)^2$

$U(\beta) = \left(\sum \frac{2}{2}(y_i - \beta_1 x_i - \beta_2 x_i^2)x_i, \sum x_i^2(y_i - \beta_1 x_i - \beta_2 x_i^2)\right)$

$U'(\beta) = \begin{pmatrix}-\sum x_i^2 & -\sum x_i^3\\ -\sum x_i^3 & -\sum x_i^4\end{pmatrix} \qquad J_m(\beta) = m\begin{pmatrix}EX^2 & EX^3\\ EX^3 & EX^4\end{pmatrix}$

$H_0 : \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$R_m = \left(\sum(Y_i - \beta_1 X_i - \beta_2 X_i^2)\cdot X_i, \sum X_i^2 Y_i\right)\frac{1}{m}\begin{pmatrix}EX^2 & EX^3\\ EX^3 & EX^4\end{pmatrix}^{-1}\begin{pmatrix}\sum Y_i X_i\\ \sum Y_i X_i^2\end{pmatrix}$

porovnáme s $\chi^2_2(1-\alpha)$

85) $X \sim Mult(4, p_1, p_2, p_3, p_4)$ z Pr 67

$H_0: (p_1, p_2, p_3, p_4) = (1/4, 1/4, 1/4, 1/4)$

$\hat{p} = \sum_{i=1}^{m} \frac{X_{ij}}{m}$ vieme že $\sqrt{m}(\hat{p} - p) \xrightarrow{D} N_4\left(0, \underbrace{\begin{pmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 & -p_1p_4 \\ & \cdots & & \\ & & \ddots & p_4(1-p_4) \end{pmatrix}}_{\Sigma(p)}\right)$

$T := m(\hat{p} - p_0)' \underbrace{\Sigma(\hat{p})^{-1}}_{\text{neexistuje}} (\hat{p} - p_0) \xrightarrow{H_0} \chi^2_3$ lebo rank $\Sigma = 3$ (veta 12.3, Anděl)

zamietam ak $T > \chi^2_3(1-\alpha)$

def $V^- = diag\left(\frac{1}{mp}\right)$. Potom podľa V12.3 z Anděla je $\Sigma V^- \Sigma = \Sigma$, t.j. V^- je pseudoinverzia ku Σ a podľa V4.15 z Anděla je

(viz V12.5)

$T_m = m(\hat{p} - p_0)' V^-(\hat{p})(\hat{p} - p_0) = \sum_{i=1}^{4} \frac{(X_i - mp_i)^2}{mp_i} \xrightarrow{H_0} \chi^2_3 \Rightarrow \chi^2$-test Multinom. rozdelenia

86) $Y|N \sim Bi(N, p)$ $N \sim Po(\lambda)$ z Pr 69

92) a) $\hat{p} = \frac{\sum m_i}{\sum y_i}$ $\hat{\lambda} = \frac{1}{m}\sum m_i$

94 b) $\sqrt{m}\left(\begin{pmatrix}\hat{p}\\ \hat{\lambda}\end{pmatrix} - \begin{pmatrix}p\\ \lambda\end{pmatrix}\right) \xrightarrow{D} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix} p(1-p)/\lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$

c) $W_m = (\hat{p} - p, \hat{\lambda} - \lambda) \begin{pmatrix} \frac{m\hat{\lambda}}{\hat{p}(1-\hat{p})} & 0 \\ 0 & m/\hat{\lambda} \end{pmatrix} \begin{pmatrix} \hat{p} - p \\ \hat{\lambda} - \lambda \end{pmatrix}$

z Pr 50 a 49

$R_m = \left(\sum Y_i/p_0 - \sum \frac{N_i - Y_i}{1-p_0}, -m + \frac{\sum N_i}{\lambda_0}\right) \begin{pmatrix} \frac{p_0(1-p_0)}{m\lambda_0} & 0 \\ 0 & (m/\lambda_0)^{-1} \end{pmatrix} \begin{pmatrix} \sum Y_i/p_0 - \sum \frac{N_i - Y_i}{1-p_0} \\ -m + \sum N_i/\lambda_0 \end{pmatrix}$

$LR_m = 2m\Big(\frac{1}{m}\sum Y_i \log \hat{p}_0 - \frac{1}{m}\sum (N_i - Y_i)\log(1-\hat{p}_0) - 1 + \sum \frac{N_i}{\hat{\lambda}}\frac{1}{m} - \frac{1}{m}\sum Y_i \log p_0 +$
$\frac{1}{m}\sum (N_i - Y_i)\log(1-p_0) + 1 - \sum \frac{N_i}{\lambda_0}\frac{1}{m}\Big)$

porovnám s $\chi^2_2(1-\alpha)$

87) $X \sim N(\mu, \sigma^2)$ $\quad H_0: \mu = \mu_0$ $\quad \tau = \mu,\ \psi = \sigma^2$ Pr 63

94

$$L(\mu,\sigma^2) = \frac{1}{(\sqrt{2\pi\sigma^2})^m} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\} \qquad \ell(\mu,\sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$$

$$U(\mu,\sigma^2) = \Big(\underbrace{\frac{\sum(x_i-\mu)}{\sigma^2}}_{U_1}, -\frac{m}{2\sigma^2} + \frac{\sum(x_i-\mu)^2}{2\sigma^4}\Big) \qquad \hat{\mu} = \bar{X} \quad \hat{\sigma}^2 = \frac{1}{m}\sum(x_i-\bar{x})^2$$

$$\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$$

- pre $\tau = \tau_0$: $\quad \tilde{\sigma}^2 = \frac{1}{m}\sum(X_i-\mu_0)^2 \qquad \tilde{\theta} = (\mu_0, \tilde{\sigma}^2)$

$$-E\frac{\partial U}{\partial \theta^T}(\mu,\sigma^2) = \begin{pmatrix} +\frac{m}{\sigma^2} & 0 \\ 0 & \frac{m}{2\sigma^4}\end{pmatrix} = J_m(\theta) \qquad J(\theta) = \left(\begin{array}{c|c} \frac{1}{\sigma^2} & 0 \\ \hline 0 & \frac{1}{2\sigma^4}\end{array}\right)$$

$$J^{-1}(\theta) = \left(\begin{array}{c|c} \sigma^2 & 0 \\ \hline 0 & 2\sigma^4\end{array}\right) \qquad J^{11}(\theta) = \sigma^2$$

