3-Manifolds: Triangulations, Algorithms and Topological Obstructions

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What Is a Manifold?

Informally, ***d*-manifolds** are the *d*-dimensional analogues of surfaces.

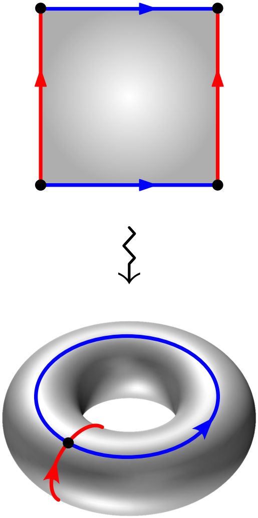
At any point they locally “look like” the *d*-dimensional Euclidean space.

We consider two manifolds **equivalent** if they are **homeomorphic**.

In this talk we are mostly concerned with **compact 3-manifolds**.

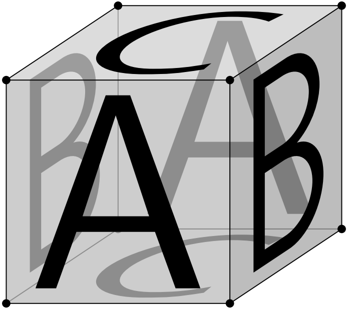
Source: PxHere (CC0 Public Domain)

Example: The 2- and 3-Dimensional Tori



3-dimensional torus

$\mathbb{T}^3 = \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$



The Classification Problem of Manifolds

Many fundamental questions in topology are **decision problems**

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Homeomorphism Problem (HP$_d$). Given two triangulations $\mathcal{T}_1$ and $\mathcal{T}_2$ of d-manifolds, algorithmically decide whether they are homeomorphic.

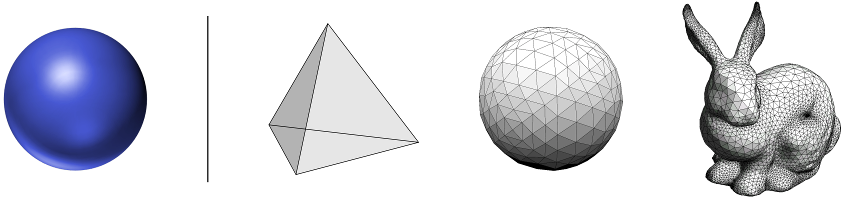

Image Credits: Wikimedia Commons (tetrahedron), Eeo Jun (triangulated sphere) and Daniel Rypl (Stanford bunny)

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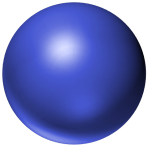

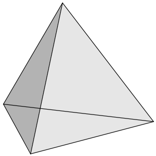

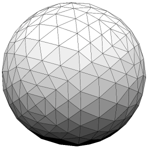

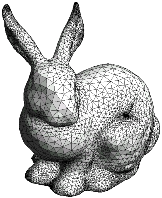

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$\boldsymbol{d} = \mathbf{2}$: ✓ (easy). $\boldsymbol{d} = \mathbf{3}$: ✓ (very complicated). $\boldsymbol{d} \geqslant \mathbf{4}$: Undecidable.

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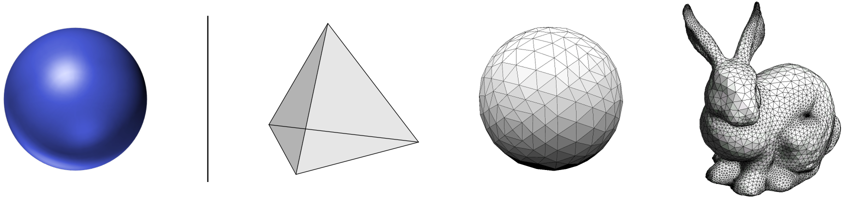

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$\boldsymbol{d = 2}$: ✓ (easy). $\boldsymbol{d = 3}$: ✓ (very complicated). $\boldsymbol{d \geqslant 4}$: Undecidable.

The decidability of HP_2 follows from the classification of closed, orientable surfaces via the **Euler characteristic**, an easily computable invariant.

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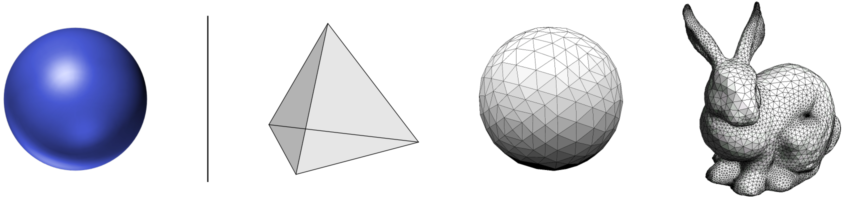

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Undecidability of HP_d ($d \geqslant 4$) follows from the **undecidability of GROUP TRIVIALITY** [Adyan, 1957; Rabin, 1958] via a reduction [Markov, 1958].

