Density of Cross-correlation measure and Stationarity

Jiří Lhotský

7th March 2006

Abstract

Cross-correlation measure is one of second order characteristics of spatial stochastic processes. It measures a spatial dependence of two given random processes. A density of such a measure and condition of existence of this density are studied. Further a particular attention is paid to so called semi-stationarity of spatial random process.

1 Introduction

The cross-correlation measure of two spatial random processes was first introduced in [3] to characterise geometrical relationship between two compound stationary spatial random processes. In the same publication some estimators of this characteristic were presented.

By compound stationarity we mean an invariance of compound distribution of given processes. Further for existence of a density of the cross-correlation measure, the semi-stationarity in needed. Thus a brief introduction of several kinds of stationarity is made in the second section.

The third section is devoted to the cross-correlation measure and its properties. Especially conditions for existence of density of this measure are deeply studied.

Basic notions

We use a stochastic approach to random spatial objects build on principles developed by G. Matheron [1].

Let (X, τ_X) be a locally compact, separable topological space, equipped with its Borel σ-algebra $\mathfrak{X}$. A measure on $(X, \mathfrak{X})$ is called locally finite, if it is finite on every compact subset of X.

Then by a *random point process* on the space $(X, \mathfrak{X})$ we understand a measurable map from some probability space $(\Omega, \mathfrak{A}, \Pr)$ to a measurable space $(\mathcal{N}(X), \mathfrak{N}(\mathfrak{X}))$ of all locally finite, integer-valued measures on $(X, \mathfrak{X})$ (see [4]).

Here the space $\mathcal{N}(X)$ is equipped with usual σ-algebra $\mathfrak{N}(\mathfrak{X})$ — the smallest σ-algebra, for which all the maps $f_Z \colon \mathcal{N}(X) \to \mathbb{N} \cup \{0\}$ defined by $\psi \to \psi(Z)$, are measurable for every $Z \in \mathfrak{X}$.

Note that a translation of an arbitrary measure μ is defined by $t_z(\mu)(\cdot) = \mu_z(\cdot) = \mu(t_{-z}(\cdot))$. Particarly the map t_z defined on $(\mathcal{N}(X), \mathfrak{N}(\mathfrak{X}))$ and its inverse t_{-z} are measurable if a translation t_z is defined on $(X, \mathfrak{X})$ and measurable for every $z \in \mathbb{R}^d$. To see that, it is enough to check that the set $t_z(\{\varphi \in \mathcal{N}(X) \colon t_{-z}(\varphi)(Z) = k\}$ is $\mathfrak{N}(\mathfrak{X})$-measurable where $Z \in \mathfrak{X}$ and $k \in \mathbb{N} \cup \{0\}$ are arbitrary. But this is easy to see since the above set is equal to measurable set $\{\varphi \in \mathcal{N}(X) \colon \varphi(t_z Z) = k\}$.

By a *(spatial) random process* on $(\mathbb{R}^d, \mathfrak{B})$ we understand a random point process on the hyperspace $(\mathcal{F}, \mathfrak{F})$ of all nonempty closed subset of $\mathbb{R}^d$ provided with a Borel σ-algebra generated by the Vietoris topology[1].

Note that according to [1] the space $(\mathcal{F}, \mathfrak{F})$ is a locally compact, separable Hausdorff topological space and that the usual set valued maps are measurable with respect to $(\mathcal{F}, \mathfrak{F})$. Namely motions, intersection, union and s-dimensional Hausdorff measure (cf. [1] and [5]).

[1]sometimes called myope topology or hit-or-miss topology

Recall some basic notions of geometric measure theory. Set some integer $m \leq d$. By $\mathscr{H}^m$ and $\mathscr{L}$ the m-dimensional Hausdorff measure (see [2, 2.10.2]) and Lebesque measure on $\mathbb{R}^d$ will be denoted respectively. Note that if $A \subset \mathbb{R}^d$ is Borel, then $\mathscr{H}^d(A) = \mathscr{L}(A)$ by [2, 2.10.35].

- A subset $E \subset \mathbb{R}^d$ is called *m-rectifiable* if there exists a Lipschitz function mapping some bounded subset of $\mathbb{R}^m$ onto E.
- A subset $E \subset \mathbb{R}^d$ is called *$\mathscr{H}^m$-rectifiable* whenever $\mathscr{H}^m(E \cap K) < \infty$ for any bounded set $K \subset \mathbb{R}^d$ and there exists a sequence $\{E_j\}_{j=1}^{\infty}$ of m-rectifiable subsets of $\mathbb{R}^d$ such that $\mathscr{H}^m(E \setminus \bigcup_{j=1}^{\infty} E_j) = 0$.

