NMST 434, Exercise session 1: Asymptotics

February 21, 2020

Example 5: Asymptotic distribution of Pearson's correlation coefficient

Derivation directly using a single function g: $R^5 \to R$ in the Δ-method.

$a = \frac{1}{n}\Sigma X_i$; $b = \frac{1}{n}\Sigma Y_i$; $c = \frac{1}{n}\Sigma X_i^2$; $d = \frac{1}{n}\Sigma Y_i^2$; $e = \frac{1}{n}\Sigma X_i Y_i$.

Assumption: WLOG $E X_1 = E Y_1 = 0$ and $\text{Var } X_1 = \text{Var } Y_1 = 1$.

```
Clear[μ];
g[a_, b_, c_, d_, e_] = (e - a b) / ((c - a^2) (d - b^2))^(1/2);
Dg =
  D[g[a, b, c, d, e], {{a, b, c, d, e}}] /. {a → 0, b → 0, c → 1, d → 1, e → ρ} // Simplify;
MatrixForm[Dg] (* gradient of g at E(a,b,c,d,e) *)
Σ = {{1, ρ, μ[3, 0], μ[1, 2], μ[2, 1]}, {ρ, 1, μ[2, 1], μ[0, 3], μ[1, 2]},
   {μ[3, 0], μ[2, 1], μ[4, 0] - 1, μ[2, 2] - 1, μ[3, 1] - μ[1, 1]},
   {μ[1, 2], μ[0, 3], μ[2, 2] - 1, μ[0, 4] - 1, μ[1, 3] - μ[1, 1]},
   {μ[2, 1], μ[1, 2], μ[3, 1] - μ[1, 1], μ[1, 3] - μ[1, 1], μ[2, 2] - μ[1, 1]^2}};
(* original variance matrix, μ[i,j] is a way of writing the expectation E (X^i Y^j) *)
MatrixForm[Σ]
Dg . Σ . Dg // Simplify(* asymptotic variance of ρ̂ *)
Dg . Σ . Dg /. ρ → 0 (* asymptotic variance under independence *)
```

$$\begin{pmatrix} 0 \\ 0 \\ -\frac{\rho}{2} \\ -\frac{\rho}{2} \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & \rho & \mu[3,0] & \mu[1,2] & \mu[2,1] \\ \rho & 1 & \mu[2,1] & \mu[0,3] & \mu[1,2] \\ \mu[3,0] & \mu[2,1] & -1+\mu[4,0] & -1+\mu[2,2] & -\mu[1,1]+\mu[3,1] \\ \mu[1,2] & \mu[0,3] & -1+\mu[2,2] & -1+\mu[0,4] & -\mu[1,1]+\mu[1,3] \\ \mu[2,1] & \mu[1,2] & -\mu[1,1]+\mu[3,1] & -\mu[1,1]+\mu[1,3] & -\mu[1,1]^2+\mu[2,2] \end{pmatrix}$$

$$2\rho\,\mu[1,1] - \mu[1,1]^2 + \mu[2,2] - \rho\left(\mu[1,3]+\mu[3,1]\right) + \frac{1}{4}\rho^2\left(-4+\mu[0,4]+2\,\mu[2,2]+\mu[4,0]\right)$$

$$-\mu[1,1]^2 + \mu[2,2]$$

Under normality

```
μ[i_, j_] = Moment[MultinormalDistribution[{0, 0}, {{1, ρ }, {ρ , 1}}], {i, j}];
Dg . Σ . Dg /. {a → 0, b → 0, c → 1, d → 1, e → ρ} //
 Simplify (* asymptotic variance of ρ̂ *)
```

$$\left(-1+\rho^2\right)^2$$

Example 7 : Asymptotic distribution of $\hat{\theta}$ in MA (1) process

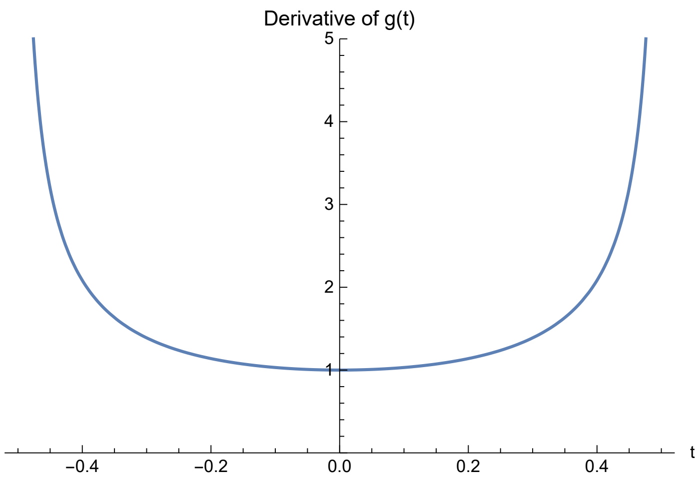
The direct approach.