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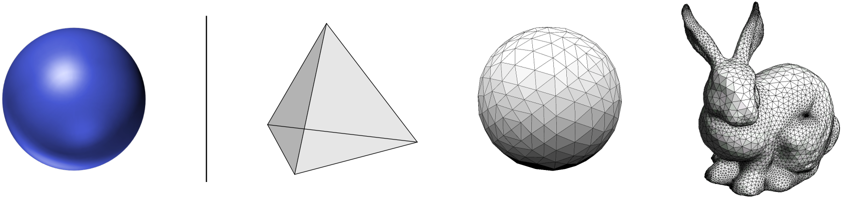

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Follows from **Geometrization** [Perelman, 2002].

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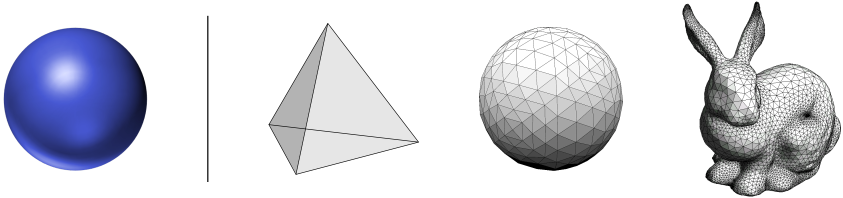

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Follows from **Geometrization** [Perelman, 2002]. At least as hard as *Graph Isomorphism* [Lackenby, 2017], but *elementary recursive* [Kuperberg, 2019].

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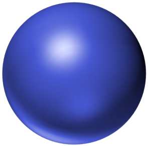

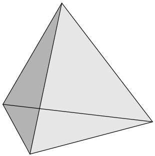

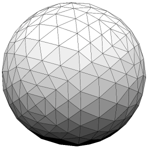

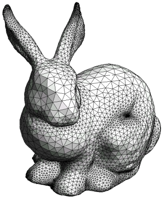

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- If $\mathcal{T}_1 \cong \mathcal{T}_2$: Try to relate them via a sequence of *Pachner moves*.
- If $\mathcal{T}_1 \ncong \mathcal{T}_2$: Try to distinguish them via **computable invariants**.

3-Manifolds: Triangulations and Their Dual Graphs

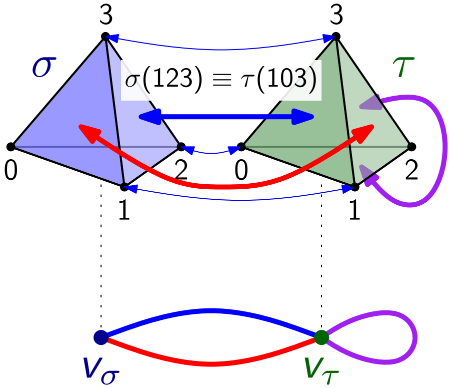
Theorem (Moise; 1952). Every 3-manifold has a **triangulation**.

$\mathcal{T}$	Finitely many **tetrahedra** glued along **triangular faces**.

Dual graph (dual 1-skeleton)

$\Gamma(\mathcal{T})$	**vertices**: tetrahedra of $\mathcal{T}$ **edges**: face gluings

(multigraph, vertex degrees ≤ 4)



Note. Most 3-manifolds **can't** be embedded in the 3-dimensional space!

(*) $13,399$ compact orientable 3-manifolds can be triangulated with $\leqslant 11$ tetrahedra.

Motivation: Compute Invariants Efficiently from "simple" input

$\mathcal{T}$: n-tetrahedron triangulation, $t = \mathrm{tw}(\Gamma(\mathcal{T}))$ is the **treewidth** of $\Gamma(\mathcal{T})$

ALGORITHM	RUNNING TIME	CITATION
taut angle structures of ideal triangulations	$O(7^t \cdot n)$	Burton–Spreer 2013
Turaev–Viro invariants for parameter $r \geq 3$	$O((r-1)^{6(t+1)} t^2 \log r \cdot n)$	Burton–Maria–Spreer 2015
optimal Morse matchings in the Hasse diagram of $\mathcal{T}$	$O(4^{t^2+t} t^3 \log t \cdot n)$	Burton–Lewiner–Paixão–Spreer 2016
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$\Downarrow$

linear in n

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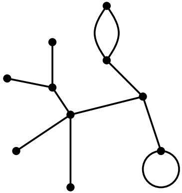
$\operatorname{tw}(\Gamma(\mathcal{T})) \leq t$ fixed $\Downarrow$ linear in n

Guiding Question. Given a 3-manifold $\mathcal{M}$, how **small** can $\operatorname{tw}(\Gamma(\mathcal{T}))$ be?

