

Models of nonlinear elasticity: Questions and progress

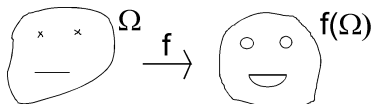
Stanislav HencI

Charles University, Prague, Czech Republic

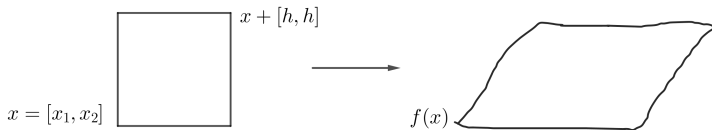
12.12. 2023, Colloquim MFF

Models in Nonlinear Elasticity - Deformation

Object of study: $\Omega \subset \mathbf{R}^n$ is a body, $n = 2, 3, \dots$, $f : \Omega \rightarrow \mathbf{R}^n$ is a mapping (deformation of the body)

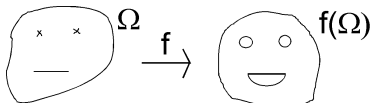


$\exists Df(x)$: f close to linear $f(y) \approx f(x) + Df(x)(y - x)$.

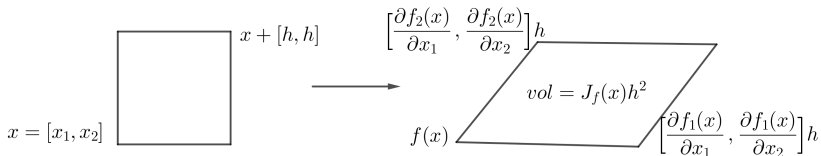


Models in Nonlinear Elasticity - Deformation

Object of study: $\Omega \subset \mathbf{R}^n$ is a body, $n = 2, 3, \dots$, $f : \Omega \rightarrow \mathbf{R}^n$ is a mapping (deformation of the body)



$\exists Df(x)$: f close to linear $f(y) \approx f(x) + Df(x)(y - x)$.



$Df(x)$ is $n \times n$ matrix of derivatives - deformation of segments

$J_f(x) = \det Df(x)$ is Jacobian - deformation of volume

$\int_A |J_f(x)| dx = |f(A)|$ if f is 1-1.

Models in Nonlinear Elasticity - Assumptions

Object of study: $\Omega \subset \mathbf{R}^n$ is a body, $n = 2, 3, \dots$, $f : \Omega \rightarrow \mathbf{R}^n$ is a mapping (deformation of the body)

$Df(x)$ is $n \times n$ matrix of derivatives - deformation of segments
 $J_f(x) = \det Df(x)$ is Jacobian - deformation of volume

Motivation: J. Ball, V. Šverák - mathematical model for Nonlinear Elasticity. The mapping minimizes the elastic energy

$$\min_f \int_{\Omega} W(Df(x)) \, dx$$

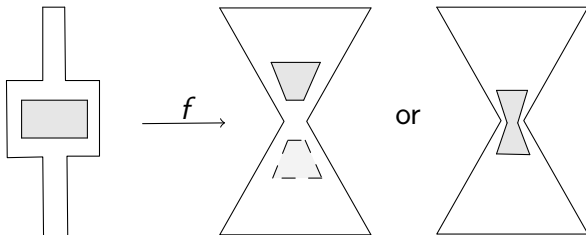
$W(A) \rightarrow \infty$ for $|A| \rightarrow \infty$, $W(A) \rightarrow \infty$ for $\det A \rightarrow 0$.

Naturally $|W(A)| \geq |A|^p$, i.e. $\int_{\Omega} |Df(x)|^p dx < \infty$ and $J_f(x) > 0$ a.e. (=mapping does not change orientation).

Sobolev space $W^{1,p}(\Omega, \mathbf{R}^n) = \{f : \int_{\Omega} |Df(x)|^p dx < \infty\}$.

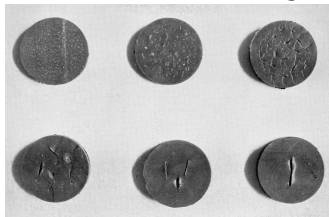
Some natural questions

- Is there a minimizer of energy? Is it unique?



Some natural questions

- Is there a minimizer of energy? Is it unique?
- Is f continuous? (Does the material break or are there any cavities created during the deformation?)