* not done at the exercise session - run the script yourself.

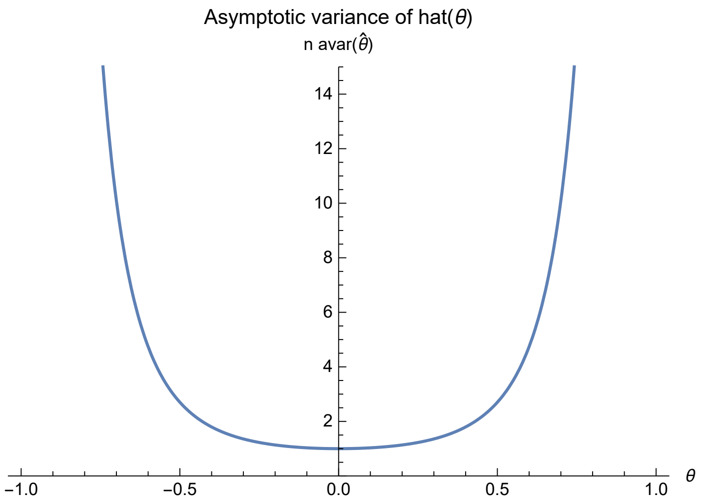
```
$Assumptions = -1 < θ < 1;
Solve[θ/(1 + θ^2) == t, θ] // Simplify
g[t_] = (1 - √(1 - 4 t^2))/(2 t);
D[g[t], t] // Simplify
Plot[%, {t, -1, 1}, PlotRange → {0, 5},
 PlotLabel → "Derivative of g(t)", AxesLabel → {"t"}]
Bartlett = 1 - 3 ρ^2 + 4 ρ^4 /. ρ → θ/(1 + θ^2);
(* The variance from the Bartlett formula - which version is correct? *)
(D[g[t], t])^2 * Bartlett /. t → θ/(1 + θ^2) // Simplify
Plot[% // Evaluate, {θ, -1, 1}, PlotRange → {0, 15},
 PlotLabel → "Asymptotic variance of hat(θ)", AxesLabel → {"θ", "n avar(θ̂)"}]
```

$$\left\{\left\{\theta \to -\frac{-1+\sqrt{1-4t^2}}{2t}\right\}, \left\{\theta \to \frac{1+\sqrt{1-4t^2}}{2t}\right\}\right\}$$

$$\frac{-1+\frac{1}{\sqrt{1-4t^2}}}{2t^2}$$



$$\frac{1+\theta^2+4\theta^4+\theta^6+\theta^8}{\left(-1+\theta^2\right)^2}$$



The approach using the inverse function.

```
m[θ_] = θ/(1 + θ^2);
D[m[θ], θ]
Bartlett / (%)^2 // Simplify
$Assumptions = True;
```

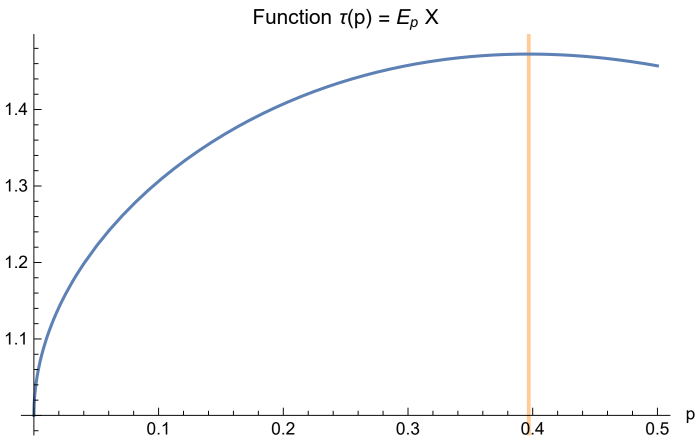
$$-\frac{2\,\theta^2}{\left(1+\theta^2\right)^2}+\frac{1}{1+\theta^2}$$