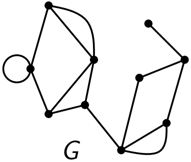
A variant of this question: [Makowsky–Mariño, 2003] and [Burton, 2015].

The Treewidth of Graphs and 3-Manifolds

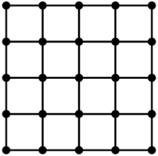
The **treewidth** $\operatorname{tw}(G)$ quantifies the similarity of G to any tree.



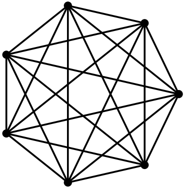
$\operatorname{tw}(\text{tree}) = 1$



$\operatorname{tw}(G) = 2$



$\operatorname{tw}(k \times k\text{-grid}) = k$



$\operatorname{tw}(K_n) = n - 1$

- Key concept in **graph minor theory** developed by Robertson and Seymour between 1983–2004 (20 papers, 500+ pages).
- Cornerstone of **parametrized complexity theory** (since the 1970s).
- A **zoo of width parameters for graphs**: cutwidth, pathwidth, etc.

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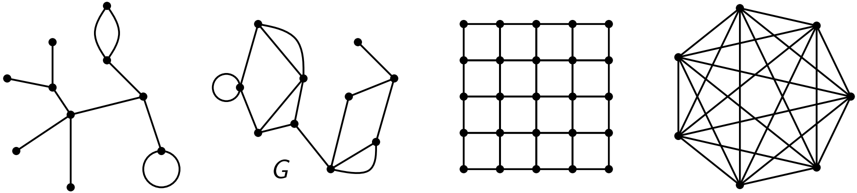

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Let us define the **treewidth of a 3-manifold $\mathcal{M}$** as

$$\text{tw}(\mathcal{M}) = \min\{\text{tw}(\Gamma(\mathcal{T})) : \mathcal{T} \text{ is a triangulation of } \mathcal{M}\}.$$

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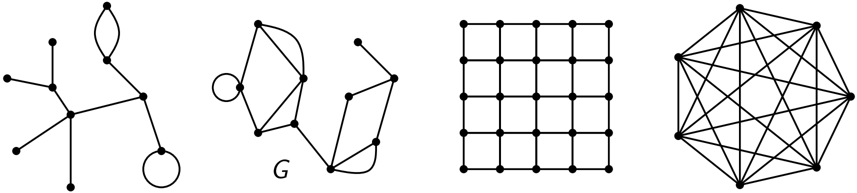

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Goal. Understand the **quantitative relationship** between the *treewidth*, *pathwidth*, etc. and classical topological invariants of 3-manifolds.

Results, I.

Treewidth versus Heegaard genus

Joint work with **Jonathan Spreer** and **Uli Wagner**
University of Sydney IST Austria

Treewidth versus Heegaard Genus

Theorem 1 (H–Spreer–Wagner, 2019**).** Let $\mathcal{M}$ be closed, orientable, *irreducible*, *non-Haken*. Its Heegaard genus and treewidth satisfy

$$\mathfrak{g}(\mathcal{M}) \leqslant 18\,(\mathrm{tw}\,(\mathcal{M}) + 1)\,.$$

Theorem (Agol, 2003**).** There exist closed, orientable, irreducible, and non-Haken 3-manifolds of arbitrary large Heegaard genus.

Corollary There exist 3-manifolds with arbitrary large treewidth.

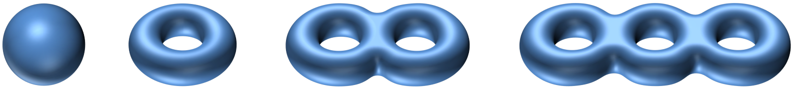
Theorem 2 (H–Spreer, 2019**).** For $\mathcal{M}$ closed and orientable we have

$$\mathrm{tw}\,(\mathcal{M}) \leqslant 4\mathfrak{g}(\mathcal{M}) - 2.$$

Corollary For non-Haken 3-manifolds we have $\mathrm{tw}\,(\mathcal{M}) \approx \mathfrak{g}(\mathcal{M})$.