- $LR := 2\cdot\left[\cancel{c} - \frac{m}{2}\log\hat{\sigma}^2 - \cancel{\frac{1}{2\hat{\sigma}^2}\sum(X_i-\bar{X})^2} - \cancel{c} + \frac{m}{2}\log\tilde{\sigma}^2 + \cancel{\frac{1}{2\tilde{\sigma}^2}\sum(X_i-\mu_0)^2}\right]$

$$= m\log\frac{\tilde{\sigma}^2}{\hat{\sigma}^2} = m\log\frac{\sum(X_i-\mu_0)^2}{\sum(X_i-\bar{X})^2}$$

- $W = m(\bar{X}-\mu_0)^2(\hat{\sigma}^2)^{-1}$
- $R = \frac{1}{m}\left(\frac{\sum(X_i-\mu_0)}{\tilde{\sigma}^2}\right)^2 \tilde{\sigma}^2 = m\frac{(\bar{X}-\mu_0)^2}{\tilde{\sigma}^2}$

Každý test zamietame ak $T_m > \chi^2_1(1-\alpha)$

88) $X \sim \log N(\mu,\sigma^2)$ $\quad H_0: \mu = \mu_0$ $\quad \tau = \mu,\ \psi = \sigma^2$ Pr 64

95

$$\ell(\mu,\sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(\log x_i - \mu)^2$$

$$U = \left(\sum(\log x_i-\mu)/\sigma^2, -m/2\sigma^2 + \sum(\log x_i-\mu)^2/2\sigma^4\right) \qquad \hat{\mu} = \frac{1}{m}\sum\log X_i \quad \hat{\sigma}^2 = \frac{1}{2}\sum(\log X_i - \hat{\mu})^2$$

$$\tilde{\sigma}^2 = \frac{1}{m}\sum(\log X_i - \mu_0)^2$$

$$J(\theta) = \begin{pmatrix} 1/\sigma^2 & 0 \\ 0 & 1/2\sigma^4\end{pmatrix} \Rightarrow J^{11}(\theta) = \sigma^2$$

- $LR = \dots = m\log\left[\frac{\sum(\log X_i-\mu_0)^2}{\sum(\log X_i-\hat{\mu})^2}\right]$
- $W = m(\hat{\mu}-\mu_0)^2(\hat{\sigma}^2)^{-1}$
- $R = \frac{1}{m}\left[\frac{\sum(\log X_i-\mu_0)}{\tilde{\sigma}^2}\right]^2\cdot\tilde{\sigma}^2 = m\frac{(\hat{\mu}-\mu_0)^2}{\tilde{\sigma}^2}$

zamietame ak $T_m > \chi^2_1(1-\alpha)$

89) $Y|N \sim Bi(\tilde{m}, p)$ pre $\tilde{m} = N$, $N \sim Po(\lambda)$ $\quad H_0: p = p_0 \quad \tau = p, \psi = \lambda$ Gn 69

96

$$L(p,\lambda) = \prod \binom{m_i}{y_i} p^{y_i}(1-p)^{m_i - y_i} \frac{\lambda^{m_i} e^{-\lambda}}{m_i!} = c \cdot p^{\sum y_i} (1-p)^{\sum m_i - \sum y_i} \lambda^{\sum m_i} e^{-m\lambda}$$

$$\ell(p,\lambda) = c + \sum y_i \log p + (\sum m_i - \sum y_i) \log(1-p) + \sum m_i \log \lambda - m\lambda$$

$$U(p,\lambda) = \left(\frac{\sum y_i}{p} - \frac{\sum m_i - \sum y_i}{1-p}, \frac{\sum m_i}{\lambda} - m \right) \qquad \hat{\lambda} = \frac{\sum m_i}{m} \qquad \hat{p} = \frac{\sum y_i}{\sum m_i}$$

$\hat{\lambda}$ nezávisí na $p \Rightarrow \tilde{\lambda} = \hat{\lambda}$

$$J(p,\lambda) = \begin{pmatrix} \frac{\lambda}{p} + \frac{\lambda}{1-p} & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \qquad J^{11}(\theta) = \frac{1/1}{\frac{(1-p)\lambda + p\lambda}{p(1-p)}} = \frac{p(1-p)}{\lambda}$$

- $LR = 2\left[\cancel{c} + \sum Y_i \log \hat{p} + \sum (N_i - Y_i) \log(1-\hat{p}) + \cancel{\sum N_i \log \hat{\lambda}} - \cancel{m\hat{\lambda}} \right.$
  $\left. - \cancel{c} - \sum Y_i \log p_0 - \sum (N_i - Y_i) \log(1-p_0) - \cancel{\sum N_i \log \hat{\lambda}} + \cancel{m\hat{\lambda}} \right]$
  $= 2\left[\sum Y_i \log \hat{p}/p_0 + \sum (N_i - Y_i) \log[(1-\hat{p})/(1-p_0)] \right]$
- $W = m(\hat{p} - p_0)^2 \left[\frac{\hat{p}(1-\hat{p})}{\hat{\lambda}} \right]^{-1} = m\frac{(\hat{p} - p_0)^2}{\hat{p}(1-\hat{p})} \cdot \hat{\lambda}$
- $R = \frac{1}{m} \left(\frac{\sum Y_i}{p_0} - \frac{\sum (N_i - Y_i)}{1-p_0} \right)^2 \frac{p_0(1-p_0)}{\hat{\lambda}} = \frac{1}{m} \left(\frac{\sum Y_i - \cancel{p_0 \sum Y_i} - p_0 \sum N_i + \cancel{p_0 \sum Y_i}}{p_0(1-p_0)} \right)^2 \frac{p_0(1-p_0)}{\hat{\lambda}}$
  $= \frac{1}{m} \left(\sum N_i \left(\frac{\sum Y_i}{\sum N_i} - p_0 \right) \right)^2 \frac{1}{\hat{\lambda} p_0(1-p_0)}$

zamietam ak $T_n > \chi^2_1(1-\alpha)$

97

90) $X \sim M(1, p_1, p_2, p_3, p_4)$ P. 67+1

→ • štandardná parametrizácia $(p_1, p_2, p_3, p_4)^T = p \qquad y_j = \sum_{i=1}^m x_{ij}$ (počet j v súbore)

i) $L(p) = \prod p_j^{y_j} \qquad \ell(p) = \sum y_j \log p_j \qquad p_1 = \tau \quad (p_2, p_3, p_4) = \psi \qquad H_0: p_1 = 1/4$

