Ex 14) $X_1, \dots, X_n$ i.i.d $U(0,\theta)$, $\theta > 0$

$$f(x;\theta) = \frac{1}{\theta} \mathbb{1}\{x \in \langle 0,\theta \rangle\}$$

$$L_n(\theta) = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}\{X_i \in \langle 0,\theta \rangle\} \quad \searrow \text{ in } \theta$$

$$\rightsquigarrow \hat{\theta}_n = \underset{\theta>0}{\arg\max}\, L_n(\theta) = \max_{1 \le i \le n}\{X_i\}$$

$$P(n(\hat{\theta}_n - \theta) \le x) = P\left(\hat{\theta}_n \le \theta + \frac{x}{n}\right)$$

$$= P\left(\max_{1\le i\le n}\{X_i\} \le \theta + \frac{x}{n}\right)$$

$$= P\left(X_1 \le \theta + \frac{x}{n}, \dots, X_n \le \theta + \frac{x}{n}\right) = \left[F_{X_1}\left(\theta + \frac{x}{n}\right)\right]^n$$

$$\left[F_{X_1}(x) = P(X_1 \le x) = \frac{x}{\theta}, \; x \in (0,\theta)\right]$$

$$\underset{x<0}{=} \left[\frac{\theta + \frac{x}{n}}{\theta}\right]^n = \left[1 + \frac{x}{n\theta}\right]^n \xrightarrow[n\to\infty]{} e^{\frac{x}{\theta}}$$

$$\rightsquigarrow n(\hat{\theta}_n - \theta) \xrightarrow[n\to\infty]{d} Z, \text{ where } Z \text{ has a cum. d.f.}$$

$$F_Z(x) \begin{cases} e^{\frac{x}{\theta}}, & x < 0 \\ 1, & x \ge 0 \end{cases}$$

Ex 15) $X_1, \dots, X_n$ i.i.d. $N(\theta, 1)$; $\hat{\theta}_n^{(S)} \begin{cases} 0, & |\overline{X}_n| \le n^{-1/4} \\ \overline{X}_n, & |\overline{X}_n| > n^{-1/4} \end{cases}$

FROM MS2: $\hat{\theta}_n = \overline{X}_n$ is MLE & $\overline{X}_n$ is the best unb. estimator

$\sqrt{n}(\overline{X}_n - \theta) \xrightarrow[n\to\infty]{d} N(0,1)$

Let $\underline{\theta_X \neq 0}$: $\overline{X}_n \xrightarrow[n\to\infty]{P} \theta_X \neq 0 \rightsquigarrow P(|\overline{X}_n| \le n^{-1/4}) \xrightarrow[n\to\infty]{} 0$

$$\rightsquigarrow P(\hat{\theta}_n^{(S)} = \overline{X}_n) \xrightarrow[n\to\infty]{} 1$$

$$\rightarrow \sqrt{n}(\hat{\theta}_n^{(S)} - \theta_X) \xrightarrow[n\to\infty]{d} N(0,1)$$

Let $\underline{\theta_X = 0}$:

$$P(|\overline{X}_n| \le n^{-1/4}) = P(\sqrt{n}|\overline{X}_n - 0| \le n^{+1/4})$$

$$\xrightarrow[n\to\infty]{} 1 \quad \rightsquigarrow P(\hat{\theta}_n^{(S)} = 0) \xrightarrow[n\to\infty]{} 1$$

$$\rightarrow n^a \hat{\theta}_n^{(S)} \xrightarrow[n\to\infty]{P} 0 \quad \text{for } \forall a > 0$$

$\rightsquigarrow$ FOR $\theta_X = 0$ $\hat{\theta}_n^{(S)}$ IS „BETTER" THAN $\overline{X}_n$

BUT CONSIDER $\theta_X^{(n)} = n^{-1/4}$

$$\rightsquigarrow \overline{X}_n \sim N\left(n^{-1/4}, \frac{1}{n}\right)$$

$$\rightarrow P(|\overline{X}_n| \le n^{-1/4}) = P(\overline{X}_n \in (-n^{-1/4}, n^{-1/4}))$$

$$= P(\sqrt{n}(\overline{X}_n - n^{-1/4}) \in (-2n^{1/4}, 0)) \xrightarrow[n\to\infty]{} \frac{1}{2}$$

$\rightsquigarrow \forall K > 0$:

$$P(|\sqrt{n}(\hat{\theta}_n^{(S)} - n^{-1/4})| \ge K) \ge P(|\sqrt{n}(\hat{\theta}_n^{(S)} - n^{-1/4})| \ge K, \hat{\theta}_n^{(S)} = 0)$$

$$= P(n^{1/4} \ge K, \hat{\theta}_n^{(S)} = 0) \xrightarrow[n\to\infty]{} \frac{1}{2}$$

$\rightsquigarrow \sqrt{n}(\hat{\theta}_n^{(S)} - n^{-1/4})$ DOES NOT CONVERGE IN DISTRIB.