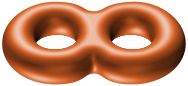
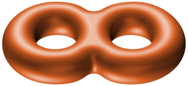
The Heegaard Genus of a 3-Manifold

A **handlebody of genus g** is a solid body with g holes.

 . . .

Theorem. Every* 3-manifold can be obtained as a **Heegaard splitting** i.e., two handlebodies of the same genus with their boundaries identified.

(*) Every compact, orientable 3-manifold can be obtained as a Heegaard splitting.

$\mathcal{M} =$  $\cup_{f_1}$ 

$\mathcal{M} =$  $\cup_{f_2}$ 

The **Heegaard genus** $\mathfrak{g}(\mathcal{M})$ is the minimum genus of any Heegaard splitting of $\mathcal{M}$.

Proof of Theorem 1

Theorem 1 (H–Spreer–Wagner, 2019**).** Let $\mathcal{M}$ be closed, orientable, irreducible, non-Haken. Then we have $\mathfrak{g}(\mathcal{M}) \leqslant 18\,(\mathrm{tw}\,(\mathcal{M}) + 1)$.

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Reminder. $\mathrm{tw}\,(\mathcal{M}) = \min\{\mathrm{tw}\,(\Gamma(\mathcal{T})) : \mathcal{T}$ is a triangulation of $\mathcal{M}\}$.

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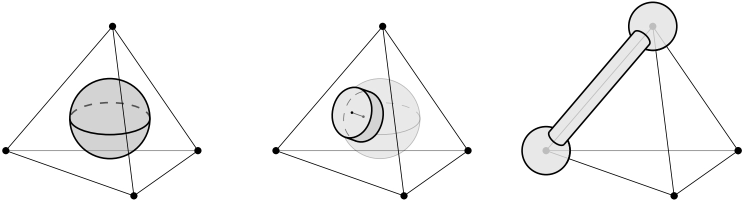

0-handle 1-handle 2- and 3-handles

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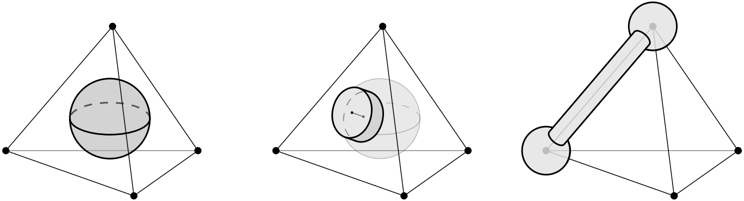

0-handle 1-handle 2- and 3-handles

$\mathcal{H}_1 = \{\text{0-handles}\} \cup \{\text{1-handles}\}$
$\mathcal{H}_2 = \{\text{2-handles}\} \cup \{\text{3-handles}\}$ $\rightsquigarrow$ $\mathcal{M} = \mathcal{H}_1 \cup \mathcal{H}_2,\ \partial\mathcal{H}_1 = \partial\mathcal{H}_2 = \mathcal{S}$

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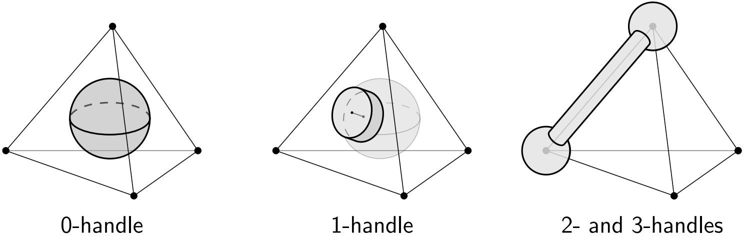

Problem If $\mathcal{T}$ has n tetrahedra, then $g(\mathcal{S}) = n + 1 \Rightarrow$ Too large!

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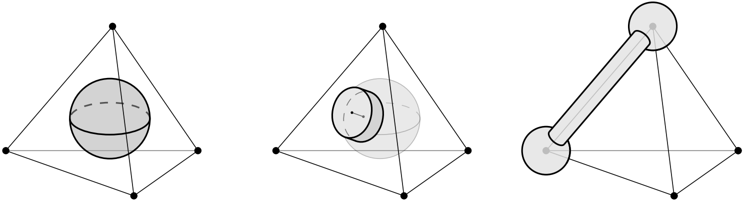

0-handle 1-handle 2- and 3-handles

Step 2 We rebuild $\mathcal{M}$ from these handles, attaching them in a specific order, so that the genus of each *intermediate bounding surface* is small.