$\hat{p}_j = y_j/m$ pre $\tilde{p}$ maximalizuj $\sum y_j \log p_j$ na podm. $p_1 = 1/4$, $\sum_{j=2}^4 p_j = \frac{3}{4}$

$\mathcal{L}$ $\quad f(p_2, p_3, p_4, \lambda) = y_1 \log 1/4 + y_2 \log p_2 + y_3 \log p_3 + y_4 \log p_4 + \lambda\left(\frac{3}{4} - \sum p_j\right)$

$\partial/\partial p_j \, f = y_j/p_j - \lambda = 0 \qquad \partial/\partial\lambda \, f = 3/4 - \sum p_j = 0$

$\Rightarrow p_j = y_j/\lambda \qquad \rightarrow \qquad 3/4 - 1/\lambda \sum y_j = 0 \Rightarrow \lambda = 4/3 \sum y_j$

$\Rightarrow \tilde{p}_j = \frac{y_j}{\sum_{j=2}^4 y_j} \cdot \frac{3}{4} \qquad j = 2,3,4$

- $LR = 2\left(\sum_{j=1}^4 Y_j \log(Y_j/m) - Y_1 \log 1/4 - \sum_{j=2}^4 Y_j \log\left(\frac{Y_j \cdot \frac{3}{4}}{\sum_{i=2}^4 Y_i} \right) \right)$

- Fisher. inf. nemôžeme počítať lebo nemáme ako zahrnúť do inverzie podm. $\sum_{j=1}^4 p_j = 1$

→ • parametrizácia $(p_1, p_2, p_3, 1 - p_1 - p_2 - p_3) \qquad p = (p_1, p_2, p_3)^T$

$L(p) = \prod_{j=1}^3 p_j^{y_j} \cdot (1 - p_1 - p_2 - p_3)^{m - y_1 - y_2 - y_3}$ Mathematica $\hat{p}_j = y_j/m$

$1 - \sum_{j=1}^3 \hat{p}_j = y_4/m$

$\tilde{p} = \left(1/4, \frac{3}{4}\frac{y_2}{m - y_1}, \frac{3}{4}\frac{y_3}{m - y_1}\right)^T \xRightarrow{\text{same}} 1 - \sum_{j=1}^3 \tilde{p}_j = \frac{3/4 \, y_4}{\sum_{i=2}^4 Y_i}$

- $LR = 2\left(\sum_{j=1}^3 Y_j \log(Y_j/m) + Y_4 \log(Y_4/m) - Y_1 \log 1/4 - \sum_{j=2}^3 Y_j \log\left(\frac{Y_j \, 3/4}{m - Y_1} \right) - Y_4 \log\left(\frac{3/4 \, Y_4}{m - Y_1} \right) \right)$

91) z Př 90 a R skriptu dostáváme $\hat{p} \doteq (0{,}240;\ 0{,}258;\ 0{,}267;\ 0{,}235)$

98 a) $LR = 60{,}08$ $H_0: p_1 = 1/4$ $\chi^2_1(1-\alpha) = 3{,}84$ p-val $= 9\cdot 10^{-15}$ $\Rightarrow$ zamítáme

b) $H: p_1 = p_2$ $LR = 76{,}41$ p-val $= 2\cdot 10^{-18}$ $\Rightarrow$ zamítáme

c) $H: p_3 = 1{,}1\, p_1$ $LR = 1{,}39$ p-val $= 0{,}239$ $\Rightarrow$ nezamítáme

92) $(X_i, Y_i)^T \sim$ mnoh. výběr kde $Y_i \mid X_i \sim N(\beta_0 + \beta_1 x_i, \sigma^2)$ X_i nezáv. na β, σ^2

99 pro $\sigma = 1$ jsme řešili v Př 47 (Př 22)

$H_0: \beta_1 = 0 \quad H_1: \beta_1 \neq 0$

$$L(\beta, \sigma^2) = c \cdot \prod \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right\} = c(\sigma^2)^{-m/2} \exp\left\{-\frac{1}{2\sigma^2}\sum (y_i - \beta_0 - \beta_1 x_i)^2\right\}$$

$$\ell(\beta, \sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i - \beta_0 - \beta_1 x_i)^2$$

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{m}{2\sigma^2} + \frac{1}{2\sigma^4}\sum (y_i - \beta_0 - \beta_1 x_i)^2 \overset{!}{=} 0 \Rightarrow \hat{\sigma}^2 = \frac{1}{m}\sum (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \qquad \hat{\beta}_1 = \frac{\sum (Y_i - \bar{Y})^2 (x_i - \bar{X})^2}{\sum (x_i - \bar{x})^2}$$

pro $\hat{\beta}_0$ a $\hat{\beta}_1$ v Př 47 (Př 22) (Dupač, Hušková) Věta 7.1

na H_0: $\ell(\beta_0, \sigma^2) = c - \frac{m}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum (y_i - \beta_0)^2$

$\Rightarrow$ standardní normální model $\Rightarrow$ $\tilde{\beta}_0 = \bar{Y}$ $\tilde{\sigma}^2 = \frac{1}{m}\sum (Y_i - \tilde{\beta}_0)^2$ $\tilde{\beta}_1 = 0$

$$\cdot\ LR = 2\left[\cancel{c} - \frac{m}{2}\log\hat{\sigma}^2 - \underbrace{\frac{1}{2\hat{\sigma}^2}\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}_{m/2} - \cancel{c} + \frac{m}{2}\log\tilde{\sigma}^2 + \underbrace{\frac{1}{2\tilde{\sigma}^2}\sum (Y_i - \tilde{\beta}_0)^2}_{m/2}\right]$$

$$= \frac{\cancel{2}\, m}{\cancel{2}}\left[\log\left[\frac{\tilde{\sigma}^2}{\hat{\sigma}^2}\right]\right]$$

$$= m \log \frac{\sum (Y_i - \tilde{\beta}_0)^2}{\sum (Y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}$$

zamítáme když $LR > \chi^2_1(1-\alpha)$

93) mnohorozměrný výběr (X_i, Y_i) jako v Př 41 kde

100 $$P(Y = 1 \mid X = x) = \frac{e^{\alpha + \beta x}}{1 + e^{\alpha + \beta x}} \qquad P(Y = 0 \mid X = x) = 1 - P(Y = 1 \mid X = x)$$

$$\ell(\alpha, \beta) = \sum y_i(\alpha + \beta x_i) - \sum \log(1 + e^{\alpha + \beta x_i})$$