Ex 12) $X_1, \dots, X_n$ i.i.d $N(\mu, 1)$, $\mu \in [0, \infty)$

$\rightsquigarrow$ MLE $\hat{\mu}_n = \max\{\overline{X}_n, 0\}$

Let $\underline{\mu_* > 0}$: $P(\hat{\mu}_n = \overline{X}_n) \xrightarrow[n\to\infty]{} 1$

$\rightsquigarrow \sqrt{n}(\hat{\mu}_n - \mu_*) \xrightarrow[n\to\infty]{d} N(0,1)$

Let $\underline{\mu_* = 0}$: $P(\overline{X}_n < 0) = \frac{1}{2} \rightsquigarrow P(\hat{\mu}_n = \overline{X}_n) = \frac{1}{2}$

$\rightsquigarrow \sqrt{n}(\hat{\mu}_n - 0) \xrightarrow[n\to\infty]{d} Z$,

where $F_Z(z) = P(Z \leq z) = \begin{cases} 0, & z < 0 \\ \frac{1}{2}, & z = 0 \\ \Phi(z), & z > 0 \end{cases}$

Ex 18) $X_1, \dots, X_n$ i.i.d. $\frac{1}{2} N(0,1) + \frac{1}{2} N(\theta, \exp\{-\frac{2}{\theta^2}\})$, $\theta > 0$

$$\ell_n(\theta) = \sum_{i=1}^{n} \log\left(\frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{X_i^2}{2}} + \frac{1}{2} \frac{e^{-\frac{(X_i-\theta)^2}{2e^{-2/\theta^2}}}}{\sqrt{2\pi e^{-\frac{2}{\theta^2}}}} \right)$$

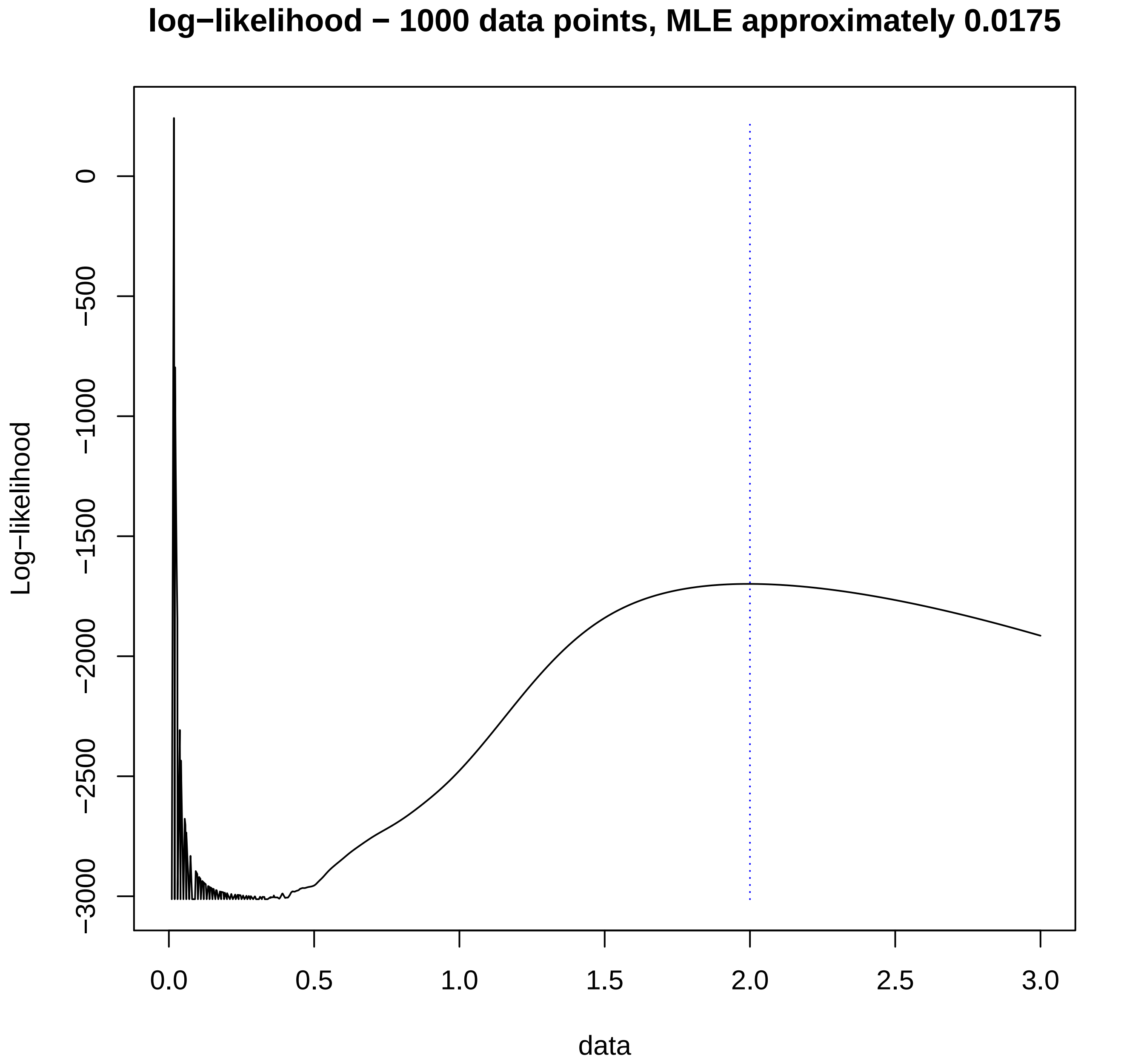
$$\hat{\theta}_n^{(ML)} = \underset{\theta > 0}{\arg\max}\ \ell_n(\theta)$$