Cavities in rubber

Some natural questions

- Is there a minimizer of energy? Is it unique?
- Is f continuous? (Does the material break or are there any cavities created during the deformation?)
- Is the mapping one-to-one? Does there exist inverse map f^{-1} ? (interpenetration of the matter)

Some natural questions

- Is there a minimizer of energy? Is it unique?
- Is f continuous? (Does the material break or are there any cavities created during the deformation?)
- Is the mapping one-to-one? Does there exist inverse map f^{-1} ? (interpenetration of the matter)
- Does f map sets of zero measure to sets of zero measure? (Is a new material created from 'nothing'? Is some material 'lost' during the deformation?). Does $\int_A J_f = |f(A)|$ hold?

Some natural questions

- Is there a minimizer of energy? Is it unique?
- Is f continuous? (Does the material break or are there any cavities created during the deformation?)
- Is the mapping one-to-one? Does there exist inverse map f^{-1} ? (interpenetration of the matter)
- Does f map sets of zero measure to sets of zero measure? (Is a new material created from 'nothing'? Is some material 'lost' during the deformation?). Does $\int_A J_f = |f(A)|$ hold?
- Does f preserve orientation, i.e. $J_f \geq 0$ a.e.? (Can the body turn over?)

Some natural questions

- Is there a minimizer of energy? Is it unique?
- Is f continuous? (Does the material break or are there any cavities created during the deformation?)
- Is the mapping one-to-one? Does there exist inverse map f^{-1} ? (interpenetration of the matter)
- Does f map sets of zero measure to sets of zero measure? (Is a new material created from 'nothing'? Is some material 'lost' during the deformation?). Does $\int_A J_f = |f(A)|$ hold?
- Does f preserve orientation, i.e. $J_f \geq 0$ a.e.? (Can the body turn over?)
- Can we approximate it by piecewise linear homeomorphisms?

Some natural questions

- Is there a minimizer of energy? Is it unique?
- Is f continuous? (Does the material break or are there any cavities created during the deformation?)
- Is the mapping one-to-one? Does there exist inverse map f^{-1} ? (interpenetration of the matter)
- Does f map sets of zero measure to sets of zero measure? (Is a new material created from 'nothing'? Is some material 'lost' during the deformation?). Does $\int_A J_f = |f(A)|$ hold?
- Does f preserve orientation, i.e. $J_f \geq 0$ a.e.? (Can the body turn over?)
- Can we approximate it by piecewise linear homeomorphisms?
- What are the properties of f^{-1} ? (Can we deform the body back to its original state?)

Hajlasz problem (~ 2000)

Problem: Let $\Omega \subset \mathbf{R}^n$ be a domain, $f : \Omega \rightarrow \mathbf{R}^n$ be a homeomorphism such that $f \in W^{1,1}(\Omega, \mathbf{R}^n)$. Is it true that $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.?

Motivation: a) change of variables formula - replace $|J_f|$ by J_f
b) assumption $J_f \geq 0$ in models superfluous
c) approximation

Hajlasz problem (~ 2000)

Problem: Let $\Omega \subset \mathbf{R}^n$ be a domain, $f : \Omega \rightarrow \mathbf{R}^n$ be a homeomorphism such that $f \in W^{1,1}(\Omega, \mathbf{R}^n)$. Is it true that $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.?

YES if f differentiable, i.e. $n = 2$ or $f \in W^{1,p}$, $p > n - 1$

Hajlasz problem (~ 2000)

Problem: Let $\Omega \subset \mathbf{R}^n$ be a domain, $f : \Omega \rightarrow \mathbf{R}^n$ be a homeomorphism such that $f \in W^{1,1}(\Omega, \mathbf{R}^n)$. Is it true that $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.?

YES if f differentiable, i.e. $n = 2$ or $f \in W^{1,p}$, $p > n - 1$

Obstacles: $\exists f$ homeomorphism and Lipschitz, but $J_f = 0$ on a set of positive measure.

Hajlasz problem (~ 2000)

Problem: Let $\Omega \subset \mathbf{R}^n$ be a domain, $f : \Omega \rightarrow \mathbf{R}^n$ be a homeomorphism such that $f \in W^{1,1}(\Omega, \mathbf{R}^n)$. Is it true that $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.?

YES if f differentiable, i.e. $n = 2$ or $f \in W^{1,p}$, $p > n - 1$

Obstacles: $\exists f$ homeomorphism and Lipschitz, but $J_f = 0$ on a set of positive measure.

Theorem (H., Malý (2010))

Let $\Omega \subset \mathbf{R}^n$ be an open set, $n \geq 2$. Suppose that $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ is a homeomorphism for some $p > [n/2]$ ($p \geq 1$ for $n = 2, 3$). Then $J_f \geq 0$ a.e. or $J_f \leq 0$ a.e.

