ASYMPTOTIC PROPERTIES OF HORVITZ-THOMPSON TYPE EMPIRICAL DISTRIBUTION FUNCTIONS IN GERM-GRAIN MODELS AND THEIR APPLICATIONS

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Abstract

We consider a single observation of randomly placed random compact sets (called grains) in a bounded, convex subset W_n of the d-dimensional Euclidean space (sampling window) which expands unboundedly in all directions as $n \to \infty$. We assume that the grains are independent copies of a so-called typical grain Ξ_0 which are shifted by the atoms of a homogeneous point process Ψ in such a way that each individual grain lying within W_n can be observed. We define an appropriate estimator $\hat{F}_n(t)$ for the distribution function $F(t)$ of some m-dimensional vector $f(\Xi_0) = (f_1(\Xi_0), \ldots, f_m(\Xi_0))$ (describing shape and size of Ξ_0) based on the corresponding data vectors of those grains which are completely observable in W_n. As main results we prove a Glivenko-Cantelli type theorem for $\hat{F}_n(t)$ and the weak convergence of the multivariate empirical processes $\sqrt{\Psi(W_n)}\,(\hat{F}_n(t) - F(t))$ to an m-dimensional Brownian bridge process as $n \to \infty$. It is possible for the particular case where $m = 1$, to examine the the goodness-of-fit of observed data to a hypothesized continuous distribution function F, analogous to the Kolmogorov-Smirnov test.

Keywords: germ-grain model, marked point process, typical grain, Glivenko-Cantelli theorem, weak convergence, Skorohod space $D(\mathbb{R}^m)$, Horvitz-Thompson type estimator, multivariate empirical distribution, Kolmogorov-Smirnov test

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1 Introduction and Preliminaries

We consider a stationary d-dimensional *germ-grain model*

$$\Xi = \bigcup_{i \geq 1} (\Xi_i + X_i)\,, \tag{1.1}$$

which consists of two independent random components defined on a common probability space $[\Omega, \mathfrak{A}, \mathsf{P}]$ — a (weakly) stationary point process $\Psi = \sum_{i \geq 1} \delta_{X_i}$ on $\mathbb{R}^d$ with intensity $\lambda = \mathsf{E}\Psi([0,1)^d)$ and a sequence

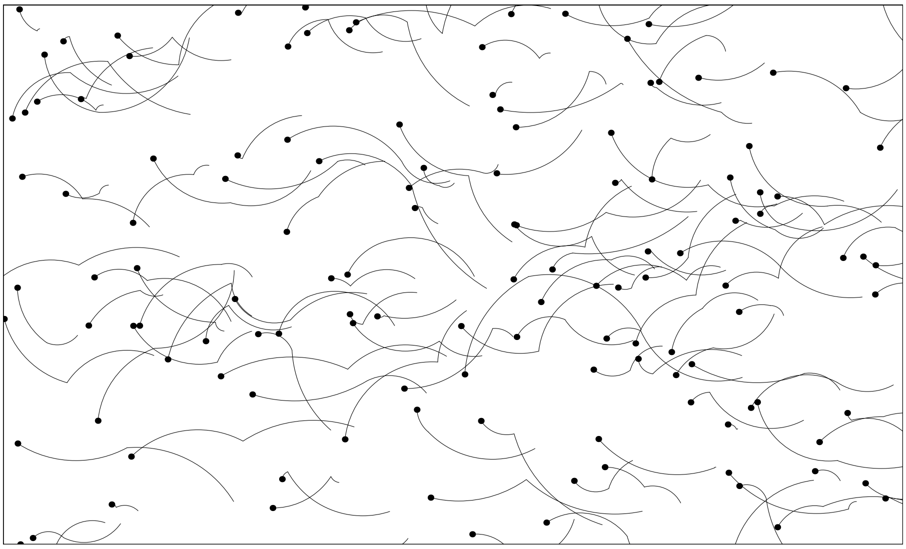

Figure 1: Realization of a fibre process as particular germ-grain model.

$\{\Xi_i, i \geq 1\}$ of independent copies of a random compact set Ξ_0 (called *typical grain*). To identify the points X_i in (1.1) we require additionally that $\mathsf{P}(c(\Xi_0) = o) = 1$, where $c(K) \in K$ is a canonical point assigned to each $K \in \mathcal{K}'$ (= family of non-empty compact sets in $\mathbb{R}^d$). For more details and further background of point process theory and germ-grain models we refer the reader to [3], [19], [7].

Throughout we assume that only a single observation of Ξ in a *sampling window* $W_n \subseteq \mathbb{R}^d$ is given, where the sequence of convex compact sets (W_n) expands unboundedly in all directions such that $H^{d-1}(\partial W_n)/|W_n| \to 0$ as $n \to \infty$. Here and below $H^k(\cdot)$ designates the k-dimensional Hausdorff measure and $|\cdot| = H^d(\cdot)$ the Lebesgue measure on $\mathbb{R}^d$.

Let $f(\Xi_0) = (f_1(\Xi_0), \ldots, f_m(\Xi_0))$ be an m-dimensional random vector describing various shape and size parameters of Ξ_0 , e.g. geometric functionals, direction of normal unit vectors at fixed points on the surface $\partial\Xi_0$ etc. For example, if Ξ_0 is a random segment or more generally a random rectifiable curve in $\mathbb{R}^2$, then nonparametric testing of the (joint) distribution function (df) of length $H^1(\Xi_0)$ and angle between the tangent in $c(\Xi_0)$ and the x-axis turns out to be a non-trivial statistical issue.

In order to estimate the m-variate df

$$F(t) := \mathsf{P}(f(\Xi_0) \leq t) = \mathsf{P}(f_1(\Xi_0) \leq t_1, \ldots, f_m(\Xi_0) \leq t_m), \quad t = (t_1, \ldots, t_m) \in \mathbb{R}^m \,, \tag{1.2}$$

we suppose that the m-dimensional data vectors $f(\Xi_i)$ of those (shifted) grains $\Xi_i + X_i$ lying completely within W_n are available. In other words, the random set (1.1) is a union of non-overlapping grains or the set of points in W_n covered by more than one grain is neglegible and does not prevent the exact measurement of the data vectors $f(\Xi_i)$, see Fig. 1. Fibre, surface and manifold processes (see [19], Chapt. 9) are typical examples of such germ-grain models. In these examples Ξ_0 is a random k-dimensional compact manifold with $\mathsf{P}(0 < H^k(\Xi_0) < \infty) = 1$ for some $k \in \{1, \ldots, d-1\}$. In case of $\mathsf{P}(|\Xi_0| > 0) > 0$ the above restriction means that the grains $\Xi_i + X_i$ in (1.1) are pairwise disjoint, e.g. if Ψ is a hard-core point process with hard-core distance $h > 0$ and $\|\Xi_0\| := \sup\{\|x\| : x \in \Xi_0\}$ is bounded and less than $h/2$, see Fig. 2.

The key question we address in this paper is : How to define a suitable empirical df $\hat{F}_n(t)$ for (1.2) by using only data vectors $f(\Xi_i)$ of those grains $\Xi_i + X_i$ lying completely within W_n such that the limit distribution of the maximum discrepancy $\sup_{t\in\mathbb{R}^m} |\hat{F}_n(t) - F(t)|$ after blowing up with $\sqrt{\Psi(W_n)}$ can be shown to exist quite similar to the classical situation of i.i.d. random vectors in $\mathbb{R}^m$, see [1], [4] (and [2] for $m = 1$).

Our sampling procedure (called *minus-sampling*) leads necessarily to weighted estimators of the form

$$\hat{M}_n(g) := \sum_{i\geq 1} \mathbf{1}_{\{\Xi_i + X_i \subseteq W_n\}} \frac{g(\Xi_i)}{|\, W_n \ominus \check{\Xi}_i \,|} \tag{1.3}$$

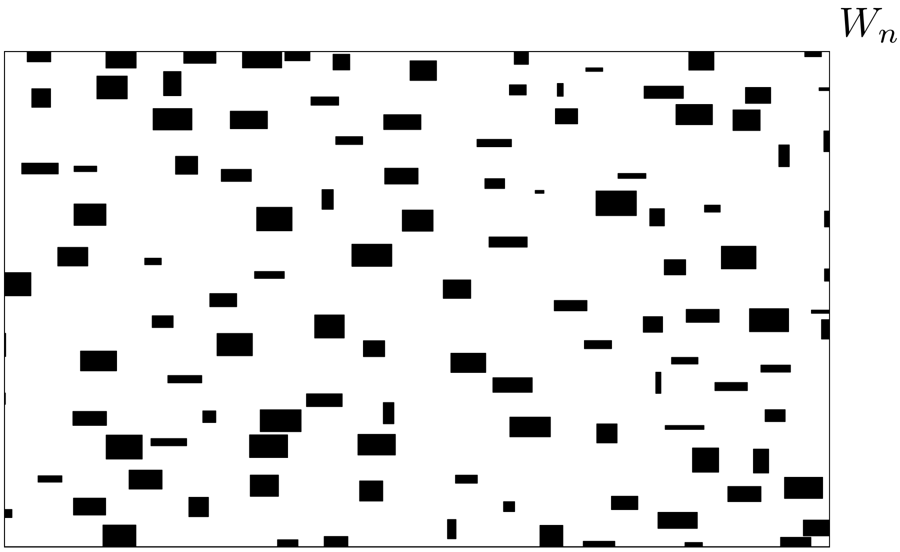

Figure 2: Realization of a germ-grain model with non-overlapping grains in the window W_n.

being unbiased for the mean value $\lambda\,\mathsf{E}g(\Xi_0)$ for any Q-integrable functional $g\,:\,\mathcal{K}'\mapsto\mathbb{R}^1$, see Chapt. 6 in [19]. Here Q denotes the distribution induced by Ξ_0 on $\mathcal{K}'$, $\mathbf{1}_B$ stands for the indicator function of a set or event B and we have used the set operations $\check{B}=\{-x : x\in B\}$ (reflection) and $A\ominus B=\cap_{y\in B}(A+y)=\{x : x+\check{B}\subseteq A\}$ (Minkowski-subtraction). Further, let $A\oplus B:=\cup_{y\in B}(A+y)$ (Minkowski-addition) and $b(x,r)$ denotes the ball with radius r centred at $x\in\mathbb{R}^d$.