Proof of Theorem 1 (cont'd)

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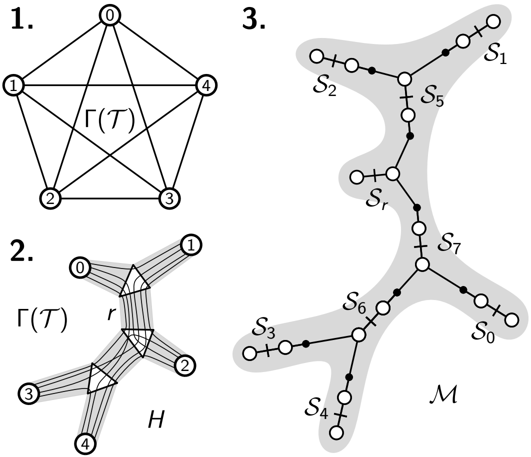

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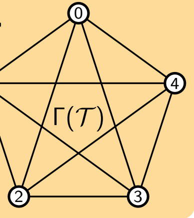

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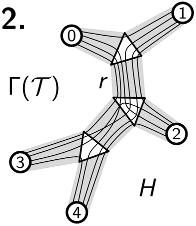
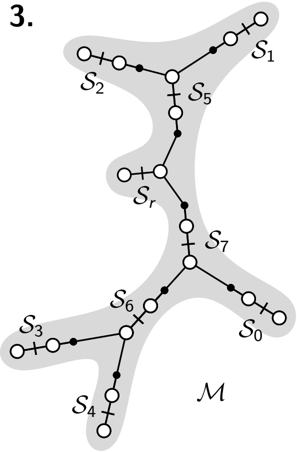


3.



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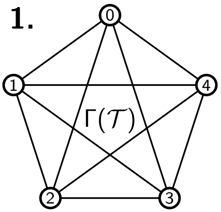

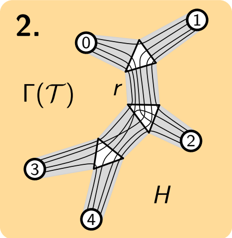

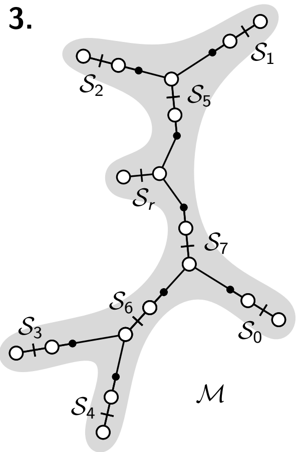

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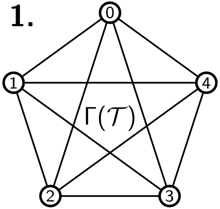

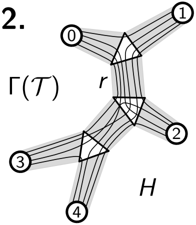

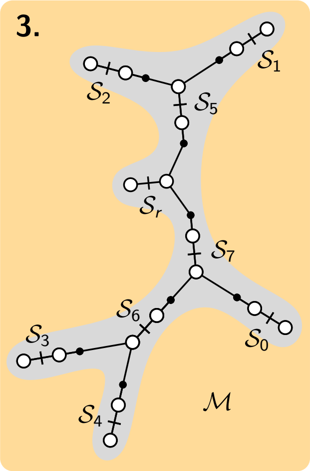

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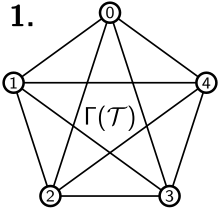

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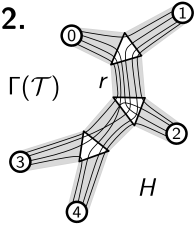

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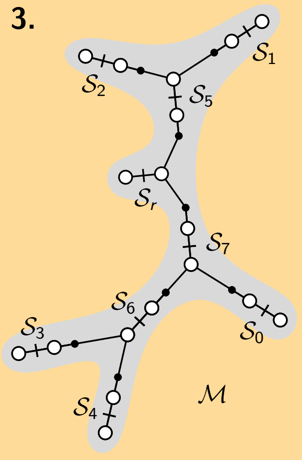

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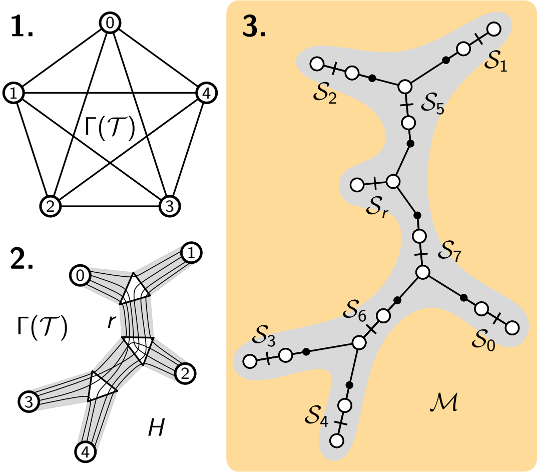

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Results, II.