$\hat{\alpha}, \hat{\beta}$ numericky v R skriptu

$$U(\alpha, \beta) = \begin{pmatrix} \sum y_i - \sum e^{\alpha + \beta x_i}/(1 + e^{\alpha + \beta x_i}) \\ \sum x_i y_i - \sum x_i e^{\alpha + \beta x_i}/(1 + e^{\alpha + \beta x_i}) \end{pmatrix} \Big\} U_2$$

$\cdot$ na $H_0: \beta = 0$
$$\ell(\alpha) = \sum y_i \alpha - \sum \log(1 + e^{\alpha})$$
$$U(\alpha) = \sum y_i - \sum \frac{e^{\alpha}}{1 + e^{\alpha}} \overset{!}{=} 0 \Rightarrow \bar{Y} = e^{\alpha}/(1 + e^{\alpha}) \qquad \tilde{\alpha} = \log\frac{\bar{Y}}{1 - \bar{Y}}, \quad \tilde{\beta} = 0$$

$$LR = 2\left[\sum Y_i(\hat{\alpha} + \hat{\beta} x_i) - \sum \log(1 + e^{\hat{\alpha} + \hat{\beta} x_i}) - \sum Y_i \tilde{\alpha} + \sum \log(1 + e^{\tilde{\alpha}})\right]$$

Informační matice: z Př 41 máme pro $X = \begin{pmatrix} 1 & x_1 \\ 1 & x_m \end{pmatrix}$ a $W = \mathrm{diag}\left(\underbrace{\frac{e^{\alpha + \beta x_i}}{(1 + e^{\alpha + \beta x_i})^2}}_{w_i}\right)$

že $$J_m(\alpha, \beta) = X' W X = \begin{pmatrix} \sum w_i & \sum w_i x_i \\ \sum w_i x_i & \sum w_i x_i^2 \end{pmatrix}$$

$\tau = \beta$, $H_0: \beta = 0$ – pro J^{11} prvek (2,2) matice $[J_m(\alpha, \beta)/m]^{-1}$

stejné pozor

$$R = \frac{1}{m}[U_2(\tilde{\alpha}, 0)]^2 J^{11}(\tilde{\alpha}, 0) \qquad W = m\left((\hat{\alpha}, \hat{\beta}) - (\hat{\alpha}, 0)\right)^2 / J^{11}(\hat{\alpha}, \hat{\beta}) \qquad \sim \chi^2_1(1-\alpha)$$

90) pokrač. $\Rightarrow$ LR testy sú v oboch parametrizáciách rovnaké.

(94) Fisherova informácia Mathematica

$$J(p) = \begin{pmatrix} 1/p_1 + 1/(1-p_1-p_2-p_3) & 1/(1-p_1-p_2-p_3) & 1/(1-p_1-p_2-p_3) \\ \cdot & 1/p_2 + 1/(1-p_1-p_2-p_3) & \cdot \\ \cdot & \cdot & 1/p_3 + 1/(1-p_1-p_2-p_3) \end{pmatrix}$$

$$J^{-1}(p) = \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & -p_1p_3 \\ -p_1p_2 & p_2(1-p_2) & -p_2p_3 \\ -p_1p_3 & -p_2p_3 & p_3(1-p_3) \end{pmatrix}$$

$$W = m\left(Y_1/m - p_0\right)^2\left[\hat{p}_1(1-\hat{p}_1)\right]^{-1} = \left[\frac{\sqrt{m}(\hat{p}_1 - p_0)}{\sqrt{\hat{p}_1(1-\hat{p}_1)}}\right]^2$$

b) v štandardnej parametrizácii $H_0: p_1 = p_2$ $\quad \ell(p) = \sum y_j \log p_j$

na H_0: $\ell(p_2,p_3,p_4) = (y_1+y_2)\log p_2 + y_3 \log p_3 + y_4 \log p_4$ $\Rightarrow$ ako v a) na podm. $2p_2 + p_3 + p_4 = 1$

$$\tilde{p} = \left(\frac{Y_1+Y_2}{2m}, \frac{Y_1+Y_2}{2m}, \frac{Y_3}{m}, \frac{Y_4}{m}\right)$$

$$\partial f/\partial p_j = y_j/p_j - \lambda = 0, \; j>2$$
$$= (y_1+y_2)/p_j - 2\lambda = 0 \quad j=2$$
$$\Rightarrow p_j = \begin{cases} (y_1+y_2)/2\lambda & j=2 \\ y_j/\lambda & j>2 \end{cases}$$

$$\partial f/\partial \lambda \Rightarrow (y_1+y_2)/2\lambda \cdot 2 + y_3/\lambda + y_4/\lambda = 1 \Rightarrow \lambda = \sum_{i=1}^{4} y_i$$

$$\tilde{p}_2 = (y_1+y_2)/2\cdot\sum y_i = \frac{y_1+y_2}{2m} \quad \tilde{p}_3 = y_3/m \quad \tilde{p}_4 = y_4/m$$

- $$LR = 2\left[\sum_{j=1}^{4} Y_j \log Y_j/m - Y_1 \log\left(\frac{Y_1+Y_2}{2m}\right) - Y_2 \log\left(\frac{Y_1+Y_2}{2m}\right) - Y_3 \log \frac{Y_3}{m} - Y_4 \log \frac{Y_4}{m}\right]$$
$$= 2\left[Y_1 \log \frac{Y_1}{m} + Y_2 \log \frac{Y_2}{m} - (Y_1+Y_2)\log\frac{Y_1+Y_2}{2m}\right]$$

c) ako v b) $\quad \ell(p_1,p_2,p_4) = y_1 \log p_1 + y_2 \log p_2 + y_3 \log 1{,}1p_1 + y_4 \log p_4 \; \left(+\lambda(2{,}1p_1 + p_2 + p_4 - 1)\right)$

$$\partial f/\partial p_j = \begin{cases} y_j/p_j - \lambda, & j = 2,4 \\ (y_1+y_3)/p_1 - 2{,}1\cdot\lambda & j=1 \end{cases} \qquad p_j = \begin{cases} y_j/\lambda & j=2,4 \\ \dfrac{(y_1+y_3)}{2{,}1\lambda} & j=1 \end{cases}$$

$$\partial f/\partial\lambda \Rightarrow \lambda = \sum_{i=1}^{4} y_i = m \qquad (\tilde{p}_1, \tilde{p}_2, \tilde{p}_4) = \left(\frac{Y_1+Y_3}{2{,}1\cdot m}, \frac{Y_2}{m}, \frac{Y_4}{m}\right)$$

$$\tilde{p}_3 = 1{,}1\cdot\tilde{p}_1 = \frac{1{,}1}{2{,}1}\cdot\frac{Y_1+Y_3}{m}$$