$$\frac{1+\theta^2+4\,\theta^4+\theta^6+\theta^8}{\left(-1+\theta^2\right)^2}$$

Example 9 : MoM in multinomial distribution

The “direct” approach.

```
$Assumptions = 0 < p < 1/2;
τ[p_] = 1 - p^2 + √p;  (* E X *)
Plot[τ[p], {p, 0, 1/2}, PlotLabel → "Function τ(p) = E_p X",
 AxesLabel → {"p"}, GridLines → {{{4^(-2/3), {Orange, Thick}}}, {}}]
Solve[τ[p] == m, p] (* hat(p) *)
VarX = 1 - p^2 - √p + 4 √p - τ[p]^2 // Simplify (* variance of X *)
VarX (D[τ[p], p])^-2 // Simplify (* asymptotic variance of p̂ *)
Plot[%, {p, 0, 1/2}, PlotLabel → "Asymptotic variance of p̂",
 AxesLabel → {"p", "n avar(p̂)"}]
Solve[D[τ[p], p] == 0, p] (* avar(p̂) explodes at this point *)
```



$$
\begin{aligned}
&\Bigg\{\Bigg\{p \to -\frac{1}{2}\sqrt{\Bigg(-\frac{4}{3}(-1+m) + \Big(16\times 2^{1/3}(1-2m+m^2)\Big)\Big/} \\
&\quad \Big(3\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big) + \\
&\quad \frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Bigg) - \\
&\quad \frac{1}{2}\sqrt{\Bigg(-\frac{8}{3}(-1+m) - \Big(16\times 2^{1/3}(1-2m+m^2)\Big)\Big/} \\
&\quad \Big(3\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big) - \\
&\quad \frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3} - \\
&\quad 2\Big/\Bigg(\sqrt{\Bigg(-\frac{4}{3}(-1+m) + \Big(16\times 2^{1/3}(1-2m+m^2)\Big)\Big/\Big(3\Big(155-384\,m+384\,m^2 -}} \\
&\quad 128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\Big)^{1/3}\Big) + \frac{1}{3\times 2^{1/3}} \\
&\quad \left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\}, \\
&\Bigg\{p \to -\frac{1}{2}\sqrt{\Bigg(-\frac{4}{3}(-1+m) + \Big(16\times 2^{1/3}(1-2m+m^2)\Big)\Big/} \\
&\quad \Big(3\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big) + \\
&\quad \frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Bigg) + \\
&\quad \frac{1}{2}\sqrt{\Bigg(-\frac{8}{3}(-1+m) - \Big(16\times 2^{1/3}(1-2m+m^2)\Big)\Big/} \\
&\quad \Big(3\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big) - \\
&\quad \frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3} - \\
&\quad 2\Big/\Bigg(\sqrt{\Bigg(-\frac{4}{3}(-1+m) + \Big(16\times 2^{1/3}(1-2m+m^2)\Big)\Big/\Big(3\Big(155-384\,m+384\,m^2 -}} \\
&\quad 128\,m^3+3\sqrt{3}\sqrt{283-768\,m+768\,m^2-256\,m^3}\Big)^{1/3}\Big) + \frac{1}{3\times 2^{1/3}}
\end{aligned}
$$

$$\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big)\Big)\Big)\Big\},$$

$$\Big\{p\to\frac{1}{2}\sqrt{\Big(-\frac{4}{3}\left(-1+m\right)+\left(16\times 2^{1/3}\left(1-2\,m+m^2\right)\right)\Big/}$$

$$\left(3\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\right)+$$

$$\frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big)-$$

$$\frac{1}{2}\sqrt{\Big(-\frac{8}{3}\left(-1+m\right)-\left(16\times 2^{1/3}\left(1-2\,m+m^2\right)\right)\Big/}$$

$$\left(3\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\right)-$$

$$\frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}+$$

$$2\Big/\Big(\sqrt{\Big(-\frac{4}{3}\left(-1+m\right)+\left(16\times 2^{1/3}\left(1-2\,m+m^2\right)\right)\Big/\Big(3\Big(155-384\,m+384\,m^2-}$$

$$128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\Big)^{1/3}\Big)+\frac{1}{3\times 2^{1/3}}$$

$$\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big)\Big)\Big)\Big\},$$

$$\Big\{p\to\frac{1}{2}\sqrt{\Big(-\frac{4}{3}\left(-1+m\right)+\left(16\times 2^{1/3}\left(1-2\,m+m^2\right)\right)\Big/}$$

$$\left(3\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\right)+$$

$$\frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big)+$$

$$\frac{1}{2}\sqrt{\Big(-\frac{8}{3}\left(-1+m\right)-\left(16\times 2^{1/3}\left(1-2\,m+m^2\right)\right)\Big/}$$

$$\left(3\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\right)-$$

$$\frac{1}{3\times 2^{1/3}}\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}+$$

