Program Synthesis for the OEIS

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October 24, 2023

Discovering patterns in mathematical objects

- Discovery in 1995 of a more efficent formula for generating the digits of π by Simon Plouffe.
- In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie published their construction of a Hadamard matrix of order 428.
- In 2012, Geoffrey Exoo has found an edge colorings of K35 that have no complete graphs of order 4 in the first color, and no complete graphs of order 6 in the second color proving that $R(4, 6) \geqslant 36$.
- Discovery in 2023 of a chiral aperiodic monotile by David Smith.

OEIS: $\geqslant$ 350000 sequences

The OEIS is supported by the many generous donors to the OEIS Foundation.

THE ON-LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

2 3 5 7 11 | Search | Hints

(Greetings from The On-Line Encyclopedia of Integer Sequences!)

Search: **seq:2,3,5,7,11**

Displaying 1-10 of 1163 results found. page 1 2 3 4 5 6 7 8 9 10 ... 117

Sort: relevance | references | number | modified | created Format: long | short | data

A000040 The prime numbers. +30
(Formerly M0652 N0241) 10150

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271 (list; graph; refs; listen; history; text; internal format)

OFFSET 1,1

COMMENTS See A065091 for comments, formulas etc. concerning only odd primes. For all information concerning prime powers, see A000961. For contributions concerning "almost primes" see A002808.
A number p is prime if (and only if) it is greater than 1 and has no positive divisors except 1 and p.
A natural number is prime if and only if it has exactly two (positive) divisors.
A prime has exactly one proper positive divisor, 1.

A **synthesize and test** approach

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$f(x) = (x \times x + x) \div 2$$

Test/Filter:

$$f(0) = 0,\ f(1) = 1,\ f(2) = 3,\ f(3) = 6,\ \ldots,\ f(53) = 1431$$

Test: criteria for selecting programs

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

An undesirable large program

```
if x = 0 then 0 else
if x = 1 then 1 else
if x = 2 then 3 else
if x = 3 then 6 else ...
if x >= 53 then 1431
```

Small program (Occam's Razor)

$$f(x) = \sum_{i=1}^{x} i$$

Fast program (efficiency criterion)

$$f(x) = (x \times x + x) \div 2$$

Possible other criteria: usefulness criterion ?

Synthesize: a Turing-complete language

- Constants: $0, 1, 2$
- Variables: x, y
- Arithmetical operators: $+, -, \times, div, mod$
- Condition: if $\ldots \leqslant 0$ then ... else ...
- $loop(f, a, b) := u_a$ where $u_0 = b$,

$$u_n = f(u_{n-1}, n)$$

- Two other loop constructs: *loop2*, a while loop

Example:

$$2^{\mathbf{x}} = \prod_{y=1}^{x} 2 = loop(2 \times x, \mathbf{x}, 1)$$
$$\mathbf{x}! = \prod_{y=1}^{x} y = loop(y \times x, \mathbf{x}, 1)$$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$($$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(\, x$$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(\; x \; \times$$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(\, x \times x$$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(\; x \times x +$$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

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Synthesize: tokens by tokens

OEIS sequence

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Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(\ x \ \times \ x \ + \ x \ ) \ \div$$

Synthesize: tokens by tokens

OEIS sequence

$$S = 0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(\; x \; \times \; x \; + \; x \;) \; \div \; 2$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3} \quad \times_{0.12}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3} \quad \times_{0.12} \quad x_{0.99}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3} \quad \times_{0.12} \quad x_{0.99} \quad +_{0.1}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3} \quad \times_{0.12} \quad x_{0.99} \quad +_{0.1} \quad x_{0.25}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3} \quad \times_{0.12} \quad x_{0.99} \quad +_{0.1} \quad x_{0.25} \quad)_{0.48}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \quad x_{0.3} \quad \times_{0.12} \quad x_{0.99} \quad +_{0.1} \quad x_{0.25} \quad)_{0.48} \quad \div_{0.02}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2}\ \ x_{0.3}\ \ \times_{0.12}\ \ x_{0.99}\ \ +_{0.1}\ \ x_{0.25}\ \ )_{0.48}\ \ \div_{0.02}\ \ 2_{0.09}$$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \dots, 1431$$

Synthesized program

$$(_{0.2}\ \ x_{0.3}\ \ \times_{0.12}\ \ x_{0.99}\ \ +_{0.1}\ \ x_{0.25}\ \ )_{0.48}\ \ \div_{0.02}\ \ 2_{0.09}$$

The probability of generating this program is:
$0.2 \times 0.3 \times 0.12 \times 0.99 \times 0.1 \times 0.25 \times 0.48 \times 0.02 \times 0.09 = 1.54... \times 10^{-7}$

Synthesize: probabilistically

OEIS sequence

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

Synthesized program

$$(_{0.2} \;\; x_{0.3} \;\; \times_{0.12} \;\; x_{0.99} \;\; +_{0.1} \;\; x_{0.25} \;\;)_{0.48} \;\; \div_{0.02} \;\; 2_{0.09}$$

The probability of generating this program is:
$0.2 \times 0.3 \times 0.12 \times 0.99 \times 0.1 \times 0.25 \times 0.48 \times 0.02 \times 0.09 = 1.54... \times 10^{-7}$

In general, we are given a probability function $\mathcal{P}(S, P, T)$.