Treewidth versus Torus Decompositions

Joint work with **Jonathan Spreer**
University of Sydney

Prime Decomposition of Surfaces and 3-Manifolds

Theorem (Classification of Surfaces**).** A closed, connected, orientable surface $\mathcal{S}$ is either homeomorphic to $\mathbb{S}^2$ or to a *connected sum* of tori.

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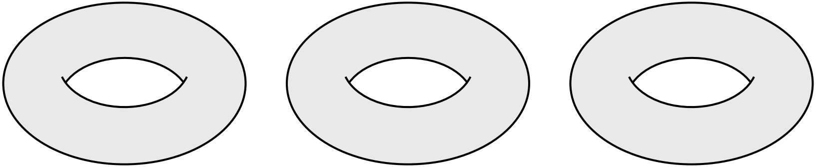

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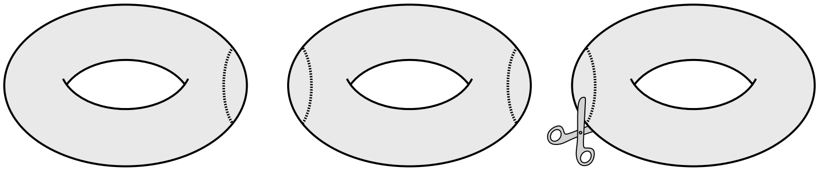

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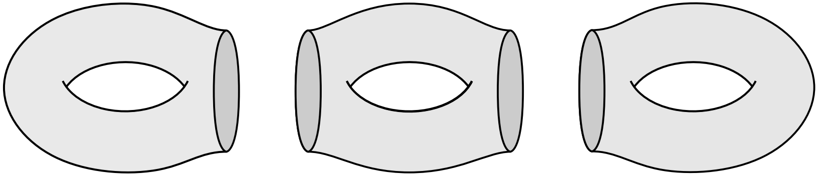

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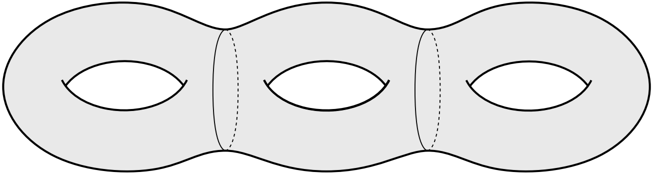

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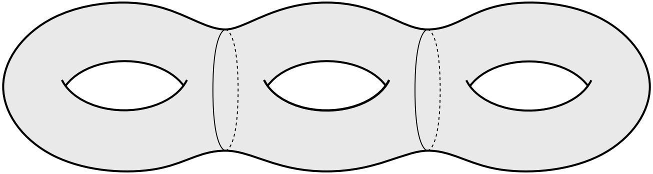

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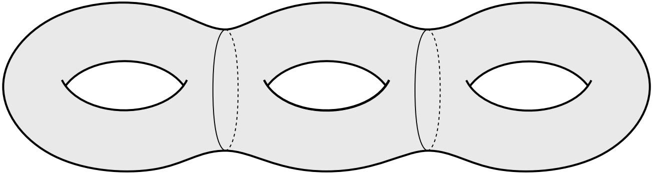

In the realm of 3-manifolds, the **connected sum** is taken along 2-spheres.

Theorem (Prime Decomposition of 3-Manifolds; Kneser '29, Milnor '62**).** Every closed, connected and oriented 3-manifold $\mathcal{M}$ can be decomposed as a connected sum $\mathcal{M} = \mathcal{M}_1 \# \cdots \# \mathcal{M}_k$ of *prime 3-manifolds* $\mathcal{M}_i$. Moreover, the summands of this decomposition are uniquely determined.

The Torus Decomposition of Prime 3-Manifolds

Theorem (Torus Decomposition Theorem; Jaco–Shalen, Johannson '79**).**
Given a prime 3-manifold $\mathcal{M}$, there is a **canonical family $\mathbb{T}$ of pairwise disjoint tori** embedded in $\mathcal{M}$, decomposing $\mathcal{M}$ into "simpler pieces."