Note that the proof of the unbiasedness of (1.3) is a straightforward application of Campbell's theorem for a stationary marked point process

$$\Psi_{mark}=\sum_{i\geq 1}\delta_{[X_i,\Xi_i]} \tag{1.4}$$

(which defines (1.1)) with intensity λ and mark distribution Q, see Chapt. 10.5 in [3] or [19]:

$$\mathsf{E}\hat{M}_n(g)=\lambda\int_{\mathcal{K}}\int_{\mathbb{R}^d}\mathbf{1}_{W_n\ominus\check{K}}(x)\,\frac{g(K)}{|\,W_n\ominus\check{K}\,|}\,\mathrm{d}x\,Q(\mathrm{d}K)=\lambda\int_{\mathcal{K}}g(K)\,Q(\mathrm{d}K)\,.$$

In Section 2 we will define an empirical df $\hat{F}_n(t)$, $t\in\mathbb{R}^m$, and state an analogue to Glivenko's theorem (Theorem 1) even for the more general case of germ-grain models (1.1) governed by a stationary ergodic marked point process Ψ_{mark}.

Our principal asymptotic result formulated (Theorem 2) and proved in Section 3 states the weak convergence of the m-parameter empirical processes

$$Y_n(t):=\sqrt{\Psi(W_n)}\,\Big(\hat{F}_n(t)\,-\,F(t)\Big),\quad t\in\mathbb{R}^m,\quad n\geq 1. \tag{1.5}$$

It turns out that the weak limit of (1.5) (as $n\to\infty$) in the Skorohod space $D(\mathbb{R}^m)$ can be identified with a mean zero Gaussian process $Y(t)$, $t\in\mathbb{R}^m$, having the covariance function $\mathsf{E}Y(s)Y(t)=F(s\wedge t)-F(s)F(t)$, where $s\wedge t=\big(\min(s_1,t_1),\ldots,\min(s_m,t_m)\big)$. Consequently, for $m=1$ a Kolmogorov-Smirnov test can be established just as for i.i.d. samples provided the df $F(\cdot)$ is continuous.

2 Empirical Distribution Functions and Glivenko's Theorem

A quite natural empirical counterpart of the df (1.2) is given by

$$\tilde{F}_n(t) = \frac{1}{N_n} \sum_{i\ge1} \mathbf{1}_{\{\Xi_i + X_i \subseteq W_n\}} \, \mathbf{1}_{(-\infty,t]}(f(\Xi_i)), \quad t = (t_1,\dots,t_m) \in \mathbb{R}^m \,, \tag{2.1}$$

where $(-\infty,t] = \prod_{j=1}^m (-\infty,t_j]$ and $N_n = \sum_{i\ge1} \mathbf{1}_{\{\Xi_i+X_i\subseteq W_n\}}$ equals the number of grains which are completely observable in W_n. Obviously, $\tilde{F}_n(t)$ is a discrete m-variate df which can be shown to converge P-a.s. (as $n\to\infty$) to $F(t)$ uniformly in $t\in\mathbb{R}^m$ provided the stationary marked point process Ψ_{mark} is ergodic. However, the relation

$$\frac{1}{\sqrt{|W_n|}} \, \mathsf{E} N_n \Big(\tilde{F}_n(t) - F(t) \Big) = \frac{\lambda}{\sqrt{|W_n|}} \int_{\mathcal{K}'} |W_n \ominus \check{K}| \, \Big(\mathbf{1}_{(-\infty,t]}(f(K)) - F(t) \Big) \, Q(\mathrm{d}K)$$

reveals that, for $d\ge 2$, a zero mean weak limit of $\sqrt{|W_n|}\,\big(\tilde{F}_n(t) - F(t)\big)$ cannot exist. In other words, the empirical value $\tilde{F}_n(t)$ is not close enough to $F(t)$. To remove this shortcoming we suggest the following empirical df which is based on the weighted estimator (1.3):

$$\hat{F}_n(t) = \frac{1}{\hat{\lambda}_n} \sum_{i\ge1} \frac{\mathbf{1}_{\{X_i+\Xi_i\subseteq W_n\}}}{|W_n \ominus \check{\Xi}_i|} \, \mathbf{1}_{(-\infty,t]}(f(\Xi_i)), \tag{2.2}$$

where

$$\hat{\lambda}_n = \sum_{i\ge1} \frac{\mathbf{1}_{\{X_i+\Xi_i\subseteq W_n\}}}{|W_n \ominus \check{\Xi}_i|} \tag{2.3}$$

is an unbiased estimator for the intensity λ implying that $\mathsf{E}\Big(\hat{\lambda}_n\,\big(\hat{F}_n(t) - F(t)\big)\Big) = 0$ for any $t\in\mathbb{R}^m$ and $n\ge1$. Here, $\hat{F}_n(\cdot)$ is again a discrete m-variate df with random jumps depending on the size of the grains Ξ_i. In the next section we will see under which circumstances $\hat{\lambda}_n$ can be substituted by the more convenient unbiased estimator $\lambda_n^* := \Psi(W_n)/|W_n|$ for λ which considers all points X_i belonging to W_n.

Next, we formulate Glivenko's theorem for $\hat{F}_n(t)$ even in the more general situation of stationary ergodic germ-grain models with not necessarily independent grains. To avoid too large weights $|W_n \ominus \check{\Xi}_i|^{-1}$ in the (2.2) and (2.3) we put an additional moment condition on the diameter $\|\Xi_0\|$.

Theorem 1: *Let the germ-grain model* (1.1) *be defined by a stationary ergodic marked point process* (1.4) *with positive and finite intensity λ and mark distribution Q satisfying $\int_{\mathcal{K}'} \|K\|^q \, Q(\mathrm{d}K) < \infty$ for some $q\ge d$. Further, let (W_n) be an increasing sequence (i.e. $W_n\subseteq W_{n+1}$ for $n\ge1$) of convex, bounded sets in $\mathbb{R}^d$ such that, for $n\ge1$,*

$$\frac{H^{d-1}(\partial W_n)}{|W_n|^{1-1/q}} \le c_0 < \infty \quad \textit{and} \quad \rho(W_n) := \sup\{r>0 : b(x,r)\subseteq W_n\,,\ x\in W_n\} \underset{n\to\infty}{\longrightarrow} \infty\,. \tag{2.4}$$

Then

$$\sup_{t\in\mathbb{R}^m} |\hat{F}_n(t) - F(t)| \underset{n\to\infty}{\longrightarrow} 0 \quad \mathsf{P}\text{-}a.s.\,, \quad \textit{where} \quad F(t) = Q(\{K\in\mathcal{K}' : f(K)\le t\})\,. \tag{2.5}$$

Proof: For notational ease, let Ξ_0 denote a $\mathcal{K}'$-valued random element on $[\Omega,\mathfrak{A},\mathsf{P}]$ having the mark distribution Q. The conditions put on the sequence (W_n) are sufficient to hold the spatial individual ergodic theorem, see [3], p. 333 or [15], which, applied to the stationary ergodic marked point process Ψ_{mark}, implies that

$$\frac{1}{|W_n|} \sum_{i\ge1} \mathbf{1}_{W_n}(X_i) \, \mathbf{1}_{(-\infty,t]}(f(\Xi_i)) \underset{n\to\infty}{\longrightarrow} \lambda\, F(t) \quad \mathsf{P}\text{-a.s.} \tag{2.6}$$

for any $t\in\mathbb{R}^m$. Since the function $\rho\mapsto V(\rho) := |W_n \ominus b(o,\rho)|$ is differentiable for $0\le\rho<\rho(W_n)$ with continuous derivative $V'(\rho) = H^{d-1}(\partial(W_n\ominus b(o,\rho))$, see [6] (p. 207), we get the identity

$$V(0) - V(r) = |W_n \setminus (W_n \ominus b(o,r))| = \int_0^r H^{d-1}\big(\partial(W_n\ominus b(o,\rho))\big)\,\mathrm{d}\rho$$

for $0 \le r \le \rho(W_n)$, which leads to the inequality

$$|W_n \setminus (W_n \ominus b(o,r))| = |\{x \in W_n : b(x,r) \cap W_n^c \neq \emptyset\}| \le r\, H^{d-1}(\partial W_n) \tag{2.7}$$

for any $r > 0$. Thus, setting $r_n^\delta := \delta |W_n| / H^{d-1}(\partial W_n)$ ($\underset{n\to\infty}{\longrightarrow} \infty$) and $W_n^\delta := W_n \ominus b(o, r_n^\delta)$, we immediately deduce that

$$\frac{|W_n|}{|W_n^\delta|} \le \frac{1}{1-\delta} \quad \text{resp.} \quad \frac{|W_n^\delta|}{|W_n|} \ge 1-\delta \quad \text{for} \quad 0 \le \delta < 1\,. \tag{2.8}$$

Further, it is easily seen that, for any $n \ge 1$ and $\delta \in [0,1)$, W_n^δ is a non-empty, convex subset of W_n with $\rho(W_n^\delta) \underset{n\to\infty}{\longrightarrow} \infty$. Hence, according to the definition given in [15], (W_n^δ) is a *regular generalized sequence* of convex sets in $\mathbb{R}^d$ to which the spatial individual ergodic theorem, see [15], p. 143, can be extended. This means that in (2.6) the sequence (W_n) can be replaced by (W_n^δ) for any $0 \le \delta < 1$.