- $$LR = 2\left[\sum_{j=1}^{4} Y_j \log \frac{Y_j}{m} - Y_1 \cdot \log \frac{Y_1+Y_3}{2{,}1m} - Y_2 \log \frac{Y_2}{m} - Y_3 \log\left[\frac{1{,}1}{2{,}1}\left(\frac{Y_1+Y_3}{m}\right)\right] - Y_4 \log \frac{Y_4}{m}\right]$$

zamietam ak $LR > \chi^2_1(1-\alpha)$

93) cont. b) výsledky z R skriptu

(100)

$LR = 1{,}138$
$R = 1{,}078$ porovnávam s $\chi^2_1(0{,}95) = 3{,}84$ $\Rightarrow$ p-value 0,286
$W = 0{,}979$ 0,299
0,323

nezamietam $H_0: \beta = 0$ voči $H_1: \beta \neq 0$.
Ohr interval spoľahlivosti nájdem z $W = \sqrt{m}\,\hat\beta / \sqrt{J^{11}(\hat\alpha,\hat\beta)} \overset{as}{\sim} N(0,1)$

$$\left[\hat\beta \mp u_{1-\alpha/2}\sqrt{J^{11}(\hat\alpha,\hat\beta)}/\sqrt{m}\right] \approx [-0{,}055;\ 0{,}168]$$

(101)

94) podobne ako v Pr 78 (87) $X \sim R(0,1)$ $Y|X \sim Exp(\lambda(\alpha,\beta,x))$ pre $\lambda(\alpha,\beta,x) = e^{\alpha+\beta x}$

a) $H_0: \beta = 0$ $L(\alpha,\beta) = \prod e^{\alpha+\beta x_i}\exp\{-e^{\alpha+\beta x_i}\cdot y_i\} = e^{\alpha m + \beta\sum x_i}\exp\{-\sum y_i e^{\alpha+\beta x_i}\}$

$\ell(\alpha,\beta) = \alpha m + \beta\sum x_i - \sum y_i e^{\alpha+\beta x_i}$

$U = \left(m - \sum y_i e^{\alpha+\beta x_i},\ \sum x_i - \sum x_i y_i e^{\alpha+\beta x_i}\right)$ $\hat\alpha,\hat\beta$ iba numericky

$$\frac{\partial U}{\partial\theta'} = \begin{pmatrix} -\sum y_i e^{\alpha+\beta x_i} & -\sum y_i x_i e^{\alpha+\beta x_i} \\ -\sum y_i x_i e^{\alpha+\beta x_i} & -\sum y_i x_i^2 e^{\alpha+\beta x_i}\end{pmatrix} \qquad J(\alpha,\beta) = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}$$

$$E[X^k Y e^{\alpha+\beta x}] = E[E(X^k Y e^{\alpha+\beta X}|X)] = E\,X^k \cancel{e^{\alpha+\beta X}}\cdot\frac{1}{\cancel{e^{\alpha+\beta X}}} = EX^k = \begin{cases} 1 & k=0 \\ 1/2 & k=1 \\ 1/3 & k=2 \end{cases}$$

$\tau = \beta$ $J^{-1} = 12\cdot\begin{pmatrix} 1/3 & -1/2 \\ -1/2 & 1\end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -6 & \boxed{12}\end{pmatrix}$ $J^{11} = 12$

- za $H_0: \beta = 0$ $\ell(\alpha) = \alpha m - \sum y_i e^{\alpha}$
  $U(\alpha) = m - \sum y_i e^{\alpha}$ $\Rightarrow$ $\tilde\alpha = \log\frac{m}{\sum Y_i} = -\log\bar Y$
- $LR = 2\left[\hat\alpha m + \hat\beta\sum X_i - \sum Y_i e^{\hat\alpha+\hat\beta X_i} - \tilde\alpha m + \sum Y_i e^{\tilde\alpha}\right]$
- $R = \frac{1}{m}\left(\sum X_i - \sum X_i Y_i e^{\tilde\alpha}\right)^2\cdot 12$ porovnávam s $\chi^2_1(1-\alpha)$
- $W = m(\hat\beta - \beta_0)^2/12 = m\hat\beta^2/12$

b) pre obecné X meníme iba $J(\alpha,\beta) = \begin{pmatrix} EX^0 & EX^1 \\ EX^1 & EX^2\end{pmatrix}$ inak je všetko rovnaké
alebo odhadnem J pozorovanou inf. maticou

(102)

95) preusporiadané multinomické rozdelenie: $p = (p_{11}, p_{12}, p_{21}, p_{22})^T$ ako v Pr 90

$H_0: p_{12} = p_{21}$, Pr 90 b) p_{21} a p_{12} ako p_1 a p_2 tam
$H_1: p_{12} \neq p_{21}$

$$\hat p = \left(\frac{Y_{11}}{m}, \frac{Y_{12}}{m}, \frac{Y_{21}}{m}, \frac{Y_{22}}{m}\right)^T \qquad \tilde p = \left(\frac{Y_{11}}{m}, \frac{Y_{12}+Y_{21}}{2m}, \frac{Y_{12}+Y_{21}}{2m}, \frac{Y_{22}}{m}\right)^T$$

$$LR = 2\left[Y_{12}\log\frac{Y_{12}}{m} + Y_{21}\log\frac{Y_{21}}{m} - (Y_{12}+Y_{21})\log\frac{Y_{12}+Y_{21}}{2m}\right]$$ zamietam $> \chi^2_1(1-\alpha)$

(103)