It was proved in [5, 2.2.1] that a class $\mathcal{X}_m$ of all closed $\mathscr{H}^m$-rectifiable sets is measurable i.e. $\mathcal{X}_m \in \mathfrak{F}$. Thus we can define *$\mathscr{H}^m$-process* as a random point process on the measurable space $(\mathcal{X}_m, \mathfrak{X}_m)$, where $\mathfrak{X}_m$ denotes the σ-algebra $\mathfrak{F} \cap \mathcal{X}_m$. Here $\mathfrak{F} \cap \mathcal{X}_m$ denotes a trace of σ-algebra that is the class

$$\mathfrak{F} \cap \mathcal{X}_m = \{B \cap \mathcal{X}_m \colon B \in \mathfrak{F}\}.$$

Note that every $\mathscr{H}^m$-process is also a spatial random process.

To every random point process $\mathbf{\Phi}$ we assign its *intensity measure*

$$\Lambda_{\Phi}(\cdot) = \mathrm{E}\mathbf{\Phi}(\cdot).$$

If $\mathbf{\Phi}$ is point process on $\mathbb{R}^d$, we call a possible density $\lambda_{\Phi}(\cdot)$ of its intensity measure Λ_{Φ} with respect to Lebesque measure *an intensity function.*

Further for $\varphi \in \mathcal{N}(\mathscr{X}_m)$ we define a measure on $\mathbb{R}^d$ by

$$\mu^m(\varphi, \cdot) = \int \mathscr{H}^m(X \cap \cdot)\, \varphi(dX),$$

and finally for a random $\mathscr{H}^m$-process $\mathbf{\Phi}$ we can define its *intensity measure* by

$$\Lambda_{\Phi}^m(\cdot) = \mathrm{E}\mu^m(\mathbf{\Phi}, \cdot).$$

To complete the list of our notions, let us finelly introduce a *random $\mathscr{H}^m$-set $\boldsymbol{X}$* as a measurable map from probability space $(\Omega, \mathcal{A}, \Pr)$ to the measurable space $(\mathcal{X}_m, \mathfrak{X}_m)$ and *intesity measure* Λ_X^m of the random $\mathscr{H}^m$-set $\boldsymbol{X}$ by

$$\Lambda_X^m(\cdot) = \mathrm{E}\mu^m(\boldsymbol{X}, \cdot).$$

2 Stationarity

Definition 1 *A random process $\mathbf{\Phi}$ is called* stationary*, if its distribution is invariant under translations, that is*

$$\Pr[t_z \mathbf{\Phi} \in (\cdot)] = \Pr[\mathbf{\Phi} \in (\cdot)]$$

for an arbitrary $z \in \mathbb{R}^d$.

If $\mathbf{\Phi}$ is a random stationary $\mathscr{H}^m$-process, then its intensity measure Λ_{Φ}^m coincides with Lebesque measure on Borel sets up to a constant $\lambda_{\Phi}^m \in \langle 0, \infty)$ called *intesity.*

Semi-stationarity

Let Λ be a measure on $(\mathscr{X}_m, \mathfrak{X}_m)$ and $S \subset \mathscr{X}_m$ a translation invariant set with $\Lambda(\mathcal{X}_m \setminus S) = 0$. We say that $p \colon S \to \mathbb{R}^d$ is *a location function* on $\mathscr{X}_m$ with respect to the measure Λ if

$$p(X + z) = p(X) + z, \ z \in \mathbb{R}^d, \ X \in S.$$

Definition 2 *Let be* $\mathbf{\Phi}$ *some* $\mathscr{H}^m$*-process. Then we say that* a process of locations $N_{\mathbf{\Phi}}$ and a location function p correspond to $\mathbf{\Phi}$ *whenever* p *is a location function with respect to the intensity measure* Λ_Φ^m *and*

$$N_{\mathbf{\Phi}} = \mathbf{\Phi}(p^{-1}(\cdot)).$$

The process $\mathbf{\Phi}$ *will be called* semi-stationary, *if it posseses a location function* p *which fulfills*

$$\mathrm{E}\int \mathbf{1}_B(p(X))\mathbf{1}_Z(X-p(X))t_z\mathbf{\Phi}(dX) = \mathrm{E}\int \mathbf{1}_B(p(X))\mathbf{1}_Z(X-p(X))\mathbf{\Phi}(dX) \qquad (1)$$

for an arbitrary $z \in \mathbb{R}^d$, $B \in \mathfrak{B}$ *and* $Z \in \mathfrak{X}_m$.

The notion of semi-stationarity was defined in [5]. A general stationary process does not need to be semi-stationary. There exists many stationary processes, whose grains do not posses any significant point needed for a correct definition of location function (e.g. process of lines). Clearly, semi-stationarity does not imply stationarity (see Remark 11).

Some conditions sufficient for stationary processes to be semi-stationary will be listed at the end of this section.