In order to prove (2.5) we first show that, for all fixed $t \in \mathbb{R}^m$,

$$\hat\lambda_n\, \hat F_n(t) = \sum_{i\ge 1} \frac{\mathbf{1}_{\{X_i + \Xi_i \subseteq W_n\}}}{|W_n \ominus \check\Xi_i|}\, \mathbf{1}_{(-\infty,t]}(f(\Xi_i)) \underset{n\to\infty}{\longrightarrow} \lambda\, F(t) \quad \mathsf{P}\text{-a.s.} \tag{2.9}$$

or equivalently, by using the sequence of events $A_n(\varepsilon) := \{\, |\hat\lambda_n\, \hat F_n(t) - \lambda\, F(t)| \ge \varepsilon\,\}$,

$$\mathsf{P}\Big(\bigcup_{k\ge n} A_k(\varepsilon)\Big) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for any} \quad \varepsilon > 0\,.$$

Define a further sequence of events $B_n(\delta) := \bigcap_{i: X_i \in W_n} \{\|\Xi_i\| \le r_n^\delta\}$ with $\delta \in [0,1)$. Then

$$\mathsf{P}\Big(\bigcup_{k\ge n} A_k(\varepsilon)\Big) \le \mathsf{P}\Big(\bigcup_{k\ge n} A_k(\varepsilon) \cap B_k(\delta)\Big) + \mathsf{P}\Big(\bigcup_{k\ge n} B_k^c(\delta)\Big)\,.$$

The first part of (2.4) yields the estimate $r_k^\delta \ge \delta\, |W_k|^{\frac{1}{q}} / c_0 =: \rho_k^\delta$ for $k \ge 1$. By exploiting the monotonicity of the sequence (ρ_k^δ) and applying Campbell's theorem we obtain that

$$\begin{aligned}
\mathsf{P}\Big(\bigcup_{k\ge n} B_k^c(\delta)\Big) &\le \mathsf{P}\Big(\bigcup_{k\ge n} \bigcup_{i: X_i \in W_k} \{\|\Xi_i\| \ge \rho_k^\delta\}\Big) \\
&= \mathsf{P}\Big(\bigcup_{i: X_i \in W_n} \{\|\Xi_i\| \ge \rho_n^\delta\} \cup \bigcup_{k\ge n} \bigcup_{i: X_i \in W_{k+1}\setminus W_k} \{\|\Xi_i\| \ge \rho_{k+1}^\delta\}\Big) \\
&\le \mathsf{E} \sum_{i\ge 1} \mathbf{1}_{W_n}(X_i)\, \mathbf{1}_{[\rho_n^\delta,\infty)}(\|\Xi_i\|) + \sum_{k\ge n} \mathsf{E} \sum_{i\ge 1} \mathbf{1}_{W_{k+1}\setminus W_k}(X_i)\, \mathbf{1}_{[\rho_{k+1}^\delta,\infty)}(\|\Xi_i\|) \\
&= \lambda \sum_{k\ge n} |W_k|\; \mathsf{P}(\rho_k^\delta \le \|\Xi_0\| < \rho_{k+1}^\delta) \\
&\le \frac{\lambda\, c_0^q}{\delta^q}\, \mathsf{E}\|\Xi_0\|^q\, \mathbf{1}_{[\rho_n^\delta,\infty)}(\|\Xi_0\|) \downarrow 0 \quad \text{as} \quad n \to \infty\,.
\end{aligned} \tag{2.10}$$

Obviously, if $\|\Xi_i\| \le r_n^\delta$ then $W_n^\delta \subseteq W_n \ominus \check\Xi_i \subseteq W_n$. Hence, using the abbreviation $X(G,t) := |G|^{-1} \sum_{i\ge 1} \mathbf{1}_G(X_i)\, \mathbf{1}_{(-\infty,t]}(f(\Xi_i))$ for $G \subseteq \mathbb{R}^d$ and choosing $\delta := \varepsilon/(2\varepsilon + 2\lambda)$ in (2.8) we find that, for any $k \ge 1$,

$$\begin{aligned}
A_k(\varepsilon) \cap B_k(\delta) &= \Big\{\, |\,\hat\lambda_k\, \hat F_k(t) - \lambda\, F(t)\,| \ge \varepsilon \Big\} \cap \bigcap_{i: X_i \in W_k} \{\|\Xi_i\| \le r_k^\delta\} \\
&\subseteq \Big\{ \frac{|W_k|}{|W_k^\delta|}\, X(W_k,t) \ge \lambda\, F(t) + \varepsilon \Big\} \cup \Big\{ \frac{|W_k^\delta|}{|W_k|}\, X(W_k^\delta,t) \le \lambda\, F(t) - \varepsilon \Big\} \\
&\subseteq \Big\{ X(W_k,t) - \lambda\, F(t) \ge \lambda\Big(\frac{|W_k^\delta|}{|W_k|} - 1\Big) + \varepsilon\, \frac{|W_k^\delta|}{|W_k|} \Big\} \cup
\end{aligned}$$

$$\Big\{ X(W_k^\delta,t) - \lambda F(t) \le \lambda\Big(\frac{|W_k|}{|W_k^\delta|} - 1\Big) - \varepsilon \frac{|W_k|}{|W_k^\delta|} \Big\}$$
$$\subseteq \quad \Big\{ X(W_k,t) - \lambda F(t) \ge \frac{\varepsilon}{2} \Big\} \cup \Big\{ X(W_k^\delta,t) - \lambda F(t) \le -\frac{\varepsilon}{2} \Big\}.$$

As stated above we have $X(W_n^\delta,t) \underset{n\to\infty}{\longrightarrow} \lambda F(t)$ P-a.s. for any $\delta \in [0,1)$. Thus, from the previous relation it follows that $\mathsf{P}\Big(\bigcup_{k\ge n} A_k(\varepsilon) \cap B_k(\delta)\Big) \underset{n\to\infty}{\longrightarrow} 0$ for $\delta := \varepsilon/(2\varepsilon + 2\lambda)$ and this together with $\mathsf{P}\Big(\bigcup_{k\ge n} B_k^c(\delta)\Big) \underset{n\to\infty}{\longrightarrow} 0$ proves (2.9). In the same way we get $\hat\lambda_n \underset{n\to\infty}{\longrightarrow} \lambda$ P-a.s. which in turn implies that $\hat F_n(t) \underset{n\to\infty}{\longrightarrow} F(t)$ P-a.s. for any fixed $t \in \mathbb{R}^m$.

The proof of the uniform P-a.s. convergence in (2.5) consists in applying a standard technique relying on the boundedness and monotonicity of $\hat F_n(t)$ (in each component of $t = (t_1,\dots,t_m)$). For details the reader is referred to [8], where a similar case is treated. This completes the proof of Theorem 1. □

Remark 1: If Ψ_{mark} is independently marked with an i.i.d. sequence of random compact sets $(\Xi_i)_{i\ge1}$ then it suffices to assume that the unmarked point process $\Psi = \sum_{i\ge1}\delta_{X_i}$ is stationary and ergodic, see [7].

Remark 2: If $\|\Xi_0\| \le c_1 < \infty$ P−a.s. then the first part of (2.4) can be dropped. Note that $\rho(W_n) \underset{n\to\infty}{\longrightarrow} \infty$ is equivalent to $|W_n|/H^{d-1}(\partial W_n) \underset{n\to\infty}{\longrightarrow} \infty$, since for any convex compact set $W_n \subset \mathbb{R}^d$ with $|W_n| > 0$ the inclusion

$$\rho(W_n)/d \le |W_n|/H^{d-1}(\partial W_n) \le \rho(W_n)$$

holds. (The first inequality is a direct consequence of a result proved by J. M. Wills [20] and the second one follows from (2.7) with $r = \rho(W_n)$ and the obvious fact that $|W_n \ominus b(o,\rho(W_n))| = 0$).

3 Weak Convergence of Empirical Distribution Functions

In this section we will prove the announced weak convergence of the centered and normalized sequence $Y_n(t)$ (see (1.5)) of random processes on $\mathbb{R}^m$, where the empirical df $\hat F_n(t)$ is defined by (2.2). This result can be formulated for the germ-grain model (1.1) with all the independence assumptions made at the beginning of Section 1. On the other hand, in contrast to Theorem 1 the assumptions of strict stationarity and ergodicity can be considerably weakened in proving the weak convergence results stated subsequently.

A point process $\Psi = \sum_{i\ge1}\delta_{X_i}$ on $\mathbb{R}^d$ is called *weakly* or (*second-order*) stationary with intensity $\lambda > 0$, if $\mathsf{E}\Psi^2([0,1]^d) < \infty$, $\mathsf{E}\Psi(\cdot) = \lambda|\cdot|$ and the second-order factorial moment measure $\alpha^{(2)}(A\times B) = \mathsf{E}\sum^*_{i,j\ge1} \mathbf{1}_A(X_i)\,\mathbf{1}_B(X_j)$ (where $A, B \subseteq \mathbb{R}^d$ are bounded Borel sets and the sum $\sum^*$ runs over pairwise distinct indices) is invariant against diagonal shifts, i.e. $\alpha^{(2)}((A+x)\times(B+x)) = \alpha^{(2)}(A\times B)$ for any $x \in \mathbb{R}^d$.

Obviously, the (factorial) *covariance measure* $\gamma^{(2)}(A\times B) := \alpha^{(2)}(A\times B) - \lambda^2|A||B|$ has the same invariance property which, by desintegration (see [3], Chapt. 10.4), provides the existence of a unique signed measure $\gamma^{(2)}_{red}(\cdot)$ on $\mathbb{R}^d$ – called *reduced covariance measure* of Ψ – such that

$$\gamma^{(2)}(A\times B) = \lambda \int_A \gamma^{(2)}_{red}(B-x)\,\mathrm{d}x\,.$$

Further, let $|\gamma^{(2)}_{red}|(B)$ be the total variation of $\gamma^{(2)}_{red}(\cdot)$ over the Borel set $B \subseteq \mathbb{R}^d$, see [3], A1.3.

In order to prepare the proof of the Theorem 2 below we first study the weak consistency of the estimators $\hat\lambda_n$, see (2.3), and $\lambda_n^* = \Psi(W_n)/|W_n|$ to the intensity λ. In what follows, let $\underset{n\to\infty}{\overset{\mathsf{P}}{\longrightarrow}}$ indicate convergence in probability.