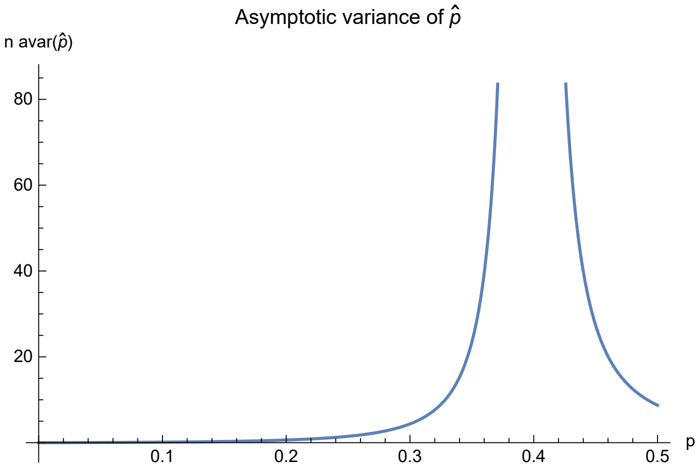
$$2\Big/\Big(\sqrt{\Big(-\frac{4}{3}\left(-1+m\right)+\left(16\times 2^{1/3}\left(1-2\,m+m^2\right)\right)\Big/\Big(3\Big(155-384\,m+384\,m^2-}$$

$$128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\Big)^{1/3}\Big)+\frac{1}{3\times 2^{1/3}}$$

$$\left(155-384\,m+384\,m^2-128\,m^3+3\,\sqrt{3}\,\sqrt{283-768\,m+768\,m^2-256\,m^3}\right)^{1/3}\Big)\Big)\Big)\Big\}\Big\}$$

$$\sqrt{p}-p+p^2+2\,p^{5/2}-p^4$$

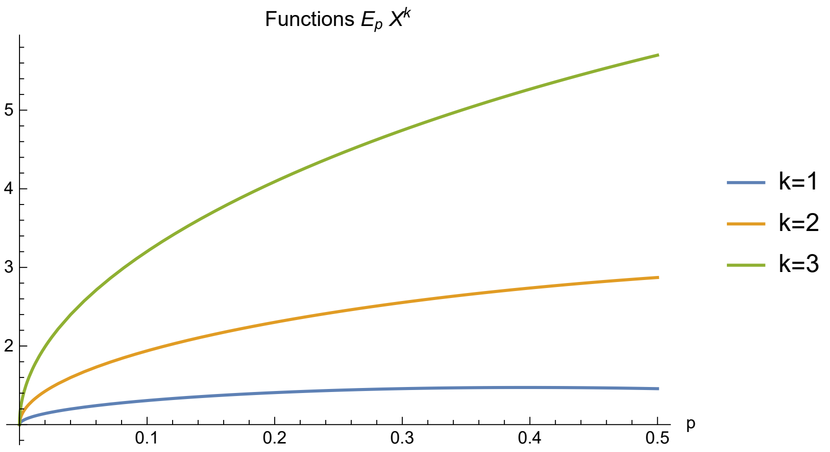
$$\frac{4\,p\left(\sqrt{p}-p+p^2+2\,p^{5/2}-p^4\right)}{\left(1-4\,p^{3/2}\right)^2}$$