96) zložitejšia situácia ako v Pr 60 a Pr 79 $(X_i', Y_i)'$ náh. výber

$Y|X=x \sim Po(e^{\alpha+\beta' x})$, X nezávisí na α,β $H_0: \beta = 0_q$ $H_1: \beta \neq 0_q$

$$L(\alpha,\beta) = \prod\lambda(x_i)^{y_i}\cdot\frac{e^{-\sum\lambda(x_i)}}{\prod y_i!} \qquad \ell(\alpha,\beta) = \sum y_i\log\lambda(x_i) - \sum\lambda(x_i) + c$$

$\lambda(x) = e^{\alpha+\beta' x}$ $\frac{\partial}{\partial\theta}\lambda(x) = (\lambda(x),\ x\lambda(x))$

$$U(\alpha,\beta) = \sum \frac{y_i}{\lambda(x_i)} \begin{pmatrix} \lambda(x_i) \\ x_i\lambda(x_i) \end{pmatrix} - \sum \begin{pmatrix} \lambda(x_i) \\ x_i\lambda(x_i) \end{pmatrix} = \sum \begin{pmatrix} y_i \\ y_i x_i \end{pmatrix} - \sum \begin{pmatrix} \lambda(x_i) \\ x_i\lambda(x_i) \end{pmatrix}$$

$\hat\alpha, \hat\beta$ numericky

$$\frac{\partial}{\partial\theta'} U = \begin{pmatrix} -\sum \lambda(x_i) & -\sum x_i'\lambda(x_i) \\ -\sum x_i \lambda(x_i) & -\sum x_i x_i' \lambda(x_i) \end{pmatrix} \qquad J(\alpha,\beta) = \begin{pmatrix} E\lambda(X) & EX'\lambda(X) \\ EX\lambda(X) & EXX'\lambda(X) \end{pmatrix}$$

$J^{11}(\alpha,\beta)$ = podmatica $(2..(q+1), 2..(q+1))$ matice $J(\alpha,\beta)$.

za $H_0: \beta = 0$ $\quad \ell(\alpha) = \sum \alpha y_i - n e^{\alpha} \quad U(\alpha) = \sum y_i - n e^{\alpha} \Rightarrow \tilde\alpha = \log \bar Y$, $\tilde\beta = 0$

- $LR = 2\left[\sum Y_i \log e^{\hat\alpha + \hat\beta' x_i} - \sum e^{\hat\alpha+\hat\beta' x_i} - \sum \log \bar Y \cdot Y_i + n\bar Y\right]$
- $R = \frac{1}{n}\left(\sum (Y_i X_i - X_i e^{\tilde\alpha})\right)' J^{11}(\tilde\alpha, 0)\left(\sum(Y_i X_i - X_i e^{\tilde\alpha})\right)$
- $W = n(\hat\beta - 0)' J^{11}(\hat\alpha,\hat\beta)^{-1}(\hat\beta - 0)$ $\qquad$ porovnám s $\chi^2_q(1-\alpha)$

iv) výsledky z R skriptu

		p-value
$LR = 11{,}21$		0,003
$R = 10{,}94$	porovnávam s $\chi^2_2(1-\alpha) = 5{,}99$	0,004
$W = 8{,}91$		0,011

zamietam $H_0: \beta = (0,0)$

interval spoľahlivosti pre β_1: cez W a jej 1. marginálne rozdelenie $\approx [-0{,}75;\ 1{,}44]$.

97) $X_1 \dots X_m \sim Exp(\lambda)$ $\quad P(X_{(1)} \le t) = 1-(1-P(X_1 \le t)^m)^n = 1-(1-(1-e^{-m\lambda x}))^n =$
$= 1 - e^{-m\lambda x} \sim Exp(m\lambda)$

$L(\lambda) = m\lambda e^{-m\lambda x}$ $\quad \ell(\lambda) = c + \lg m - m\lambda x + \lg \lambda$ $\quad U(\lambda) = \frac{1}{\lambda} - mx \overset{!}{=} 0$

$\frac{1}{\lambda} = mx \Rightarrow \underline{\underline{\hat\lambda = \frac{1}{m X_{(1)}}}}$ $\quad P(m X_{(1)} \le t) = 1 - e^{-m\lambda t/m} = 1 - e^{-\lambda t}$

$\Rightarrow m X_{(1)} \sim Exp(\lambda)$ a $\hat\lambda \sim [Exp(\lambda)]^{-1}$ a nie je konzistentný odhad.

98) a) $(X_i, Y_i) \sim R$ (kruh s polomerom θ) $\quad f(x,y) = \frac{1}{\pi\theta^2} I[x^2+y^2 \le \theta^2]$

$L(\theta) = \prod \frac{1}{\pi\theta^2} I[x_i^2+y_i^2 \le \theta^2] = \left(\frac{1}{\pi\theta^2}\right)^m I[x_i^2 + y_i^2 \le \theta^2 \ \forall i]$

maximalizujem pre čo najmenšie θ s podm. $x_i^2+y_i^2 \le \theta^2 \Rightarrow \underline{\hat\theta = \max_i \sqrt{X_i^2+Y_i^2}}$

b) $(X_i, Y_i) \sim R([-\theta,\theta]^2)$ $\quad f(x,y) = \frac{1}{4\theta^2} I[x_i \in [-\theta,\theta], y_i \in [-\theta,\theta]]$

$\Rightarrow \underline{\underline{\hat\theta = \max_i \{|X_i| \vee |Y_i|\}}}$

99) m = pre rodiny s m detmi $\quad X_1 \dots X_m \sim Bi(m, p)$

a) $L(p) = \prod_{i=1}^m \binom{m}{x_i} p^{x_i}(1-p)^{m-x_i} = \prod \binom{m}{x_i} \cdot p^{\sum x_i}(1-p)^{mm - \sum x_i}$

$\ell(p) = \sum \lg \binom{m}{x_i} + \sum x_i \lg p + (mm - \sum x_i)\lg(1-p)$

$U(p) = \sum x_i / p + - \frac{mm - \sum x_i}{1-p} = 0$

$\sum x_i - \cancel{p\sum x_i} - mmp + \cancel{p\sum x_i} = 0$ $\quad$ Rskript

$\underline{\underline{\hat p = \frac{1}{mm}\sum x_i}} \Rightarrow \hat p = 0{,}514$

b) rodiny s 2 alebo 6 detmi – zmes $Bi(2,p)$ a $Bi(6,p)$ s neurn. zmes.