Weak stationarity of Ψ implies immediately the unbiasedness of λ_n^* and that the variance of $\Psi(W_n)$ can be expressed by $\gamma_{red}^{(2)}(\cdot)$ in the following way:

$$\mathsf{Var}\big(\Psi(W_n)\big) = \lambda\,|W_n| + \lambda \int\limits_{\mathbb{R}^d} |W_n \cap (W_n - x)|\,\gamma_{red}^{(2)}(\mathrm{d}x).$$

Hence, by $W_n \oplus \check{W}_n = \{x \in \mathbb{R}^d : W_n \cap (W_n - x) \neq \emptyset\}$, we obtain that

$$\mathsf{E}\big(\,\lambda_n^* - \lambda\,\big)^2 = \frac{\mathsf{Var}\big(\Psi(W_n)\big)}{|W_n|^2} \leq \frac{\lambda}{|W_n|}\,\Big(1 + |\gamma_{red}^{(2)}|(W_n \oplus \check{W}_n)\,\Big) \underset{n\to\infty}{\longrightarrow} 0 \tag{3.1}$$

if the additional assumption

$$\frac{|\gamma_{red}^{(2)}|(W_n \oplus \check{W}_n)}{|W_n|} \underset{n\to\infty}{\longrightarrow} 0 \quad \text{as} \quad |W_n| \underset{n\to\infty}{\longrightarrow} \infty \quad \text{(in particular, if} \quad |\gamma_{red}^{(2)}|(\mathbb{R}^d) < \infty\,). \tag{3.2}$$

is satisfied. Thus, λ_n^* is mean-square consistent for λ and this in turn gives $\lambda_n^* \overset{\mathsf{P}}{\underset{n\to\infty}{\longrightarrow}} \lambda$ as $|W_n| \underset{n\to\infty}{\longrightarrow} \infty$.

The corresponding result for $\hat{\lambda}_n$ is formulated in the following lemma.

Lemma 1: *Let the conditions of Theorem 1 be fulfilled, where ergodicity of Ψ_{mark} can be replaced by the assumption that the (strictly) stationary unmarked point process $\Psi(\cdot) = \Psi_{mark}((\cdot) \times \mathcal{K}')$ has second moments and its reduced covariance measure $\gamma_{red}^{(2)}(\cdot)$ satisfies (3.2). Then $\hat{\lambda}_n$ is an unbiased, weakly consistent estimator for λ, i.e.*

$$\hat{\lambda}_n \overset{\mathsf{P}}{\underset{n\to\infty}{\longrightarrow}} \lambda. \tag{3.3}$$

Proof: To verify (3.3) we again apply the truncation technique used in the proof of Theorem 1. With the notation used there we may write for any $\varepsilon, \delta \in (0,1)$, that

$$\begin{aligned}\mathsf{P}(|\hat{\lambda}_n - \lambda| \geq \varepsilon) &= \mathsf{P}(\{|\hat{\lambda}_n - \lambda| \geq \varepsilon\} \cap B_n(\delta)) + \mathsf{P}(\{|\hat{\lambda}_n - \lambda| \geq \varepsilon\} \cap B_n^c(\delta)) \\ &\leq \mathsf{P}(|\lambda_n^* - \lambda| \geq \varepsilon/2) + \mathsf{P}(\{|\hat{\lambda}_n - \lambda_n^*| \geq \varepsilon/2\} \cap B_n(\delta)) + \mathsf{P}(B_n^c(\delta))\,.\end{aligned}$$

In view of the decomposition

$$\hat{\lambda}_n - \lambda_n^* = \frac{1}{|W_n|} \sum_{i\geq 1} \mathbf{1}_{W_n}(X_i)\,\frac{|W_n \setminus (W_n \ominus \check{\Xi}_i)|}{|W_n \ominus \check{\Xi}_i|} - \sum_{i\geq 1} \frac{\mathbf{1}_{W_n \setminus (W_n \ominus \check{\Xi}_i)}(X_i)}{|W_n \ominus \check{\Xi}_i|}$$

and since $\|\Xi_i\| \leq r_n^\delta$ implies the inclusion $W_n^\delta := W_n \ominus b(o, r_n^\delta) \subseteq W_n \ominus \check{\Xi}_i$ we arrive at

$$\{|\hat{\lambda}_n - \lambda_n^*| \geq \varepsilon/2\} \cap B_n(\delta) \subseteq \Big\{\frac{\Psi(W_n)\,|W_n \setminus W_n^\delta|}{|W_n|\,|W_n^\delta|} \geq \frac{\varepsilon}{4}\Big\} \cup \Big\{\frac{\Psi(W_n \setminus W_n^\delta)}{|W_n^\delta|} \geq \frac{\varepsilon}{4}\Big\}.$$

By Markov's inequality and (2.8) ,

$$\mathsf{P}(\{|\hat{\lambda}_n - \lambda_n^*| \geq \varepsilon/2\} \cap B_n(\delta)) \leq \frac{8\,\lambda\,|W_n \setminus W_n^\delta|}{\varepsilon\,|W_n^\delta|} \leq \frac{8\,\lambda\,\delta}{\varepsilon\,(1-\delta)}.$$

Finally, (3.1) and Chebyshev's inequality yield $\mathsf{P}(|\lambda_n^* - \lambda| \geq \varepsilon/2) \underset{n\to\infty}{\longrightarrow} 0$ so that (3.4) and (2.10) with $\delta = \varepsilon^2/2$ imply that

$$\overline{\lim_{n\to\infty}}\, \mathsf{P}(|\hat{\lambda}_n - \lambda| \geq \varepsilon) \leq 8\,\lambda\,\varepsilon \quad \text{for any} \quad 0 < \varepsilon < 1\,.$$

This proves the assertion (3.3) of Lemma 1. □

The previous proof reveals the following observation.

Remark 3: If in Lemma 1 Ψ_{mark} is independently marked, then it suffices to require weak stationarity of $\Psi(\cdot) = \Psi_{mark}((\cdot) \times \mathcal{K}')$.

Next we summarize the essential facts concerning weak convergence of sequences of m-parameter random processes $X_n(t)$, $t \in \mathbb{R}^m$, living in $D(\mathbb{R}^m)$, the set of real functions on $\mathbb{R}^m$ which are right continuous with finite left limits existing everywhere (for a precise definitions of the limits, see [14] or [11]) — briefly the *cádlág-functions* on $\mathbb{R}^m$. We first recall that, following [1] and [14], the set $D[s,t]$ of cádlág-functions defined on the closed hyper-rectangle $[s,t] := \prod_{j=1}^m [s_j, t_j]$ (for $s = (s_1, \ldots, s_m) \in \mathbb{R}^m$ and $t = (t_1, .., t_m) \in \mathbb{R}^m$ with $s < t$, i.e. $s_j < t_j$ for $j = 1, \ldots, m$) can be equipped with a metric $\varrho_{s,t}$ making $D[s,t]$ to a Polish space which generalizes in a natural way the one-dimensional Skorohod-space $D[a,b]$ for $-\infty < a < b < \infty$, see [2], Chapt. 3, for details. Weak convergence $X_n(\cdot) \underset{n\to\infty}{\Longrightarrow} X(\cdot)$ of random elements $X_n(\cdot)$ in $D[s,t]$ is then defined in the usual way and criteria for the convergence in terms of mixed moments of increments of $X_n(\cdot)$ over neighbouring hyper-rectangles in $[s,t]$ are given in [1].

For $m = 1$, the extension of weak convergence to random processes defined on an infinite interval goes back to papers of C. Stone, T. Lindvall and W. Whitt, see [17], Chapt. 4.4.1 or [21], Chapt. 12.9 and references therein. In the Billingsley's monograph [2] (Chapt. 16 and p. 191) the reader can find a criterion for the weak convergence in $D[0,\infty)$ and $D(\mathbb{R}^1)$, respectively, which applies almost verbatim to the higher-dimensional case. A detailed study of weak convergence in the space $D([0,\infty)^m, E)$ of cádlág-functions taking values in a Polish space E can be found in [11]. In our context we need the extension of Skorohod-space $D[s,t]$ to the corresponding space $D(\mathbb{R}^m)$. The crucial point is the introduction of a metric ϱ in $D(\mathbb{R}^m)$, see [2] (pp. 168–179, 191) for $m = 1$ and [11] (pp. 182–184) for $m \geq 2$, which is defined in such way that, for $x(\cdot)$, $x_n(\cdot) \in D(\mathbb{R}^m)$, $n = 1, 2, \ldots$,

$$\varrho(x_n(\cdot), x(\cdot)) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{iff} \quad \varrho_{s,t}(r_{s,t}x_n(\cdot), r_{s,t}x(\cdot)) \underset{n\to\infty}{\longrightarrow} 0$$

for all continuity points $s, t \in \mathbb{R}^m$ of $x(\cdot)$ satisfying $s < t$. Here $r_{s,t} : D(\mathbb{R}^m) \mapsto D[s,t]$ is defined by $r_{s,t}x(u) = x(u)$ for $u \in [s,t]$ and $[s,t] \subset \mathbb{R}^m$ will be called *continuity hyper-rectangle of* $x(\cdot)$ if the cádlág-function $x : \mathbb{R}^m \mapsto \mathbb{R}^1$ is continuous at both its 'lower-left' vertex s and its 'upper-right' vertex t.

In this way weak convergence in $D(\mathbb{R}^m)$ can be reduced to weak convergence in the more familiar space $D[s,t]$, see [14], [1]. A criterion for weak convergence in $D(\mathbb{R}^m)$ generalizing Proposition 4.18 in [17] for $m = 1$ is stated in the following proposition.

Proposition 1: If $\{X_n(\cdot),\, n \geq 1\}$ and $X(\cdot)$ are random elements of $D(\mathbb{R}^m)$ then

$$X_n(\cdot) \underset{n\to\infty}{\Longrightarrow} X(\cdot) \qquad \text{in} \quad D(\mathbb{R}^m)$$

if and only if for any continuity hyper-rectangle $[s,t]$ of $X(\cdot)$ (i.e. $\mathsf{P}(X(\cdot)$ is continuous at s and $t) = 1$) we have

$$r_{s,t}X_n(\cdot) \underset{n\to\infty}{\Longrightarrow} r_{s,t}X(\cdot) \qquad \text{in} \quad D[s,t]\,.$$

Note that the latter limiting relations are needed only for two sequences $s^{(k)} = (s_1^{(k)}, \ldots, s_m^{(k)})$ and $t^{(k)} = (t_1^{(k)}, \ldots, t_m^{(k)})$ of continuity points of $X(\cdot)$ satisfying the conditions $\max\{s_1^{(k)}, \ldots, s_m^{(k)}\} \underset{k\to\infty}{\longrightarrow} -\infty$ and $\min\{t_1^{(k)}, \ldots, t_m^{(k)}\} \underset{k\to\infty}{\longrightarrow} \infty$, which do always exist.