Open problem: 1. How about $p \leq [n/2]$.
2. Is there positively oriented f with $J_f \leq 0$?

Homeomorphisms with $J_f \equiv 0$

Area Formula : $\exists N \subset \Omega$ such that $\mathcal{L}_n(\Omega \setminus N) = \mathcal{L}_n(\Omega)$ but

$$0 = \int_{\Omega \setminus N} J_f(x) = \int_{f(\Omega \setminus N)} 1 = \mathcal{L}_n(f(\Omega \setminus N))$$

Homeomorphisms with $J_f \equiv 0$

Area Formula : $\exists N \subset \Omega$ such that $\mathcal{L}_n(\Omega \setminus N) = \mathcal{L}_n(\Omega)$ but

$$0 = \int_{\Omega \setminus N} J_f(x) = \int_{f(\Omega \setminus N)} 1 = \mathcal{L}_n(f(\Omega \setminus N))$$

$$\Rightarrow \mathcal{L}_n(N) = 0 \quad \text{but} \quad \mathcal{L}_n(f(N)) = \mathcal{L}_n(f(\Omega))$$

Homeomorphisms with $J_f \equiv 0$

Area Formula : $\exists N \subset \Omega$ such that $\mathcal{L}_n(\Omega \setminus N) = \mathcal{L}_n(\Omega)$ but

$$0 = \int_{\Omega \setminus N} J_f(x) = \int_{f(\Omega \setminus N)} 1 = \mathcal{L}_n(f(\Omega \setminus N))$$

$$\Rightarrow \mathcal{L}_n(N) = 0 \quad \text{but} \quad \mathcal{L}_n(f(N)) = \mathcal{L}_n(f(\Omega))$$

Theorem (H. (2011))

Let $n \geq 2$ and $1 \leq p < n$. There is a homeomorphism $f \in W^{1,p}((0,1)^n, (0,1)^n)$ such that $J_f(x) = 0$ a.e.

D'Onofrio, H., Schiattarella: $n \geq 3$ also $f^{-1} \in W^{1,1}$

Liu, Malý: f can be a gradient mapping, using laminates

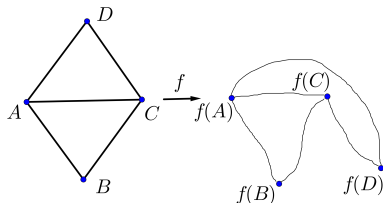
Faraco, Mora-Corall, Oliva: laminates, sharp conditions also for $f \in W^{1,p}$, $f^{-1} \in W^{1,q}$ or $(\det Df)_k = 0$ a.e.

Ball-Evans Problem

Problem [Ball-Evans]: $\Omega \subset \mathbf{R}^n$ domain, $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ homeomorphism. Can we find f_k piecewise affine (or diffeomorphisms) such that $f_k \rightarrow f$ in $W^{1,p}$?

$n = 1$ easy:

$n \geq 2$ hard:

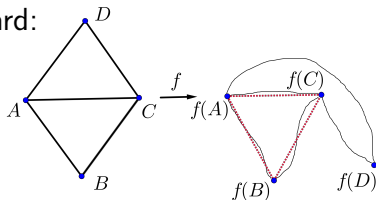


Ball-Evans Problem

Problem [Ball-Evans]: $\Omega \subset \mathbf{R}^n$ domain, $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ homeomorphism. Can we find f_k piecewise affine (or diffeomorphisms) such that $f_k \rightarrow f$ in $W^{1,p}$?

$n = 1$ easy:

$n \geq 2$ hard:

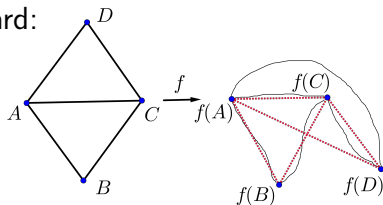


Ball-Evans Problem

Problem [Ball-Evans]: $\Omega \subset \mathbf{R}^n$ domain, $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ homeomorphism. Can we find f_k piecewise affine (or diffeomorphisms) such that $f_k \rightarrow f$ in $W^{1,p}$?

$n = 1$ easy:

$n \geq 2$ hard:



Ball-Evans Problem

Problem [Ball-Evans]: $\Omega \subset \mathbf{R}^n$ domain, $f \in W^{1,p}(\Omega, \mathbf{R}^n)$ homeomorphism. Can we find f_k piecewise affine (or diffeomorphisms) such that $f_k \rightarrow f$ in $W^{1,p}$?