$\binom{x_i}{c_i}$ $\quad x_i$ – počet chlapcov z m detí $\quad$ náhodný jedinec
$\quad c_i = I[m=6]$ inak $m=2$ $\quad m_1 = \sum c_i$ (počet 6-čl. rodín)

$L(p) = \prod_{i=1}^m \left[\binom{2}{x_i} p^{x_i}(1-p)^{2-x_i}\right]^{1-c_i}\left[\binom{6}{x_i} p^{x_i}(1-p)^{6-x_i}\right]^{c_i} = k \cdot p^{\sum x_i c_i + \sum x_i(1-c_i)}$
$\cdot (1-p)^{\sum(2-x_i)(1-c_i) + \sum(6-x_i)c_i}$

$\ell(p) = k + \sum x_i \lg p + (\underbrace{\sum (2-x_i)}_{2\text{-čl. rod.}} + \underbrace{\sum(6-x_i)}_{6\text{-čl.}})(1-p)$

$U(p) = \frac{\sum x_i}{p} + - \frac{\overbrace{\sum(2-x_i) + \sum(6-x_i)}^{a}}{1-p} \overset{!}{=} 0$ $\quad \sum x_i - \sum x_i p - \cancel{\sum x_i} ap = 0$
$p(n) = \sum x_i$
$\hat p = \frac{\sum x_i}{n} \Rightarrow \underline{\underline{\hat p = 0{,}514}}$

a = celkový počet detí – celkový počet chlapcov
$= $ celkový počet dievčat $= \underset{\text{celk. poč. detí}}{n} - \sum x_i$

$\frac{\text{celk. poč. chlapcov}}{\text{celk. poč. detí}}$

NMST332 Neyman-Pearson a LR test.

73) $X_1 \dots X_n \sim Po(\lambda)$

i) $T_n = \frac{L_n(\lambda_1)}{L_n(\lambda_0)} = \frac{e^{-n\lambda_1}\lambda_1^{\sum X_i}}{e^{-n\lambda_0}\lambda_0^{\sum X_i}} = e^{-n(\lambda_1-\lambda_0)}\left(\lambda_1/\lambda_0\right)^{\sum X_i} \geq c$

$-n(\lambda_1-\lambda_0) + \sum X_i \log(\lambda_1/\lambda_0) \geq \log c$ $\lambda_1 > \lambda_0$, $\log \lambda_1/\lambda_0 > 0$

$\sum X_i \geq [\log c + n(\lambda_1-\lambda_0)] / \log(\lambda_1/\lambda_0) =: k$

k: aby $\alpha \overset{\geq}{=} P_{H_0}(\sum X_i \geq k) \Rightarrow k =$ kvantil $(1-\alpha)$ $Po(n\lambda_0)$

$\sum X_i \overset{H_0}{\sim} Po(n\lambda_0)$

ii) $\sum X_i \leq k \Rightarrow k = \alpha$-kvantil $Po(n\lambda_0)$ pravý test

iii) $\hat\lambda = \bar X$ (str. 50) $\sqrt{n}(\bar X - \lambda) \xrightarrow{d} N(0,\lambda)$

$\sqrt{n}\,\frac{\bar X - \lambda_0}{\sqrt{\lambda_0}} \underset{as}{\overset{H_0}{\sim}} N(0,1)$ test: $\sqrt{n}\,\frac{|\bar X - \lambda_0|}{\sqrt{\lambda_0}} \overset{\text{zamítnu}}{\geq} u_{1-\alpha/2}$

resp. Slutsky: $\frac{\sqrt{n}\,|\bar X - \lambda_0|}{\sqrt{\bar X}} \geq u_{1-\alpha/2}$ resp. $\frac{n(\bar X - \lambda_0)^2}{\bar X} \geq \chi^2_{1,1-\alpha}$ asympt. test.

iv) $LR_n \Rightarrow \frac{L_n(\bar X)}{L_n(\lambda_0)} \Rightarrow LR_n = 2\left[-n(\bar X - \lambda_0) + \sum X_i \log\left(\bar X/\lambda_0\right)\right] \geq \chi^2_{1,1-\alpha}$

76) $X_1 \dots X_n \sim N(\mu_x, \sigma_x^2)$

i) MV: $\hat\mu = \bar X$, $\hat\sigma^2 = \frac{1}{n}\sum (X_i - \bar X)^2$

$\ell_n(\hat\mu, \hat\sigma^2) = c - \frac{n}{2}\log\hat\sigma^2 - \frac{1}{2\hat\sigma^2}\sum(X_i-\bar X)^2 = c - \frac{n}{2}\log\hat\sigma^2 - \frac{n}{2}$

na $H_0: \mu_x = \mu_0$

$\tilde\ell_n(\sigma^2) = c - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum(X_i-\mu_0)^2$

$\tilde U_n(\sigma^2) = \frac{n}{2\sigma^2} + \frac{\sum(X_i-\mu_0)^2}{2\sigma^4} = 0 \Rightarrow \tilde\sigma^2 = \frac{1}{n}\sum(X_i-\mu_0)^2$

$\tilde\ell(\tilde\sigma^2) = c - \frac{n}{2}\log\tilde\sigma^2 - \frac{1}{2\tilde\sigma^2}\sum(X_i-\mu_0)^2 = c - \frac{n}{2}\log\tilde\sigma^2 - \frac{n}{2}$

$LR_n = 2\left[-\frac{n}{2}\log\hat\sigma^2 + \frac{n}{2}\log\tilde\sigma^2\right] = n\log\left[\frac{\tilde\sigma^2}{\hat\sigma^2}\right] \geq \chi^2_1(1-\alpha)$

t-test: $\sqrt{n}(\bar X - \mu) \xrightarrow{d} N(0,\sigma^2)$ $\sqrt{n}\,\frac{\bar X - \mu_0}{\sqrt{S^2}} \overset{H_0}{\sim} t_{n-1}$

ii) na $H_0: \sigma_x^2 = \sigma_0^2$

$\tilde\ell_n(\mu) = c - \frac{n}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\sum(X_i-\mu)^2$

$\tilde U_n(\mu) = \frac{1}{\sigma_0^2}\sum(X_i-\mu) = 0 \Rightarrow \tilde\mu = \frac{1}{n}\sum X_i$