By the very definition of an m-variate df $F(t)$ and its empirical counterpart $\hat{F}_n(t)$, see (2.2), we immediately get the following proposition.

Proposition 2: The random processes $\{Y_n(\cdot),\, n \geq 1\}$ defined by (1.5) as well as the mean zero Gaussian process $Y(\cdot)$ with covariance function $\mathsf{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t)$ belong P-a.s. to the subspace $D_0(\mathbb{R}^m)$ containing those $x \in D(\mathbb{R}^m)$ which have finite one-sided limits $\lim_{t\to t^*} x(t)$ for any $t^* = (t_1^*, \ldots, t_m^*)$ with $t_i^* \in \{-\infty, +\infty\}$ for some $i \in \{1, \ldots, m\}$, see [11].

Our next step towards the proof of $Y_n(\cdot) \underset{n\to\infty}{\Longrightarrow} Y(\cdot)$ in $D(\mathbb{R}^m)$ consists in showing that $Y(\cdot)$ is the weak limit of the following sequence of m-parameter empirical processes

$$Z_n(t) = \frac{1}{\sqrt{\lambda\,|W_n|}} \sum_{i\ge 1} \mathbf{1}_{W_n}(X_i) \Big(\mathbf{1}_{(-\infty,t]}(f(\Xi_i)) \;-\; F(t) \Big), \quad t \in \mathbb{R}^m \,.$$

Lemma 2: *Let the germ-grain model* (1.1) *be defined by a weakly stationary point process* $\Psi = \sum_{i\ge 1} \delta_{X_i}$ *with intensity* $\lambda > 0$ *and an independent sequence* $\{\Xi_i, i \ge 1\}$ *of i.i.d. copies of the typical grain* Ξ_0 *. If in addition the reduced covariance measure* $\gamma^{(2)}_{red}(\cdot)$ *of* Ψ *satisfies condition* (3.2) *, then*

$$Z_n(\cdot) \underset{n\to\infty}{\Longrightarrow} Y(\cdot) \qquad in \quad D(\mathbb{R}^m) \tag{3.4}$$

provided that $|W_n| \underset{n\to\infty}{\longrightarrow} \infty$ *, where* $Y(t)$ *,* $t \in \mathbb{R}^m$ *, denotes the Gaussian process of Proposition 2.*

Proof: A straightforward application of the classical CLT for i.i.d. random variables shows that

$$U_N(t) := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \Big(\mathbf{1}_{(-\infty,t]}(f(\Xi_i)) \;-\; F(t) \Big) \underset{N\to\infty}{\Longrightarrow} \mathcal{N}\big(0, F(t)(1-F(t))\big) \quad \text{for any} \quad t \in \mathbb{R}^m \,,$$

where $\mathcal{N}\big(\mu, \sigma^2\big)$ denotes a Gaussian random variable with mean μ and variance σ^2 . Since, after a short calculation, $\mathsf{E} U_N(s) U_N(t) = F(s \wedge t) - F(s)F(t)$ for any $N \ge 1$, we get in the same way

$$\sum_{k=1}^{p} c_k U_N(t^{(k)}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N}\sum_{k=1}^{p} c_k \,\Big(\mathbf{1}_{(-\infty,t^{(k)}]}(f(\Xi_i)) \;-\; F(t^{(k)}) \Big) \underset{N\to\infty}{\Longrightarrow} \mathcal{N}\big(0, \sigma^2(t^{(1)},\dots,t^{(p)})\big)$$

for any $t^{(1)},\dots,t^{(p)} \in \mathbb{R}^m$ and $c_1,\dots,c_p \in \mathbb{R}^1$, where

$$\sigma^2(t^{(1)},\dots,t^{(p)}) = \sum_{k,l=1}^{p} c_k\, c_l \,\big(F(t^{(k)} \wedge t^{(l)}) - F(t^{(k)})F(t^{(l)})\big) \,.$$

A well-known so-called "transfer theorem" see [5] (Chapt. 4) or [16] (Chapt. 8.7), tells us that the previous CLT remains valid if the number of summands N is replaced by the random number $\Psi(W_n)$ ($\underset{n\to\infty}{\longrightarrow} \infty$). Here we have used that the random integers $\Psi(W_n)$ are independent of the i.i.d. sequence $f(\Xi_i), i \ge 1$. Further, by using Lemma 1 and Slutsky's theorem, we find that

$$\sum_{k=1}^{p} c_k Z_n(t^{(k)}) = \sqrt{\frac{\Psi(W_n)}{\lambda\,|W_n|}} \sum_{k=1}^{p} c_k U_{\Psi(W_n)}(t^{(k)}) \underset{n\to\infty}{\Longrightarrow} \mathcal{N}\big(0, \sigma^2(t^{(1)},\dots,t^{(p)})\big) \,.$$

Hence, by applying the Cramér-Wold device,

$$\big((Z_n(t^{(1)}),\dots,Z_n(t^{(p)})\big) \underset{n\to\infty}{\Longrightarrow} \big(Y(t^{(1)}),\dots,Y(t^{(p)})\big)$$

for any $t^{(1)},\dots,t^{(p)} \in \mathbb{R}^m$. It remains to verify the tightness of the sequence $Z_n(\cdot), n \ge 1$, by estimating the mixed fourth moment $\mathsf{E} Z_n^2((u,v]) Z_n^2((\bar{u},\bar{v}])$ for each pair of neighbouring half-open rectangles $(u,v], (\bar{u},\bar{v}]$ having a common $(m-1)$-dimensional face and lying in some continuity hyper-rectangle $[s,t]$ of $Y(\cdot)$. Note that the "increment" of $Z_n(\cdot)$ around $(u,v]$, $u < v$, is defined by

$$Z_n((u,v]) = \sum_{\varepsilon_1,\dots,\varepsilon_m \in \{0,1\}} (-1)^{m-\varepsilon_1-\dots-\varepsilon_m} Z_n(u_1 + \varepsilon_1(v_1-u_1),\dots,u_m + \varepsilon_m(v_m-u_m)) \,.$$

For brevity put $g_{(u,v]}(K) = \mathbf{1}_{(u,v]}(f(K)) - \nu_F((u,v])$ for $K \in \mathcal{K}$ and $u < v$, where $\nu_F(\cdot)$ designates the probability measure on the Borel sets of $\mathbb{R}^m$ generated by the df $F(\cdot)$.

In view of the independence assumptions made in Lemma 2 and $\mathsf{E}g_{(u,v]}(\Xi_0) = 0$ we obtain

$$
\begin{aligned}
&\mathsf{E}Z_n^2((u,v])\, Z_n^2((\bar{u},\bar{v}]) \\
&= \frac{1}{\lambda^2\,|W_n|^2} \sum_{p\geq 1} \mathsf{P}(\Psi(W_n) = p) \sum_{i,j=1}^{p} \sum_{k,l=1}^{p} \mathsf{E}g_{(u,v]}(\Xi_i) g_{(u,v]}(\Xi_j) g_{(\bar{u},\bar{v}]}(\Xi_k) g_{(\bar{u},\bar{v}]}(\Xi_l) \\
&= \frac{\mathsf{E}(\Psi(W_n))}{\lambda^2\,|W_n|^2}\, \mathsf{E}g^2_{(u,v]}(\Xi_0)\, g^2_{(\bar{u},\bar{v}]}(\Xi_0) + \frac{\alpha^{(2)}(W_n \times W_n)}{\lambda^2\,|W_n|^2} \\
&\qquad\qquad \times \Big(\mathsf{E}g^2_{(u,v]}(\Xi_0)\, \mathsf{E}g^2_{(\bar{u},\bar{v}]}(\Xi_0) \; + 2 \Big(\mathsf{E}g_{(u,v]}(\Xi_0)\, g_{(\bar{u},\bar{v}]}(\Xi_0) \Big)^2 \Big) \\
&\leq \; \frac{\nu_F((u,v])\,\nu_F((\bar{u},\bar{v}])}{\lambda\,|W_n|} \Big(1 + 3\,|\gamma^{(2)}_{red}|(W_n \oplus \check{W}_n) + 3\,\lambda\,|W_n| \Big) \; \leq \; c(\lambda)\, \nu_F((u,v])\, \nu_F((\bar{u},\bar{v}])
\end{aligned}
$$

for large enough n, which proves the desired tightness condition, see [1].

Here we have used (3.1) and the relations $\mathsf{E}g^2_{(u,v]}(\Xi_0) = \nu_F((u,v])\,\big(1 - \nu_F((u,v])\big)$,

$$\mathsf{E}g_{(u,v]}(\Xi_0) g_{(\bar{u},\bar{v}]}(\Xi_0) = -\nu_F((u,v])\,\nu_F((\bar{u},\bar{v}])$$

and

$$\mathsf{E}g^2_{(u,v]}(\Xi_0)\, g^2_{(\bar{u},\bar{v}]}(\Xi_0) = \nu_F((u,v])\,\nu_F((\bar{u},\bar{v}]) \Big(\nu_F((u,v]) + \nu_F((\bar{u},\bar{v}]) - 3\nu_F((u,v])\,\nu_F((\bar{u},\bar{v}]) \Big)\,.$$

Thus, by Theorem 4 in [1] and some additional comments given there, we have shown that

$$Z_n(\cdot) \underset{n\to\infty}{\Longrightarrow} Y(\cdot) \qquad \text{in} \quad D[s,t]$$

for any continuity hyper-rectangle $[s,t]$ of $Y(\cdot)$. This together with Proposition 1 completes the proof of (3.4). □

Remark 4: Note that the weak convergence of empirical df's for i.i.d. samples taking values in the unit cube $[0,1]^m$ and having a not necessarily continuous df $F(\cdot)$ has been already proven by G. Neuhaus [14]. Further refinements of this result can be found in [4]. Lemma 2 extends this result to a random number of i.i.d. observations in $\mathbb{R}^m$ driven by a stationary point process.