$n = 1$ easy: $n \geq 2$ hard:

It is not easy: triangulization or mollification destroy injectivity

$\exists f_k$ smooth $\xRightarrow{\text{easy}} \Pratelli \& \text{Mora-Corral} \xleftarrow{\quad} \exists f_k$ piecewise affine

Motivation

- Regularity for models in Nonlinear Elasticity
Ball models $\min \int W(Du)$ where $E(u) \rightarrow \infty$ as $J_u \rightarrow 0$
- Numerics - finite elements method
- Easier proofs of known (and new) statements

Known results

C. Mora-Corral: f smooth up to one point

Theorem (Iwaniec, Kovalev, Onninen (2011))

Let $n = 2$ and $1 < p < \infty$. Given a homeomorphism $f \in W^{1,p}(\Omega, \mathbf{R}^2)$ there are diffeomorphisms f_k with $f_k \rightarrow f$ in $W^{1,p}$, $f_k \rightrightarrows f$ and $f_k - f \in W_0^{1,p}$

Theorem (H., Pratelli (2018))

Let $n = 2$. Given a homeomorphism $f \in W^{1,1}(\Omega, \mathbf{R}^2)$ there are diffeomorphisms f_k with $f_k \rightarrow f$ in $W^{1,1}$, $f_k \rightrightarrows f$ and $f_k - f \in W_0^{1,1}(\Omega, \mathbf{R}^2)$.

Open problems:

- $n = 2, p = 2, f \in W^{1,2}, f^{-1} \in W^{1,2}$ - Can we approximate

$$\int |Df|^2 + \frac{|Df|^2}{J_f} (= \int |Df|^2 + \int |Df^{-1}|^2) ?$$

Open problems:

- $n = 2, p = 2, f \in W^{1,2}, f^{-1} \in W^{1,2}$ - Can we approximate
$$\int |Df|^2 + \frac{|Df|^2}{J_f} (= \int |Df|^2 + \int |Df^{-1}|^2) ?$$
- Anything about the approximation in $n = 3$?
Is there some improved construction by hand?

Open problems:

- $n = 2, p = 2, f \in W^{1,2}, f^{-1} \in W^{1,2}$ - Can we approximate
$$\int |Df|^2 + \frac{|Df|^2}{J_f} (= \int |Df|^2 + \int |Df^{-1}|^2) ?$$
- Anything about the approximation in $n = 3$?
Is there some improved construction by hand?
- Is there some counterexample in dimension $n \geq 4$ in $W^{1,p}, p > [n/2]$?

Dimension $n \geq 4$ - Hajlasz problem

Theorem (Campbell, H., Tengvall, Vejnar (2016, 2018))

Let $n \geq 4$ and $1 \leq p < [\frac{n}{2}]$. There is a homeomorphism in the Sobolev space $f \in W^{1,p}((0,1)^n, \mathbf{R}^n)$ such that $\mathcal{L}^4(\{x : J_f(x) > 0\}) > 0$ and $\mathcal{L}^4(\{x : J_f(x) < 0\}) > 0$.

Dimension $n \geq 4$ - Hajlasz problem

Theorem (Campbell, H., Tengvall, Vejnar (2016, 2018))

Let $n \geq 4$ and $1 \leq p < [\frac{n}{2}]$. There is a homeomorphism in the Sobolev space $f \in W^{1,p}((0,1)^n, \mathbf{R}^n)$ such that $\mathcal{L}^4(\{x : J_f(x) > 0\}) > 0$ and $\mathcal{L}^4(\{x : J_f(x) < 0\}) > 0$.

Corollary: Let $n \geq 4$. For this $f \in W^{1,p}$ there are no diffeomorphisms (or piecewise affine homeomorphisms) with $f_k \rightarrow f$ in $W_{\text{loc}}^{1,p}((-1,1)^n, \mathbf{R}^n)$.

Dimension $n \geq 4$ - Hajlasz problem

Theorem (Campbell, H., Tengvall, Vejnar (2016, 2018))

Let $n \geq 4$ and $1 \leq p < [\frac{n}{2}]$. There is a homeomorphism in the Sobolev space $f \in W^{1,p}((0,1)^n, \mathbf{R}^n)$ such that $\mathcal{L}^4(\{x : J_f(x) > 0\}) > 0$ and $\mathcal{L}^4(\{x : J_f(x) < 0\}) > 0$.