$$\mathcal{P}([0, 1, 3, 6, \ldots] \,,\; [] \,,\; \text{“(”} \;) = 0.2$$

$$\mathcal{P}([0, 1, 3, 6, \ldots] \,,\; [\text{“(”}] \,,\; x \;) = 0.3$$

$$\mathcal{P}([0, 1, 3, 6, \ldots] \,,\; [\text{“(”}, x] \,,\; \times \;) = 0.12$$

$$\mathcal{P}([0, 1, 3, 6, \ldots] \,,\; [\text{“(”}, x] \,,\; + \;) = 0.67$$

$$\mathcal{P}([2, 3, 5, 7, \ldots] \,,\; [\text{“(”}, x] \,,\; + \;) = 0.60$$

Synthesize: updating a probabilistic function

Having synthesized the program (fastest so far) $(x \times x + x) \div 2$ generating

$$0, 1, 3, 6, 10, 15, \ldots, 1431$$

how do we update a probabilistic function $\mathcal{P}$?

$$\mathcal{P}_{desired}([0, 1, 3, 6, \ldots], [], \text{“(”}) = 0.2 \rightarrow 1$$
$$\mathcal{P}_{desired}([0, 1, 3, 6, \ldots], [\text{“(”}], x) = 0.3 \rightarrow 1$$
$$\mathcal{P}_{desired}([0, 1, 3, 6, \ldots], [\text{“(”}, x], \times) = 0.12 \rightarrow 1$$
$$\mathcal{P}_{desired}([0, 1, 3, 6, \ldots], [\text{“(”}, x], +) = 0.67 \rightarrow 0$$

A neural network finds a smooth curve approximating $\mathcal{P}_{desired}$.
Then $\mathcal{P}'$ is used to sample new programs from OEIS sequences.

Self-learning: five different runs

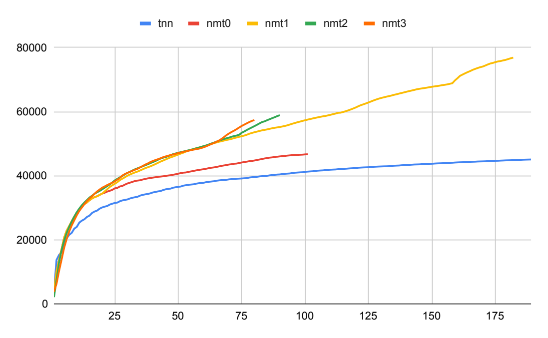

Figure: Number *y* of OEIS sequences (with at least one program) after *x* iterations

$\mathcal{P}_0$ is a random probability distribution function.
$\mathcal{P}_x$ synthesizes/samples 240 programs for each OEIS sequence.
A program may be correct for an other OEIS sequence.
$\mathcal{P}_x$ is updated to $\mathcal{P}_{x+1}$.

Examples

Let's now see some examples of synthesized programs.

A10445: squares modulo 84

OEIS sequence

$$0, 1, 4, 9, 16, 21, 25, 28, 36, 37, 49, 57, 60, 64, 72, 81$$

Synthesized program

$$\{x \mid (x^4 - x) \ mod \ 84 = 0\}$$

with $84 = 2 \times f^2(2)$ *and* $f(x) = x \times x + x$

Proof: Left to the listener.

Example: characteristic function of primes

OEIS sequence

$$0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, \ldots, 1, 0$$

Synthesized program

$$((x \times x!) \ mod \ (1 + x)) \ mod \ 2$$

Proof: Left to the listener.

Example: prime numbers

OEIS sequence

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, \ldots, 271$$

Synthesized program

$$\{x \mid 2^x \equiv 2 \mod x\}$$

Proof: 341 is a counterexample

Example: A294082 by David Cerna

$1, 1, 1, 1, 2, 1, 1, 4, 3, 1, 1, 14, 9, 4, 1, 1, 184, 75, 16, 5, 1, 1, 33674, 5553, 244 \ldots$

OEIS description: Square array read by antidiagonals:
$T(m, n) = T(m, n-1)^2 - T(m, n-2)^2 + T(m, n-2) withT(1, n) = 1, T(m, 0) = 1, andT(m, 1) = m.$

Program:

```
loop2(1 + (((x * x) - x) + y), y, 0 - (1 + loop(x - (if x <
  1, loop(loop(y, x - y, x), x, x))
```

Proof: Left to the reader.

Selection of 123 Solved Sequences

Table: Samples of the solved sequences.

`https://oeis.org/A317485`	Number of Hamiltonian paths in the n-Bruhat graph.
`https://oeis.org/A349073`	a(n) = U(2*n, n), where U(n, x) is the Chebyshev polynomial of the second kind.
`https://oeis.org/A293339`	Greatest integer k such that $k/2^n < 1/e$.
`https://oeis.org/A1848`	Crystal ball sequence for 6-dimensional cubic lattice.
`https://oeis.org/A8628`	Molien series for A_5.
`https://oeis.org/A259445`	Multiplicative with $a(n) = n$ if n is odd and $a(2^s) = 2$.
`https://oeis.org/A314106`	Coordination sequence Gal.6.199.4 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings
`https://oeis.org/A311889`	Coordination sequence Gal.6.129.2 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings.
`https://oeis.org/A315334`	Coordination sequence Gal.6.623.2 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings.
`https://oeis.org/A315742`	Coordination sequence Gal.5.302.5 where G.u.t.v denotes the coordination sequence for a vertex of type v in tiling number t in the Galebach list of u-uniform tilings.
`https://oeis.org/A004165`	OEIS writing backward
`https://oeis.org/A83186`	Sum of first n primes whose indices are primes.
`https://oeis.org/A88176`	Primes such that the previous two primes are a twin prime pair.
`https://oeis.org/A96282`	Sums of successive twin primes of order 2.
`https://oeis.org/A53176`	Primes p such that $2p + 1$ is composite.
`https://oeis.org/A267262`	Total number of OFF (white) cells after n iterations of the "Rule 111" elementary cellular automaton starting with a single ON (black) cell.

Thank you for your attention!

https://github.com/Anon52MI4/oeis-alien