Remark 5: By applying a large deviations inequality for empirical df's due to Kiefer and Wolfowitz [13] we obtain the estimate $\mathsf{P}(\sup_{t\in\mathbb{R}^m}\big|U_N(t)\big| \geq r\,\sqrt{N}) \;\leq\; A\,\mathrm{e}^{-a\,r^2\,N}$ for all $N \geq 1$ and all $r > 0$, where a and A are positive constants. This result and the independence assumption of Lemma 2 yield

$$\mathsf{P}\Big(\sup_{t\in\mathbb{R}^m} \big|Z_n(t)\big| \geq r\,\lambda_n^*\,\sqrt{|W_n|} \Big) \;\leq\; A\,\mathsf{E}\mathrm{e}^{-a\,\lambda\,r^2\,\Psi(W_n)} \quad \Big(= A\,\exp\Big\{ -\lambda\,\big(1 - \mathrm{e}^{-a\,\lambda\,r^2}\big)\,|W_n| \Big\} \Big)\,,$$

where the equality included in parenthesis holds if the point process $\Psi(\cdot)$ is Poisson.

Now we are in a position to prove our main result.

Theorem 2: *Let the germ-grain model* (1.1) *be defined by a weakly stationary point process* $\Psi = \sum_{i\geq 1} \delta_{X_i}$ *with intensity* $\lambda > 0$ *and an independent sequence* $\{\Xi_i, i \geq 1\}$ *of i.i.d. copies of the typical grain* Ξ_0. *Assume that* $|\gamma^{(2)}_{red}|(\mathbb{R}^d) < \infty$ *and* $\mathsf{E}\|\Xi_0\|^q < \infty$ *for some* $q \geq d$ *and let* (W_n) *be an increasing sequence of convex, bounded sets in* $\mathbb{R}^d$ *satisfying* (2.4). *Then the sequence of empirical processes* $\{Y_n(\cdot)\,, n \geq 1\}$ *defined by* (1.5) *and* (2.2) *converges weakly (as* $n \to \infty$*) in* $D(\mathbb{R}^m)$ *to a mean zero Gaussian process* $Y(\cdot)$ *with covariance function* $\mathsf{E}Y(s)Y(t) = F(s \wedge t) - F(s)F(t)$.

PROOF: First note that by applying Lemma 1 and Slutsky-type arguments, see e.g. Theorem 3.1 in [2], the weak limit (if it exists) of $Y_n(t)$, $t \in \mathbb{R}^m$, coincides with that of

$$\hat{Z}_n(t) = \frac{\hat{\lambda}_n}{\sqrt{\lambda\,\lambda_n^*}}\,Y_n(t) = \sqrt{\frac{|W_n|}{\lambda}}\sum_{i\ge 1}\frac{\mathbf{1}_{W_n\ominus\check{\Xi}_i}(X_i)}{|W_n\ominus\check{\Xi}_i|}\,\big(\mathbf{1}_{(-\infty,t]}(f(\Xi_i))\,-\,F(t)\big)\,.$$

This means that Lemma 2 and Theorem 3.1 in [2] imply $Y_n(\cdot)\underset{n\to\infty}{\Longrightarrow} Y(\cdot)$ in $D(\mathbb{R}^m)$ whenever we can verify that $\varrho_{s,t}(Z_n(\cdot),\hat{Z}_n(\cdot))\xrightarrow[n\to\infty]{\mathsf{P}} 0$ or slightly stronger that

$$\sup_{u\in[s,t]}|Z_n(u)-\hat{Z}_n(u)|\xrightarrow[n\to\infty]{\mathsf{P}} 0\quad\text{for any continuity hyper-rectangle}\quad [s,t]\subseteq\mathbb{R}^m\,. \tag{3.5}$$

For this we consider the difference process $\Delta_n(\cdot) := \sqrt{\lambda}\left(Z_n(\cdot)-\hat{Z}_n(\cdot)\right)$ on the event $B_n(\delta) = \bigcap_{i:X_i\in W_n}\{\|\Xi_i\|\le r_n^\delta\}$ with $r_n^\delta = \delta\,|W_n|/H^{d-1}(\partial W_n)$ and $\delta\in(0,1)$ (as in the proof of Theorem 1) which allows to replace the grains Ξ_i (with germs $X_i\in W_n$) by the truncated grains $\Xi_i^\delta = \Xi_i\cap b(o,r_n^\delta)$ such that

$$\Delta_n(t)\,\mathbf{1}_{B_n(\delta)} = \left(\Delta_n^\delta(t)-\Delta_{n,0}^\delta\,F(t)\right)\mathbf{1}_{B_n(\delta)}\quad\text{for all}\quad t\in\mathbb{R}^m\,,$$

where

$$\Delta_n^\delta(t) := \sqrt{|W_n|}\sum_{i\ge 1}\left(\frac{\mathbf{1}_{W_n}(X_i)}{|W_n|}\,-\,\frac{\mathbf{1}_{W_n\ominus\check{\Xi}_i^\delta}(X_i)}{|W_n\ominus\check{\Xi}_i^\delta|}\right)\mathbf{1}_{(-\infty,t]}(f(\Xi_i))$$

and

$$\Delta_{n,0}^\delta := \sqrt{|W_n|}\sum_{i\ge 1}\left(\frac{\mathbf{1}_{W_n}(X_i)}{|W_n|}\,-\,\frac{\mathbf{1}_{W_n\ominus\check{\Xi}_i^\delta}(X_i)}{|W_n\ominus\check{\Xi}_i^\delta|}\right).$$

For any $\varepsilon>0$ and $\delta\in(0,1)$ we have

$$\begin{aligned}\mathsf{P}(\sup_{u\in[s,t]}|\Delta_n(u)|\ge\varepsilon) &= \mathsf{P}(\sup_{u\in[s,t]}|\Delta_n(u)|\ge\varepsilon, B_n(\delta))+\mathsf{P}(\sup_{u\in[s,t]}|\Delta_n(u)|\ge\varepsilon, B_n^c(\delta))\\ &\le \mathsf{P}(\sup_{u\in[s,t]}|\Delta_n^\delta(u)|\ge\varepsilon/2)+\mathsf{P}\left(|\Delta_{n,0}^\delta|\ge\varepsilon/2\right)+\mathsf{P}(B_n^c(\delta))\,.\end{aligned}$$

The moment assumption imposed on $\|\Xi_0\|$ combined with (2.4) implies (as an immediate consequence of (2.10)) that $\mathsf{P}(B_n^c(\delta_n))\xrightarrow[n\to\infty]{} 0$ for certain sequence $\delta_n>0$ with $\delta_n\downarrow 0$ as $n\to\infty$. The second term in the latter line will be estimated using Chebyshev's inequality. Having in mind $\mathsf{E}\Delta_{n,0}^\delta = 0$, we can express and estimate the second moment $\mathsf{E}(\Delta_{n,0}^\delta)^2$ by means of the reduced second cumulant measure $\gamma_{red}^{(2)}(\cdot)$ as follows:

$$\begin{aligned}&\lambda\,|W_n|\int_{\mathbb{R}^d}\mathsf{E}\left(\frac{\mathbf{1}_{W_n}(x)}{|W_n|}\,-\,\frac{\mathbf{1}_{W_n\ominus\check{\Xi}_0^\delta}(x)}{|W_n\ominus\check{\Xi}_0^\delta|}\right)^2\mathrm{d}x\\ &+\lambda\,|W_n|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathsf{E}\left(\frac{\mathbf{1}_{W_n}(x)}{|W_n|}\,-\,\frac{\mathbf{1}_{W_n\ominus\check{\Xi}_0^\delta}(x)}{|W_n\ominus\check{\Xi}_0^\delta|}\right)\mathsf{E}\left(\frac{\mathbf{1}_{W_n}(y+x)}{|W_n|}\,-\,\frac{\mathbf{1}_{W_n\ominus\check{\Xi}_0^\delta}(y+x)}{|W_n\ominus\check{\Xi}_0^\delta|}\right)\gamma_{red}^{(2)}(\mathrm{d}y)\,\mathrm{d}x\\ \le\;&\frac{\lambda\,(1+|\gamma_{red}^{(2)}|(\mathbb{R}^d))}{|W_n|}\int_{\mathbb{R}^d}\mathsf{E}\left(\mathbf{1}_{W_n}(x)\,-\,\mathbf{1}_{W_n\ominus\check{\Xi}_i^\delta}(x)\frac{|W_n|}{|W_n\ominus\check{\Xi}_0^\delta|}\right)^2\mathrm{d}x\\ =\;&\lambda\left(1+|\gamma_{red}^{(2)}|(\mathbb{R}^d)\right)\mathsf{E}\,\frac{|W_n\setminus(W_n\ominus\check{\Xi}_0^\delta)|}{|W_n\ominus\check{\Xi}_0^\delta|}\le\lambda\left(1+|\gamma_{red}^{(2)}|(\mathbb{R}^d)\right)\frac{\delta}{1-\delta}\,.\end{aligned}$$

The last inequality follows from (2.8). Hence,

$$\mathsf{P}\Big(\,|\Delta^{\delta_n}_{n,0}| \ge \frac{\varepsilon}{2}\,\Big) \underset{n\to\infty}{\longrightarrow} 0 \quad \text{for any} \quad \varepsilon > 0\,.$$

In the same way as before we get

$$\mathsf{E}(\Delta^{\delta}_n(t))^2 \le \lambda\left(1 + |\gamma^{(2)}_{red}|(\mathbb{R}^d)\right)\frac{\delta}{1-\delta}$$

which implies $\Delta^{\delta_n}_n(t) \xrightarrow[n\to\infty]{\mathsf{P}} 0$ for all $t \in \mathbb{R}^m$.