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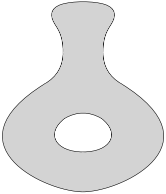

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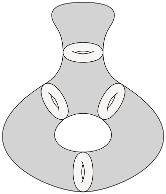

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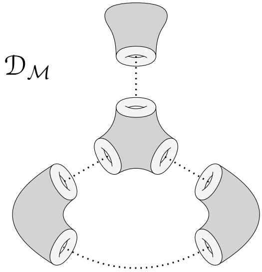

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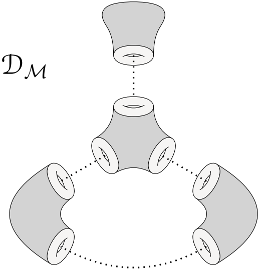

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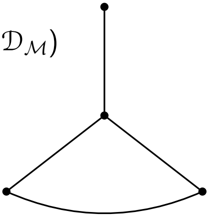

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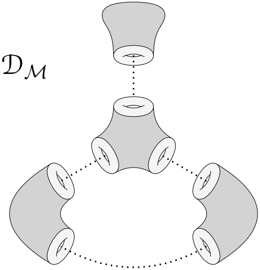
{Path,Tree}width and Torus Decompositions

Theorem 2 (H–Spreer, 2023)**.** For any closed, orientable and prime 3-manifold $\mathcal{M}$ with *"sufficiently complicated"* torus gluings in its torus decomposition $\mathcal{D}_{\mathcal{M}}$, the following inequalities are satisfied:

$$\mathrm{tw}\left(\Gamma(\mathcal{D}_{\mathcal{M}})\right) \leqslant 18(\mathrm{tw}\left(\mathcal{M}\right) + 1) \quad \text{and} \tag{1}$$

$$\mathrm{pw}(\Gamma(\mathcal{D}_{\mathcal{M}})) \leqslant 12\,\mathrm{pw}(\mathcal{M}) + 4. \tag{2}$$

$\mathcal{D}_{\mathcal{M}}$



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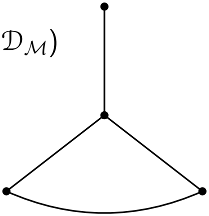

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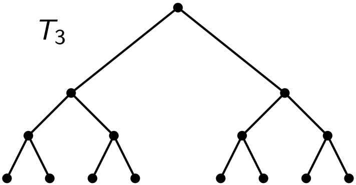
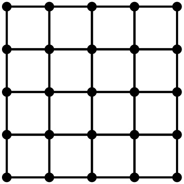
Applications

1. Family of bounded-treewidth 3-manifolds with arbitrary large pathwidth.
To our knowledge, this is the first construction of such a family of 3-manifolds.

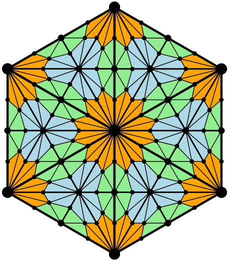
2. Haken 3-manifolds with arbitrary large treewidth.
Previously, the existence of such 3-manifolds was only known in the non-Haken case (see [H–Spreer–Wagner, 2019]).





1. $\mathrm{tw}(T_h) = 1$, $\mathrm{pw}(T_h) = \lceil h/2 \rceil$ **2.** $\mathrm{tw}(k \times k\text{-grid}) = k$

Merry Christmas! Veselé Vánoce!



https://kristofhuszar.github.io

The Treewidth of a Graph – The Precise Definition

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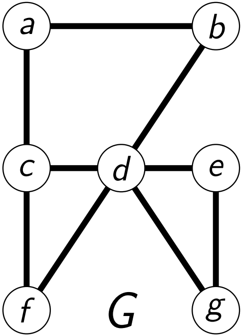

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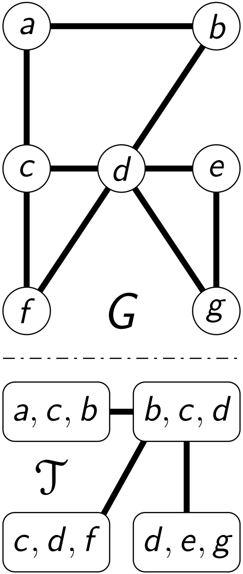

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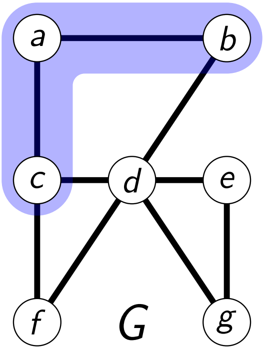