$$\leadsto \hat{\theta}_n^{(ML)} \xrightarrow[n \to \infty]{P} 0 < \theta_X$$

$\boxed{\text{Th 5}}$: $\exists\ \hat{\theta}_n : U_n(\hat{\theta}_n) \overset{!}{=} 0$

SUCH THAT $\hat{\theta}_n \xrightarrow[n\to\infty]{P} \theta_X$

$+ \sqrt{n}(\hat{\theta}_n - \theta_X) \xrightarrow[n\to\infty]{d} N\left(0, \frac{1}{J(\theta_X)}\right)$



log−likelihood − 1000 data points, MLE approximately 0.0175
Log−likelihood
0
−500
−1000
−1500
−2000
−2500
−3000
0.0
0.5
1.0
1.5
2.0
2.5
3.0
data

Ex 19) $X_1, \dots, X_n$ i.i.d. "$\frac{1}{2} N(0,1) + \frac{1}{2} N(\mu, \sigma^2)$"

$$\ell_n(\mu,\sigma^2) = \sum_{i=1}^{n} \log\left(\frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{X_i^2}{2}} + \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_i-\mu)^2}{2\sigma^2}} \right)$$

NOTE THAT FOR $\mu = X_1$:

$$\ell_n(X_1,\sigma^2) \geq \log\left(\frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(X_1-X_1)^2}{2\sigma^2}} \right) + \sum_{i=2}^{n} \log\left(\frac{1}{2} \frac{1}{\sqrt{2\pi}} e^{-\frac{X_i^2}{2}} \right)$$

$$= -\frac{1}{2} \log \sigma^2 + C \xrightarrow[\sigma^2 \searrow 0_+]{} +\infty$$

$$\leadsto \sup_{(\mu,\sigma^2) \in \mathbb{R} \times \mathbb{R}_+} \ell_n(\mu,\sigma^2) = +\infty$$

$\boxed{\text{Th 5}}$: $\exists\, \hat{\mu}_n, \hat{\sigma}_n^2$:

$$\frac{\partial \ell_n(\mu,\sigma^2)}{\partial \mu} \overset{!}{=} 0$$

$$\frac{\partial \ell_n(\mu,\sigma^2)}{\partial \sigma^2} \overset{!}{=} 0$$

SUCH THAT:

$$\begin{pmatrix} \hat{\mu}_n \\ \hat{\sigma}_n^2 \end{pmatrix} \xrightarrow[n\to\infty]{P} \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix}$$

$$+ \sqrt{n} \left[\begin{pmatrix} \hat{\mu}_n \\ \hat{\sigma}_n^2 \end{pmatrix} - \begin{pmatrix} \mu \\ \sigma^2 \end{pmatrix} \right] \xrightarrow[n\to\infty]{d} N_2\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathcal{I}^{-1}(\mu,\sigma^2) \right)$$