In order to verify the uniform convergence $\sup_{u\in[s,t]} |\Delta^{\delta_n}_n(u)| \xrightarrow[n\to\infty]{\mathsf{P}} 0$ for any continuity hyper-rectangle $[s,t]$ we rewrite $\Delta^{\delta}_n(t)$ as sum $\Delta^{\delta}_{n,1}(t) + \Delta^{\delta}_{n,2}(t)$, with

$$\Delta^{\delta}_{n,1}(t) = \frac{1}{\sqrt{|W_n|}}\sum_{i\ge 1} A^{\delta}_n(X_i,\Xi_i,(-\infty,t]) \quad \text{and} \quad \Delta^{\delta}_{n,2}(t) = \frac{1}{\sqrt{|W_n|}}\sum_{i\ge 1} a^{\delta}_n(X_i,(-\infty,t])\,,$$

where the functions $A^{\delta}_n(x,K,B)$ and $a^{\delta}_n(x,B)$ are defined for any $x \in \mathbb{R}^d$, $B \subseteq \mathbb{R}^m$ and $K \in \mathcal{K}'$ with $c(K) = o$ as follows:

$$A^{\delta}_n(x,K,B) = a^{\delta}_n(x,K,B) - a^{\delta}_n(x,B) \quad \text{with} \quad a^{\delta}_n(x,B) = \int\limits_{\mathcal{K}'} a^{\delta}_n(x,K,B)\,Q(\mathrm{d}K)$$

and

$$a^{\delta}_n(x,K,B) = \left(\mathbf{1}_{W_n}(x) \;-\; \frac{|W_n|}{|W_n \ominus (\check{K}\cap b(o,r^{\delta}_n))|}\,\mathbf{1}_{W_n\ominus(\check{K}\cap b(o,r^{\delta}_n))}(x)\right)\mathbf{1}_B(f(K))\,.$$

From (2.8) we get

$$|a^{\delta}_n(x,K,B)| \le \max\left\{1, \frac{\delta}{1-\delta}\right\}\,\mathbf{1}_{W_n}(x)\;\mathbf{1}_B(f(K))$$

and hence, for $0 < \delta \le 1/2$,

$$|a^{\delta}_n(x,B)| \le \mathbf{1}_{W_n}(x)\,\nu_F(B) \qquad \text{and} \qquad |A^{\delta}_n(x,K,B)| \le \mathbf{1}_{W_n}(x)\,\big(\,\mathbf{1}_B(f(K)) + \nu_F(B)\,\big)\,. \tag{3.6}$$

It is obvious that both random processes $\Delta^{\delta}_{n,1}(\cdot)$ and $\Delta^{\delta}_{n,2}(\cdot)$ belong to $D(\mathbb{R}^m)$ for any $\delta \in (0,1)$. Further, calculating the second moment of $\Delta^{\delta}_{n,i}(t)$ along the above lines and using Chebyshev's inequality yield $\Delta^{\delta_n}_{n,i}(t) \xrightarrow[n\to\infty]{\mathsf{P}} 0$ for any fixed $t \in \mathbb{R}^m$ and $i = 1,2$.

The proof of the weak convergence $\Delta^{\delta_n}_{n,i}(\cdot) \underset{n\to\infty}{\Longrightarrow} 0$ in $D[s,t]$ for some continuity hyper-rectangle (implying $\sup_{u\in[s,t]} |\Delta^{\delta_n}_{n,i}(u)| \xrightarrow[n\to\infty]{\mathsf{P}} 0$) relies on the tightness of the sequences $\{\Delta^{\delta_n}_{n,i}(\cdot)\,, n \ge 1\}$ in $D[s,t]$ which will be verified now.

By virtue of the independence between $\Psi = \sum_{i\ge 1}\delta_{X_i}$ and the i.i.d. sequence of grains $\{\Xi_i\,, i \ge 1\}$ together with $\mathsf{E}A^{\delta}_n(x,\Xi_0,B) = 0$ and (3.6) we obtain for any two disjoint (neighbouring) hyper-rectangles $(u,v]$ and $[\bar{u},\bar{v}]$ and $0 < \delta \le 1/2$ that

$$\mathsf{E}\big(\Delta^{\delta}_{n,1}((u,v])\big)^2\big(\Delta^{\delta}_{n,1}((\bar{u},\bar{v}])\big)^2 = \frac{\lambda}{|W_n|^2}\int\limits_{\mathbb{R}^d} \mathsf{E}\Big(A^{\delta}_n(x,\Xi_0,(u,v])\Big)^2\,\Big(A^{\delta}_n(x,\Xi_0,(\bar{u},\bar{v}])\Big)^2\,\mathrm{d}x$$

$$+\,\frac{1}{|W_n|^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \mathsf{E}\Big(A^{\delta}_n(x,\Xi_0,(u,v])\Big)^2\mathsf{E}\Big(A^{\delta}_n(y,\Xi_0,(\bar{u},\bar{v}])\Big)^2\,\alpha^{(2)}(\mathrm{d}x,\mathrm{d}y)$$

$$+\frac{2}{|W_n|^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\Big(\mathsf{E}A_n^{\delta}(x,\Xi_0,(u,v])\,A_n^{\delta}(y,\Xi_0,(\bar{u},\bar{v}])\Big)^2\,\alpha^{(2)}(\mathrm{d}x,\mathrm{d}y)$$

$$\leq \quad \frac{4\,\lambda\,\nu_F((u,v])\nu_F(\bar{u},\bar{v}])}{|W_n|}+\frac{48\,\alpha^{(2)}(W_n\times W_n)\,\nu_F((u,v])\nu_F(\bar{u},\bar{v}])}{|W_n|^2}.$$

Similarly, by making use of $\int\limits_{\mathbb{R}^d} a_n^{\delta}(x,(u,v])\,\mathrm{d}x=0$, we find that

$$\mathsf{E}\big(\Delta_{n,2}^{\delta}((u,v])\big)^2 \;=\; \frac{\lambda}{|W_n|}\int\limits_{\mathbb{R}^d}\Big(\big(a_n^{\delta}(x,(u,v])\big)^2+\int\limits_{\mathbb{R}^d} a_n^{\delta}(x,(u,v])a_n^{\delta}(y+x,(u,v])\,\gamma_{red}^{(2)}(\mathrm{d}y)\Big)\,\mathrm{d}x$$

$$\leq \quad \lambda\left(1+|\gamma_{red}^{(2)}|(\mathbb{R}^d)\right)(\nu_F((u,v]))^2\,.$$

This and the previous estimate together with $\alpha^{(2)}(W_n\times W_n)\leq\lambda\,|W_n|(\lambda\,|W_n|+|\gamma_{red}^{(2)}|(\mathbb{R}^d))$, see (3.1), yield

$$\mathsf{E}\big(\Delta_{n,i}^{\delta_n}((u,v])\big)^2\big(\Delta_{n,i}^{\delta_n}((\bar{u},\bar{v}])\big)^2\leq C\,\nu_F((u,v])\nu_F(\bar{u},\bar{v}])\quad \text{for}\quad i=1,2$$

proving the tightness of $\Delta_{n,1}^{\delta_n}(\cdot)$ and $\Delta_{n,2}^{\delta_n}(\cdot)$ in $D[s,t]$. Summarizing the above steps and (3.5) completes the proof of Theorem 2. □

In view of Proposition 2 and using the fact that the mapping $x(\cdot)\mapsto\sup_{u\in\mathbb{R}^m}|x(u)|$ is continuous on the subspace $D_0(\mathbb{R}^m)$ (which results essentially from its continuity on the spaces $D[s,t]$ for $s,t\in\mathbb{R}^m$, see [2], [14], [4], [11]), Theorem 2 and the continuous mapping theorem yield

$$\sup_{t\in\mathbb{R}^m}|Y_n(t)|\underset{n\to\infty}{\Longrightarrow}\sup_{t\in\mathbb{R}^m}|Y(t)|,$$

where the df of the limit depends on $F(\cdot)$ for $m\geq 2$, see e.g. [13].

4 Kolmogorov-Smirnov test

The case $m=1$ is of special interest for testing the goodness-of-fit of some hypothetical distribution function $F(t)=\mathsf{P}(f(\Xi_0)\leq t)$ for $t\in\mathbb{R}^1$. We know that in this case the Gaussian limit process $Y(\cdot)$ is stochastically equivalent to $W^o(F(\cdot))$, where $W^o(\cdot)$ is the Brownian bridge — a zero mean Gaussian process on $[0,1]$ with $\mathsf{E}W^o(s)W^o(t)=s\wedge t-s\,t$, see e.g. [2].

Corollary: *Under the assumptions of Theorem 2 (for $m=1$) we have*

$$\sup_{t\in\mathbb{R}^1}|Y_n(t)|\underset{n\to\infty}{\Longrightarrow}\sup_{t\in\mathbb{R}^1}|W^o(F(t))|.$$

Furthermore, if $F(\cdot)$ is continuous then the limit df $\mathsf{P}(\sup_{0\leq t\leq 1}|W^o(t)|\leq x)$ coincides with well-known Kolmogorov df

$$K(x)=1+2\sum_{k=1}^{\infty}(-1)^k\mathrm{e}^{-2k^2x^2}\quad \textit{for}\quad x>0.$$

This corollary enables us to perform a Kolmogorov-Smirnov test in analogy to the classical case of i.i.d. samples drawn from an unknown source with one or several hypothetical distribution functions. The

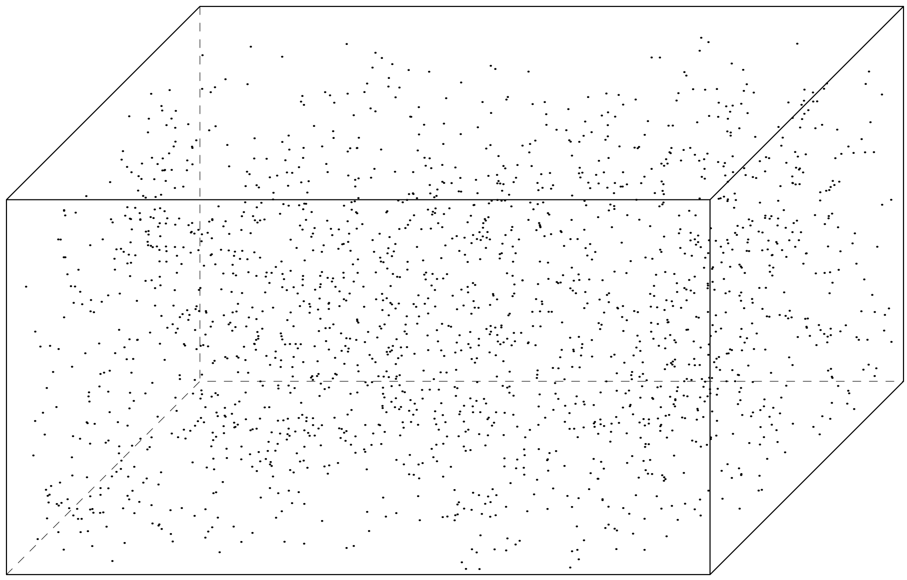

Figure 3: The point process Ψ observed in the rectangular block W_n.