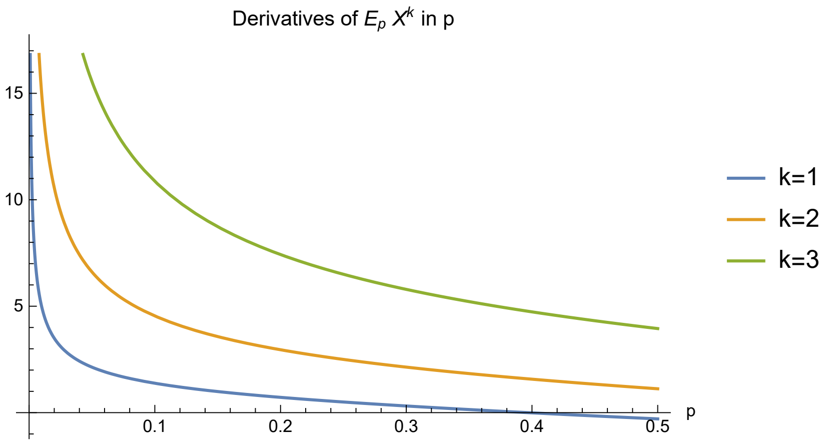


$$\left\{\left\{p \to \frac{1}{2 \times 2^{1/3}}\right\}\right\}$$

A correct approach.

```
Clear[k];
μ[p_, k_] = 0^k p^2 + 1^k (1 - p^2 - √p) + 2^k √p;
Plot[Table[μ[p, k], {k, 1, 3}] // Evaluate, {p, 0, 1/2},
 PlotLabel → "Functions E_p X^k", AxesLabel → {"p"}, PlotLegends → {"k=1", "k=2", "k=3"}]
Plot[Table[D[μ[p, k], p], {k, 1, 3}] // Evaluate,
 {p, 0, 1/2}, PlotLabel → "Derivatives of E_p X^k in p",
 AxesLabel → {"p"}, PlotLegends → {"k=1", "k=2", "k=3"}]
```






```
k = 2;
Solve[μ[p, k] == m, p]
VarX = μ[p, 2 * k] - μ[p, k]^2 // Simplify
VarX / (D[μ[p, k], p])^2 // Simplify
Plot[%, {p, 0, 1/2}, PlotLabel → "Asymptotic variance of p̂",
 AxesLabel → {"p", "n avar(p̂)"}, PlotRange → {0, 2}]
$Assumptions = True;
```

$$\Bigg\{\Bigg\{p \to -\frac{1}{2}\sqrt{\left(-\frac{4}{3}(-1+m) + \left(16\times 2^{1/3}\left(1-2m+m^2\right)\right)\Big/\left(3\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right)} -$$

$$\frac{1}{2}\sqrt{\left(-\frac{8}{3}(-1+m) - \left(16\times 2^{1/3}\left(1-2m+m^2\right)\right)\Big/\right.}$$

$$\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) - \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3} - 18\Big/\left(\sqrt{\left(-\frac{4}{3}(-1+m) + \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right)}\right)\Big)\Big\},$$

$$\Big\{p \to -\frac{1}{2}\sqrt{\left(-\frac{4}{3}(-1+m) + \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right)} + \frac{1}{2}\sqrt{\left(-\frac{8}{3}(-1+m) - \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) - \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3} - 18\Big/\left(\sqrt{\left(-\frac{4}{3}(-1+m) + \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right)}\right)\right)}\Big\},$$

$$\Big\{p \to \frac{1}{2}\sqrt{\left(-\frac{4}{3}(-1+m) + \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right)} - \frac{1}{2}\sqrt{\left(-\frac{8}{3}(-1+m) - \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) - \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3} + 18\Big/\left(\sqrt{\left(-\frac{4}{3}(-1+m) + \left(16\times 2^{1/3}\left(1 - 2\,m + m^2\right)\right)\Big/\left(3\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right) + \frac{1}{3\times 2^{1/3}}\left(2315 - 384\,m + 384\,m^2 - 128\,m^3 + 27\sqrt{3}\sqrt{2443 - 768\,m + 768\,m^2 - 256\,m^3}\right)^{1/3}\right)}\right)\right)}\Big\},$$

$$\Bigg\{\Bigg\{p \to \frac{1}{2}\sqrt{\left(-\frac{4}{3}\left(-1+m\right)+\left(16\times 2^{1/3}\left(1-2m+m^2\right)\right)\Big/\left(3\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right)+\frac{1}{3\times 2^{1/3}}\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right)}+$$

$$\frac{1}{2}\sqrt{\left(-\frac{8}{3}\left(-1+m\right)-\left(16\times 2^{1/3}\left(1-2m+m^2\right)\right)\Big/\left(3\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right)-\frac{1}{3\times 2^{1/3}}\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}+18\Big/\left(\sqrt{\left(-\frac{4}{3}\left(-1+m\right)+\left(16\times 2^{1/3}\left(1-2m+m^2\right)\right)\Big/\left(3\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right)+\frac{1}{3\times 2^{1/3}}\left(2315-384m+384m^2-128m^3+27\sqrt{3}\sqrt{2443-768m+768m^2-256m^3}\right)^{1/3}\right)}\right)\right)}\Bigg\}\Bigg\}$$

$$9\sqrt{p}-9p+p^2+6p^{5/2}-p^4$$

$$\frac{4p\left(9\sqrt{p}-9p+p^2+6p^{5/2}-p^4\right)}{\left(3-4p^{3/2}\right)^2}$$