Ex 20) $\underset{\sim}{X}_1, \dots, \underset{\sim}{X}_n$ i.i.d $\mathrm{Mult}_K(1, \underset{\sim}{p})$, $\underset{\sim}{p} = (p_1, \dots, p_K)^T$

$(X_{11}, \dots, X_{1k})^T$

$(p_1^0, \dots, p_k^0)^T$

$H_0: \underset{\sim}{p}_X = \underset{\sim}{p}^0 \qquad H_1: \underset{\sim}{p}_X \neq \underset{\sim}{p}^0$

DICE

$\underset{\sim}{p}^0 = \underbrace{\left(\frac{1}{6}, \dots, \frac{1}{6}\right)}_{6\times}^T$

MS 1: $X^2 = \sum_{k=1}^{K} \frac{(n_k - n p_k^0)^2}{n p_k^0} \underset{H_0}{\overset{as}{\sim}} \chi^2_{K-1}$,

REJECT H_0 $\Leftrightarrow X^2 \geq \chi^2_{K-1}(1-\alpha)$

where $n_k = \sum_{i=1}^{n} X_{ik}$

MLE APPROACH:

$L_n(\underset{\sim}{p}) = \prod_{i=1}^{n} \left(p_1^{X_{1i}} \cdot \dots \cdot p_K^{X_{iK}}\right) = p_1^{n_1} \cdot \dots \cdot p_K^{n_K}$

$l_n(\underset{\sim}{p}) = \sum_{k=1}^{K} n_k \log p_k$

$\frac{\partial l_n(\underset{\sim}{p})}{\partial p_k} = \frac{n_k}{p_k} \overset{!}{=} 0$, $k = 1, \dots, K$??

$\rightsquigarrow$ $\overset{!}{\nabla}$ $p_K = 1 - p_1 - \dots - p_{K-1}$ $\boxed{R3}$ $\overset{!}{\nabla}$

$\rightsquigarrow l_n(p_1, \dots, p_{K-1}) = \sum_{k=1}^{K-1} n_k \log p_k + n_K \log(1 - \underbrace{p_1 - \dots - p_{K-1}}_{p_K})$

$\frac{\partial l_n(\dots)}{\partial p_k} = \frac{n_k}{p_k} - \frac{n_K}{p_K} \overset{!}{=} 0$, $k = 1, \dots, K-1$

$\rightsquigarrow \hat{p}_k = \frac{n_k}{n}$, $k = 1, \dots, K \rightsquigarrow \hat{\underset{\sim}{p}}_n = \begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_K \end{pmatrix}$

$LR_n = 2\left(l_n(\hat{\underset{\sim}{p}}_n) - l_n(\underset{\sim}{p}^0)\right)$

$= 2\left(\sum_{k=1}^{K} n_k \log\left(\frac{n_k}{n}\right) - \sum_{k=1}^{K} n_k \log(p_k^0)\right)$

$= 2 \sum_{k=1}^{K} n_k \log\left(\frac{n_k}{n p_k^0}\right)$

$\boxed{\text{RST}}$ $$R_n = \frac{1}{n} \underset{\sim}{U}_n^T(\underset{\sim}{p}^0_{-k})\, I^{-1}(\underset{\sim}{p}^0_{-k})\, \underset{\sim}{U}_n(\underset{\sim}{p}^0_{-k}),$$

where $\underset{\sim}{p}^0_{-k} = \begin{pmatrix} p^0_1 \\ \vdots \\ p^0_{k-1} \end{pmatrix}$

CLT: $$\sqrt{n}\left(\begin{pmatrix} \hat{p}_1 \\ \vdots \\ \hat{p}_{k-1} \end{pmatrix} - \begin{pmatrix} p_1 \\ \vdots \\ p_{k-1} \end{pmatrix} \right) \xrightarrow[n\to\infty]{d} N_{k-1}\left(0, \Sigma(\underset{\sim}{p}_{-k})\right),$$

where $$\Sigma(\underset{\sim}{p}_{-k}) = \operatorname{diag}(\underset{\sim}{p}_{-k}) - \underset{\sim}{p}_{-k}\underset{\sim}{p}_{-k}^T \overset{\text{Th5}}{=} I^{-1}(\underset{\sim}{p}_{-k})$$