Kolmogorov-Smirnov test requires the continuity of the hypothesized distribution of the data implying that the critical values of the test statistic do not depend on this distribution.

In what follows we present a practical application to the microstructure of ceramic plasma-sprayed coatings. A specimen has been prepared in the Institute of Plasma Physics, Academy of Sciences of the Czech Republic, Prague. The data analysed here (kindly provided by Dr. Pavel Ctibor) consist of approximately convex pores in a three-dimensional sampling window W_n. For further details about the data set, see [12], where the spatial distribution of particles has been investigated.

The specimen is a rectangular block with dimensions $450 \times 350 \times 240\,\mu$m (see Fig. 3). The number of shifted grains $X_i + \Xi_i$ lying completely in W_n is $N_n = 1976$ and the number of reference points X_i in W_n is $\Psi(W_n) = 2085$. Note that the shape of W_n entails that each of the eroded windows $W_n \ominus \check{\Xi}_i$ is a rectangular block with dimension depending on the widths of Ξ_i measured parallel to the edges of W_n. This fact facilitates considerably the computation of the weighted estimator (2.2).

At first we consider the distribution of volume of the typical grain, i.e. we put $f(\Xi_0) = |\Xi_0|$. Since very small particles could not be detected and so are omitted in the study, there exists a lower threshold $a > 0$ such that $F(a) = 0$ and $F(t) > 0$ for $t > a$. Similarly, the absence of large pores indicates the existence of an upper bound $b > 0$. At the first glance and supported by the experience of the material scientists, the empirical df $\hat{F}_n(t)$ seems to be approximately Pareto distributed. Thus, we will examine the null hypothesis that the df $\mathsf{P}(f(\Xi_0) \le t)$ coincides with a truncated Pareto df $F_0(\cdot)$ given by

$$F_0(t) = 1 - \left(\frac{a}{t}\right)^c \cdot \frac{b^c - t^c}{b^c - a^c}, \quad a \le t \le b.$$

Correcting sampling bias effects by the weights $|W_n \ominus \check{\Xi}_i|$ we modify the maximum likelihood method leading to the following estimates of the location parameters $a > 0$, $b > 0$ and the shape parameter $c > 0$, respectively:

$$\begin{aligned} \hat{a} &= 31.462\,\mu\text{m}^3, \\ \hat{b} &= 6152.056\,\mu\text{m}^3, \\ \hat{c} &= 1.012. \end{aligned}$$

Here, it should be noted that the weak convergence of the empirical process (1.5) stated in Theorem 1 does not hold in general when parameters in $F(\cdot)$ are replaced by corresponding (maximum likelihood) estimators. For this reason we are not allowed to plug in the parameter estimates $\hat{a}$, $\hat{b}$ and $\hat{c}$ in the

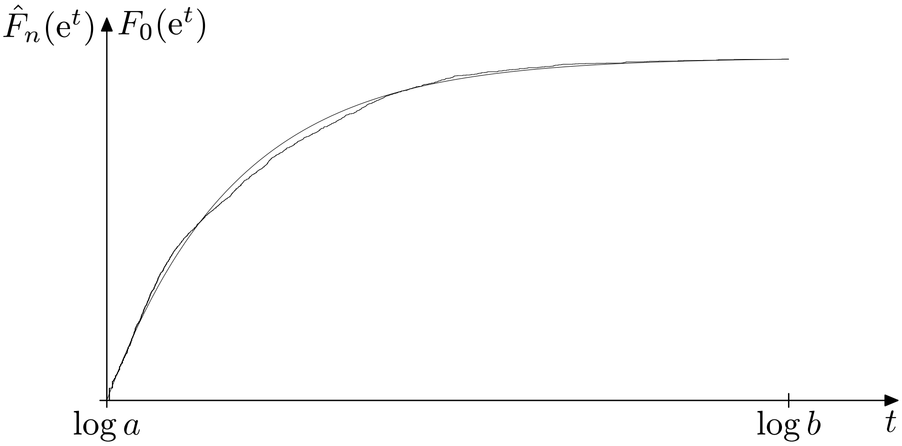

Figure 4: The empirical df of volume (dashed line) and fitted truncated Pareto df (solid line).

hypothesized df $F_0(\cdot)$. This fact is is already well-known from the classical i.i.d. case and can be interpreted as higher sensitivity of the Kolmogorov-Smirnov test against the null hypothesis.

However, the above estimates give us at least a hint about where the true parameters could be located. We perform the test for the values $a = 30.9$, $b = 6500$ and $c = 1.005$.

In Fig. 4 the plot of the empirical df $\hat{F}_n(\cdot)$ defined by (2.2) is compared with the hypothesized df $F_0(\cdot)$, where the curves are plotted in log-scale.

The maximal deviation of $\hat{F}_n(\cdot)$ from $F_0(\cdot)$ is $\sup_{t\in\mathbb{R}} |\hat{F}_n(t) - F_0(t)| = 0.025$. This means that $\sup_{t\in\mathbb{R}} |Y_n(t)| = 1.141$ is not greater than 1.358 (= 95%-quantile of the Kolmogorov df) and so the null hypothesis is not rejected at the 5%-level.

We have also computed the maximal deviation of $\tilde{F}_n(\cdot)$ (see (2.1)) from $F_0(\cdot)$ which is slightly higher than the value for $\hat{F}_n(\cdot)$: $\sup_{t\in\mathbb{R}} |\tilde{F}_n(t) - F_0(t)| = 0.030$.

As a second example we choose the shape parameter $f(\Xi_0)$ being equal to the natural logarithm of the ratio of the maximal diameter and the minimal diameter of the typical grain. Suggested by material scientists we check the null hypothesis whether the corresponding shape parameter of the pores in our specimen is Weibull distributed with parameters $\alpha = 1.04$ and $\beta = 0.31$

$$F_0(t) = 1 - e^{-(t/\beta)^\alpha}, \quad t \geq 0.$$

The functions $\hat{F}_n(t)$ and $F_0(t)$ are compared in Fig. 5, where again the log-scale is used. The maximal deviation is $\sup_{t\in\mathbb{R}} |\hat{F}_n(t) - F_0(t)| = 0.020$ implying that $\sup_{t\in\mathbb{R}} |Y_n(t)| = 0.934$. Hence, the null hypothesis is again not rejected at the 5%-level.

5 Concluding Remarks and Examples

The independence assumptions put on the germ-grain model (1.1) in Theorem 2 are crucial in proving the existence of a weak limit and guarantee that the covariance function of the Gaussian limit process $Y(\cdot)$ depends only on the df $F(\cdot)$. The above Theorem 2 suggests generalizations in various directions. One of them is to consider empirical marginal df's in germ-grain models with dependent grains and dependences between the sequence of grains and the point process of germs. Random tessellations seem to be tractable structures of this kind which give rise to future investigations. In any case these models have to satisfy certain (strong) mixing-type conditions as they have been shown and applied e.g. in [9] or [10], respectively. On the other hand, the covariance function of the expected Gaussian limit process will reflect the dependence structure of the model under consideration. There is an obvious similarity with the asymptotic behaviour of empirical processes related with strictly stationary sequences, see e.g. [18].

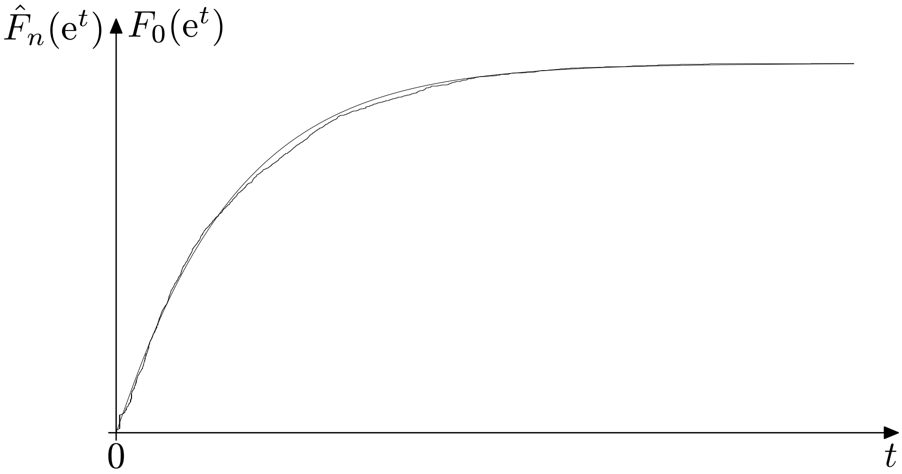

Figure 5: The empirical df of a shape parameter (dashed line) and the hypothesized df (solid line).

We have demonstrated the applicability of the results with the practical example. Our tests confirm the conjectures of the material scientists on the distributions of volume and shape of the pores. On the other hand, we should be aware that Theorem 1 and our goodness-of-fit test rely essentially on the independence assumption. There are grounds for the assumption that more or less weak dependencies between neighbouring pores exist. Mathematically spoken, the system of pores modelled by (1.1) is driven by a stationary marked point process (1.4) involving dependencies between different grains as well as between grains and germs. Under certain mixing and regularity conditions an analogue to Theorem 2 seems provable but the covariance structure of the corresponding limit process would be more complicated than that of $Y(\cdot)$. A detailed study of such weakly dependent structures with applications to testing the goodness-of-fit of certain marginal distributions of the typical grain should be a meaningful subject of future research.

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