$\tilde\ell_n(\tilde\mu) = c - \frac{n}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\sum(X_i-\bar X)^2$

$LR_n = 2\left[-\frac{n}{2}\log\hat\sigma^2 + \frac{n}{2}\log\sigma_0^2 - \frac{n}{2} + \frac{1}{2\sigma_0^2}\sum(X_i-\bar X)^2\right] = n\log\left[\frac{\sigma_0^2}{\hat\sigma^2}\right] + n\left[\frac{\hat\sigma^2}{\sigma_0^2} - 1\right] \geq \chi^2_1(1-\alpha)$

opačně tomu $\frac{(m-1)S_m^2}{\sigma_0^2} \overset{H_0}{\sim} \chi^2_{m-1}$ $\qquad \frac{(m-1)S_m^2}{\sigma_0^2} = \frac{\sum(X_i-\bar{X})^2}{\sigma_0^2} = \frac{m\hat\sigma^2}{\sigma_0^2}$

iii) na H_0: $\ell_m(\tilde\mu, \tilde\sigma^2) = c - \frac{m}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\sum(X_i-\mu_0)^2$

$LR_m = 2\left[-\frac{m}{2}(\log\hat\sigma^2 - \log\sigma_0^2) - \frac{n}{2} + \frac{1}{2\sigma_0^2}\sum(X_i-\mu_0)^2\right] = m\log\left[\frac{\sigma_0^2}{\hat\sigma^2}\right] - m + \frac{\sum(X_i-\mu_0)^2}{\sigma_0^2}$

$\geq \chi^2_2(1-\alpha)$

iv) z Př 63. $\sqrt{m}\left(\begin{pmatrix}\bar X\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma^2\end{pmatrix}\right) \xrightarrow{d} N_2\left(\begin{pmatrix}0\\0\end{pmatrix}, \overbrace{\begin{pmatrix}\sigma^2 & 0\\ 0 & 2\sigma^4\end{pmatrix}}^{J^{-1}(\mu,\sigma^2)}\right)$

Anděl Věta 4.15: $X \sim N_2(\mu, V)$, $h(V) = 2 \Rightarrow (X-\mu)'V^{-1}(X-\mu) \sim \chi^2_2$

obecně $X \sim N_m(\mu, V)$, $h(V) = r \Rightarrow (X-\mu)'V^{-}(X-\mu) \sim \chi^2_r$

$V := J^{-1}(\mu,\sigma^2)/m \qquad V^{-1} = mJ(\mu,\sigma^2)$

$m\left(\begin{pmatrix}\bar X\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu_0\\ \sigma_0^2\end{pmatrix}\right)' J(\mu_0,\sigma_0^2)\left(\begin{pmatrix}\bar X\\ \hat\sigma^2\end{pmatrix} - \begin{pmatrix}\mu_0\\ \sigma_0^2\end{pmatrix}\right) \overset{H_0}{\approx} \chi^2_2$

odhad $J(\hat\mu, \hat\sigma^2) \Rightarrow W_m = m\left[\frac{(\bar X-\mu_0)^2}{\hat\sigma^2} + \frac{(\hat\sigma^2-\sigma_0^2)^2}{2\hat\sigma^4}\right] \geq \chi^2_2(1-\alpha)$

77) $X \sim Mult(m; p_1, p_2, p_3, p_4)$ $\qquad H_0: p_1 = p_2 = p_3 = p_4 = 1/4 \qquad H_1: \neg H_0$

$L_m(p) = c\, p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$ $\qquad$ MV: Př 67 $\quad \hat p_j = \underbrace{\sum_{i=1}^m X_{ij}}_{x_j}/m \qquad \hat p = \frac{\sum X_i}{m}$

$L_m(p_0) = c\,(1/4)^m$

$LR_m = 2\left[\sum x_j \log\left(\frac{x_j}{m}\right) - m\log 1/4\right] = 2\sum x_j \log\left[\frac{4x_j}{m}\right] \overset{H_0}{\sim} \chi^2_3(1-\alpha)$

$\doteq 327{,}15 > \chi^2_3(1-\alpha) = 7{,}8$ $\qquad$ zamítáme H_0

78) $L_m(\alpha, \beta) = \prod \frac{\lambda(x_i)^{y_i} e^{-\lambda(x_i)}}{y_i!}$ $\qquad f_x(x_i) = \frac{\prod\lambda(x_i)^{y_i} \cdot e^{-\sum\lambda(x_i)}}{\prod y_i!}$ $\qquad \frac{\partial\lambda}{\partial\alpha} = \lambda \quad \frac{\partial\lambda}{\partial\beta} = \lambda x$

$\ell_m(\alpha,\beta) = \sum y_i \log\lambda(x_i) - \sum\lambda(x_i) + c$ $\qquad \lambda(x) = \cancel{\alpha+\beta' x}\ e^{\alpha+\beta' x}$

$\frac{\partial\ell_m}{\partial\alpha} = \sum\frac{y_i \cancel{x_i}}{\cancel{\lambda(x_i)}} - \sum\lambda_i = 0$

~~$\sum\frac{y_i}{\alpha+\beta x_i}$~~

numericky

na H_0: $\tilde\ell_m(\alpha, 0) = y\alpha\sum y_i - m\alpha$

$\frac{\partial\tilde\ell}{\partial\alpha} = \frac{\sum y_i}{\alpha} - m = 0$

$\tilde\alpha = \bar Y$

$\frac{\partial\ell_m}{\partial\beta} = \sum\frac{y_i \cancel{x_i} x}{\cancel{\lambda(x_i)}} - \sum\lambda x_i = 0$

$LR_m = 2\left[\sum y_i (\hat\alpha + \hat\beta' x_i) - \sum\exp(\hat\alpha + \hat\beta' x_i) - \sum y_i \log\bar y + m\bar y\right] \geq \chi^2_q(1-\alpha)$

$\hat\alpha = \cancel{-0{,}52}\ -1{,}85 \qquad \hat\beta_1 = \cancel{-0{,}51}\ 0{,}51 \qquad \hat\beta_2 = \cancel{1{,}87}\ 0{,}33$

$LR_m = 11{,}\cancel{81}\ 21 > \chi^2_2(0{,}95) = 5{,}99$ $\qquad$ zamítáme $H_0: (\beta_1, \beta_2)' = (0,0)'$

$\chi^2_1(1-\alpha)$