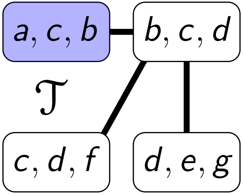

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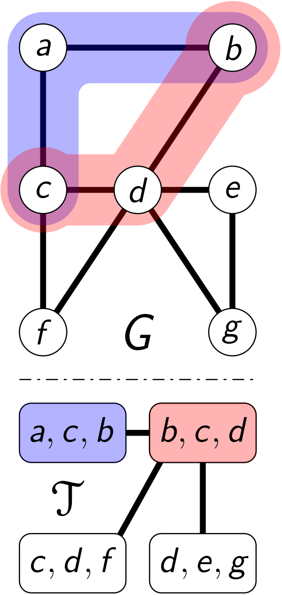

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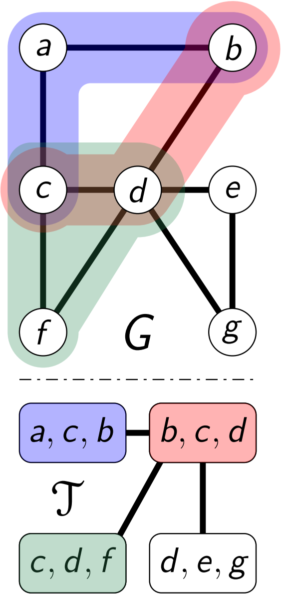

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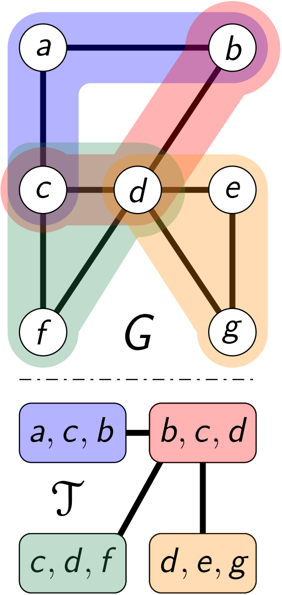

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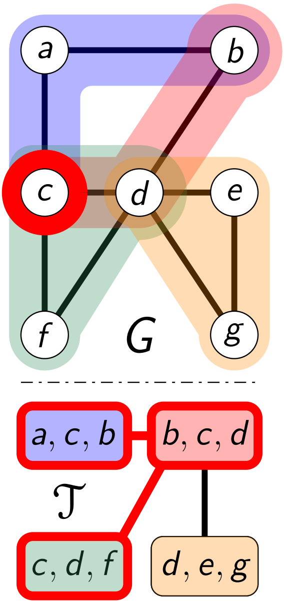
2. $\forall \{u, v\} \in E \quad \exists i : \{u, v\} \subseteq B_i$
3. $\forall v \in V$, the B_i's containing v induce a connected subgraph of T.

Treewidth $\mathrm{tw}(G) = \min_{\mathcal{T}}\{\max_i |B_i|\} - 1.$

The Treewidth of a Graph – The Precise Definition



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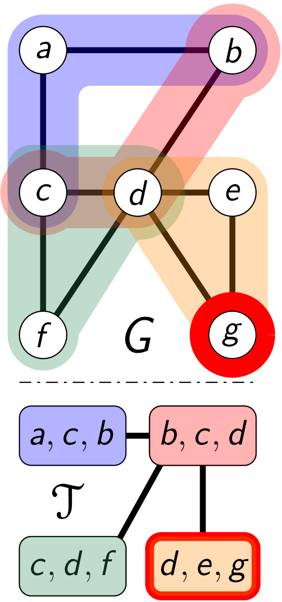

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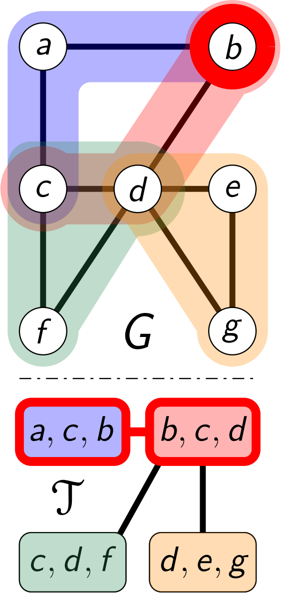

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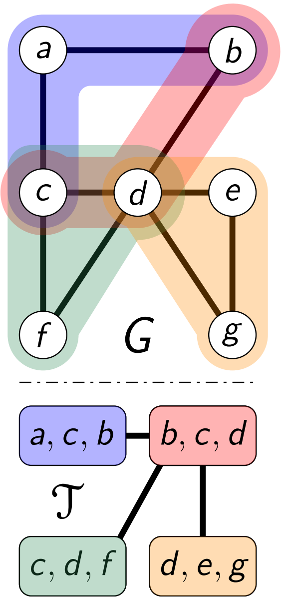

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Pathwidth $\mathrm{pw}(G)$ is defined analogously, but the min is taken over $\mathcal{T}$ where T is a path.