Corollary: Let $n \geq 4$. For this $f \in W^{1,p}$ there are no diffeomorphisms (or piecewise affine homeomorphisms) with $f_k \rightarrow f$ in $W_{\text{loc}}^{1,p}((-1,1)^n, \mathbf{R}^n)$.

Proof by contradiction: $f_k \xrightarrow{W^{1,p}} f \xRightarrow{\text{subsequence}} Df_k(x) \rightarrow Df(x)$ for a.e. $x \Rightarrow J_{f_k}(x) \rightarrow J_f(x)$ for a.e. x . As f_k smooth $\Rightarrow J_{f_k} \geq 0$ a.e. or $J_{f_k} \leq 0$ a.e. Hence their pointwise limit does not change sign - contradiction.

Dimension $n \geq 4$ - Hajlasz problem

Theorem (Campbell, H., Tengvall, Vejnar (2016, 2018))

Let $n \geq 4$ and $1 \leq p < [\frac{n}{2}]$. There is a homeomorphism in the Sobolev space $f \in W^{1,p}((0,1)^n, \mathbf{R}^n)$ such that $\mathcal{L}^4(\{x : J_f(x) > 0\}) > 0$ and $\mathcal{L}^4(\{x : J_f(x) < 0\}) > 0$.

Corollary: Let $n \geq 4$. For this $f \in W^{1,p}$ there are no diffeomorphisms (or piecewise affine homeomorphisms) with $f_k \rightarrow f$ in $W_{\text{loc}}^{1,p}((-1,1)^n, \mathbf{R}^n)$.

Proof by contradiction: $f_k \xrightarrow{W^{1,p}} f \xRightarrow{\text{subsequence}} Df_k(x) \rightarrow Df(x)$ for a.e. $x \Rightarrow J_{f_k}(x) \rightarrow J_f(x)$ for a.e. x . As f_k smooth $\Rightarrow J_{f_k} \geq 0$ a.e. or $J_{f_k} \leq 0$ a.e. Hence their pointwise limit does not change sign - contradiction.

Example: \exists homeomorphism $f(x) = x$ on $\partial[0,1]^n$ and $J_f < 0$ a.e. for $n = 4$ and $1 \leq p < 3/2$.

Sobolev regularity of the inverse

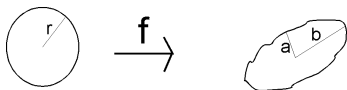
Problem: Let $f \in W^{1,p}(\Omega, \mathbf{R}^n)$, $p \geq 1$, be a homeomorphism.
When $f^{-1} \in W_{\text{loc}}^{1,1}(f(\Omega), \Omega)$? (or BV)?

Sobolev regularity of the inverse

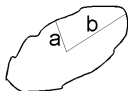
Problem: Let $f \in W^{1,p}(\Omega, \mathbf{R}^n)$, $p \geq 1$, be a homeomorphism. When $f^{-1} \in W_{\text{loc}}^{1,1}(f(\Omega), \Omega)$? (or BV)?

Elementary example: There is a Lipschitz homeomorphism $f : [0, 2] \times [0, 1]^{n-1} \rightarrow [0, 1]^n$ with $f^{-1} \notin W_{\text{loc}}^{1,1}([0, 1]^n, \mathbf{R}^n)$.
Proof ($n = 2$): $h(x) = x + C(x)$, $f(x, y) = [h^{-1}(x), y]$.

Definition: Homeomorphism $f \in W_{\text{loc}}^{1,1}(\Omega, \mathbf{R}^n)$ has finite distortion if $|Df(x)|^n \leq K(x)J_f(x)$ holds a.e., where $1 \leq K(x) < \infty$ a.e. Especially if $J_f > 0$ a.e. then f has finite distortion.



$$K = b/a$$



$$|Df| = b/r$$

$$J_f = (ab)/r^2$$

$$K_2 > K$$



$$K_3 = \infty$$

Sobolev regularity of the inverse

Problem: Let $f \in W^{1,p}(\Omega, \mathbf{R}^n)$, $p \geq 1$, be a homeomorphism. When $f^{-1} \in W_{\text{loc}}^{1,1}(f(\Omega), \Omega)$? (or BV)?

Elementary example: There is a Lipschitz homeomorphism $f : [0, 2] \times [0, 1]^{n-1} \rightarrow [0, 1]^n$ with $f^{-1} \notin W_{\text{loc}}^{1,1}([0, 1]^n, \mathbf{R}^n)$.
Proof ($n = 2$): $h(x) = x + C(x)$, $f(x, y) = [h^{-1}(x), y]$.