$$\underset{\sim}{U}_n(\underset{\sim}{p}_{-k}) = \begin{pmatrix} \frac{n_1}{p_1} - \frac{n_k}{p_k} \\ \vdots \\ \frac{n_{k-1}}{p_{k-1}} - \frac{n_k}{p_k} \end{pmatrix}$$

$$\leadsto R_n = \frac{1}{n}\left(\frac{n_1}{p^0_1} - \frac{n_k}{p^0_k}, \dots, \frac{n_{k-1}}{p^0_{k-1}} - \frac{n_k}{p_k}\right)\left[\operatorname{diag}(\underset{\sim}{p}^0_{-k}) - \underset{\sim}{p}^0_{-k}(\underset{\sim}{p}^0_{-k})^T\right]$$
$$\left(\frac{n_1}{p^0_1} - \frac{n_k}{p^0_k}, \dots, \frac{n_{k-1}}{p^0_{k-1}} - \frac{n_k}{p^0_k}\right)$$

$$= \frac{1}{n}(A_n - B_n), \text{ where}$$

$$A_n = \sum_{\ell=1}^{k-1}\left(\frac{n_\ell}{p^0_\ell} - \frac{n_k}{p^0_k}\right) p^0_\ell \left(\frac{n_\ell}{p^0_\ell} - \frac{n_k}{p^0_k}\right)$$
$$= \left(\sum_{\ell=1}^{k-1} \frac{n_\ell^2}{p^0_\ell}\right) - 2\frac{n_k}{p^0_k}(n - n_k) + \frac{n_k^2}{[p^0_k]^2}(1 - p^0_k)$$

$$B_n = \sum_{\ell=1}^{k-1}\sum_{\ell'=1}^{k-1}\left(\frac{n_\ell}{p^0_\ell} - \frac{n_k}{p^0_k}\right) p^0_\ell p^0_{\ell'} \left(\frac{n_{\ell'}}{p^0_{\ell'}} - \frac{n_k}{p^0_k}\right)$$

$$= \sum_{\ell=1}^{k-1}\sum_{\ell'=1}^{k-1} n_\ell n_{\ell'} - \frac{2n_k}{p^0_k}\sum_{\ell=1}^{k-1} p^0_\ell \sum_{\ell'=1}^{k-1} n_{\ell'}$$
$$+ \frac{n_k^2}{[p^0_k]^2}\sum_{\ell=1}^{k-1}\sum_{\ell'=1}^{k-1} p^0_\ell p^0_{\ell'}$$

$$= (n - n_k)^2 - \frac{2n_k}{p^0_k}(1 - p_k)(n - n_k)$$
$$+ \frac{n_k^2}{[p^0_k]^2}(1 - p_k)(1 - n_k)$$

$$R_n = \frac{1}{n}\left(A_n - B_n\right) =$$

$$= \frac{1}{n}\left[\sum_{l=1}^{k-1} \frac{n_l^2}{n p_l^0} - (n - n_k)^2 - 2 n_k (n - n_k) + \frac{n_k^2}{n p_k^0}(1 - p_k^0)\right]$$

$$= \frac{1}{n}\left[\sum_{l=1}^{k-1} \frac{n_l^2}{n p_l^0} - n^2 + 2 n n_k - n_k^2 - 2 n_k n + 2 n_k^2 + \frac{n_k^2}{n p_k^0} - n_k^2\right]$$

$$= \frac{1}{n} \sum_{l=1}^{K} \frac{n_l^2}{n p_l^0} - n$$

ON THE OTHER HAND

$$X^2 = \sum_{l=1}^{K} \frac{(n_l - n p_l^0)^2}{n p_l^0} =$$

$$= \sum_{l=1}^{K} \frac{n_l^2}{n p_l^0} - 2 \sum_{l=1}^{K} n_l + \sum_{l=1}^{K} n p_l^0$$

$$= \frac{1}{n} \sum_{l=1}^{K} \frac{n_l^2}{p_l^0} \underbrace{- 2n + n}_{-n} = R_n$$

SIMILARLY:

$$W_n = n\left(\hat{p}_{-k} - p_{-k}\right)^T I_n\left(\hat{p}_{-k}\right)\left(\hat{p}_{-k} - p_{-k}\right) = \ldots$$

$$= \sum_{l=1}^{K} \frac{(n_l - n p_l^0)^2}{n_l}$$