Theorem (H+Koskela ($n = 2$) 2006, Csörnyei+H+Malý 2010)

Suppose that $f \in W^{1,n-1}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion. Then $f^{-1} \in W_{\text{loc}}^{1,1}(f(\Omega), \mathbf{R}^n)$ and has finite distortion.

Sobolev regularity of the inverse

Problem: Let $f \in W^{1,p}(\Omega, \mathbf{R}^n)$, $p \geq 1$, be a homeomorphism. When $f^{-1} \in W_{\text{loc}}^{1,1}(f(\Omega), \Omega)$? (or BV)?

Elementary example: There is a Lipschitz homeomorphism $f : [0, 2] \times [0, 1]^{n-1} \rightarrow [0, 1]^n$ with $f^{-1} \notin W_{\text{loc}}^{1,1}([0, 1]^n, \mathbf{R}^n)$.
Proof ($n = 2$): $h(x) = x + C(x)$, $f(x, y) = [h^{-1}(x), y]$.

Theorem (H+Koskela ($n = 2$) 2006, Csörnyei+H+Malý 2010)

Suppose that $f \in W^{1,n-1}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion. Then $f^{-1} \in W_{\text{loc}}^{1,1}(f(\Omega), \mathbf{R}^n)$ and has finite distortion.

Theorem

Suppose that $f \in W_{\text{loc}}^{1,n-1}(\Omega, \mathbf{R}^n)$ is a homeomorphism of finite distortion and moreover we assume that $K \in L^{n-1}(\Omega)$. Then $f^{-1} \in W_{\text{loc}}^{1,n}(f(\Omega), \mathbf{R}^n)$.

BV regularity of the inverse

Theorem (Csörnyei+H.+Malý (2010))

Suppose that $f \in W^{1,n-1}(\Omega; \mathbf{R}^n)$ is a homeomorphism. Then $f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbf{R}^n)$.

Definition

We say that $h \in BV(\Omega)$ if $h \in L^1(\Omega)$ and $D_i h = \mu_i$ are signed Radon measures with finite total variation:

$$\int_{\Omega} h D_i \varphi \, dx = - \int_{\Omega} \varphi \, d\mu_i, \text{ for all } \varphi \in C_0^\infty(\Omega).$$

We say that $f \in BV(\Omega; \mathbf{R}^n)$ if $f_i \in BV(\Omega)$.

BV regularity of the inverse

Theorem (Csörnyei+H.+Malý (2010))

Suppose that $f \in W^{1,n-1}(\Omega; \mathbf{R}^n)$ is a homeomorphism. Then $f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbf{R}^n)$.

Theorem

Let $0 < \varepsilon < 1$ and $n \geq 3$. There is a homeomorphism of finite distortion $f \in W^{1,n-1-\varepsilon}((-1,1)^n; \mathbf{R}^n)$ such that $f^{-1} \notin BV_{\text{loc}}(f(\Omega); \mathbf{R}^n)$. ('because $|\nabla f^{-1}| \notin L^1_{\text{loc}}$ ')

BV regularity of the inverse

Theorem (Csörnyei+H.+Malý (2010))

Suppose that $f \in W^{1,n-1}(\Omega; \mathbf{R}^n)$ is a homeomorphism. Then $f^{-1} \in BV_{\text{loc}}(f(\Omega), \mathbf{R}^n)$.

Theorem

Let $0 < \varepsilon < 1$ and $n \geq 3$. There is a homeomorphism of finite distortion $f \in W^{1,n-1-\varepsilon}((-1,1)^n; \mathbf{R}^n)$ such that $f^{-1} \notin BV_{\text{loc}}(f(\Omega); \mathbf{R}^n)$. ('because $|\nabla f^{-1}| \notin L^1_{\text{loc}}$ ')

$$\int_{f(\Omega)} |Df^{-1}(y)| \, dy = \int_{\Omega} |Df^{-1}(f(x))| J_f(x) \, dx$$

$$\stackrel{f^{-1} \circ f = \text{id}}{=} \int_{\Omega} |(Df(x))^{-1}| J_f(x) \, dx$$

$$\stackrel{A \operatorname{adj} A = \det A I}{=} \int_{\Omega} |\operatorname{adj} Df(x)| \, dx \leq \int_{\Omega} |Df(x)|^{n-1} \, dx$$