Remark 3 *The choice of* $Z = \mathbb{R}^d$ *in (1) implies that the intensity measure of the process of locations* $N_{\mathbf{\Phi}}$ *is translation invariant (i.e.* $\mathrm{E}t_zN_{\mathbf{\Phi}}(B) = \mathrm{E}N_{\mathbf{\Phi}}(B)$*).*

Lemma 4 *If* $\mathbf{\Phi}$ *is an* $\mathscr{H}^m$*-process, then for an arbitrary nonnegative measurable function* $f\colon (\mathbb{R}^d \times \mathscr{F}, \mathfrak{B} \otimes \mathfrak{F}) \to \mathbb{R}$ *and every* $z \in \mathbb{R}^d$ *the following inequality holds*

$$\mathrm{E}f(p(X), X-p(X))\, t_z\mathbf{\Phi}(dX) = \mathrm{E}f(p(X), X-p(X))\, \mathbf{\Phi}(dX).$$

Proof The above equation holds for characteristic functions by (1). The statement then follows from approximation by a sequence of step functions. □

Proposition 5 *For an* $\mathscr{H}^m$*-process* $\mathbf{\Phi}$ *with a location function* p*, the equation*

$$\mathrm{E}\int \mathbf{1}_B(p(X))\mathbf{\Phi}(dX) = \eta_\Phi \mathscr{L}(B)$$

holds for some $\eta_\Phi \in R_+ \cup \{\infty\}$.

Moreover, if $\eta_\Phi < \infty$ *then there exists a uniquely defined distribution* Q_p^Φ *on the space* $(\mathscr{X}_m, \mathfrak{X}_m)$ *satisfying*

$$\mathrm{E}\int \mathbf{1}_B(p(X))\mathbf{1}_Z(X-p(X))\, \mathbf{\Phi}(dX) = \eta_\Phi \mathscr{L}(B) Q_p^\Phi(Z). \qquad (2)$$

Proof For a proof see 3.2.4.1 [5]. □

Definition 6 *A distribution* Q_p^Φ *satisfying (2) will be called* the distribution of a typical grain.

Corollary 7 *If* $f\colon (\mathbb{R}^d \times \mathscr{F}, \mathfrak{B} \otimes \mathfrak{F}) \to (\mathbb{R}, \mathfrak{B})$ *is nonnegative measurable function and* $\eta_\Phi < \infty$*, then*

$$\mathrm{E}f(p(X), X-p(X))\, \mathbf{\Phi}(dX) = \eta_\Phi \iint f(x, Z)\, Q_p^\Phi(dZ)dx.$$

Proof Equation (2) implies the above assertion for characteristic functions. The general version follows by a routine procedure of approximation. □

Corollary 8 *If* η_Φ *is finite then every nonnegative* $\mathfrak{F}$*-measurable function* f *on* $\mathcal{X}_m$ *fulfills*

$$\int f(X)\, \Lambda_\Phi(dX) = \eta_\Phi \iint f(z + X_0)\, Q_p^\Phi(dX_0)dz.$$

Proof The map $(z, X) \to z + X$ is measurable. Thus the function $f(X) = f(p(X) + X - p(X))$ satisfies assumptions of the Corollary 7. $\square$

Theorem 9 *The semi-stationarity property is independent of the choice of a location function.*

Proof Denote p_1 the location function of a semi-stationary process $\mathbf{\Phi}$. Further for an arbitrary other location function p_2 of the process $\mathbf{\Phi}$ denote $f(X) := p_2(X) - p_1(X)$. The function $f(X)$ is obviously translation invariant, especially $f(X) = f(X - p(X))$.

Therefore

$$\begin{aligned}
&\mathrm{E}\int \mathbf{1}_B(p_2(X))\mathbf{1}_Z(X - p_2(X))\, t_z\mathbf{\Phi}(dX) \\
&= \mathrm{E}\int \mathbf{1}_B(p_1(X) + f(X))\mathbf{1}_Z(X - p_1(X) - f(X))\, t_z\mathbf{\Phi}(dX) \\
&= \mathrm{E}\int \mathbf{1}_B(p_1(X) + f(X))\mathbf{1}_Z(X - p_1(X) - f(X))\, \mathbf{\Phi}(dX) \\
&= \mathrm{E}\int \mathbf{1}_B(p_2(X))\mathbf{1}_Z(X - p_2(X))\, \mathbf{\Phi}(dX),
\end{aligned}$$

where the second equality follows directly by Lemma 4. $\square$

Theorem 10 *Let $\mathbf{\Phi}$ be a semi-stationary point process on $\mathscr{X}_m$ with location functions p_1, p_2 and $N_{1,\mathbf{\Phi}}$, $N_{2,\mathbf{\Phi}}$ their corresponding processes of location. Then $\eta_{1,\Phi} = \eta_{2,\Phi}$.*

Proof If $\eta_{1,\Phi} = \eta_{2,\Phi} = \infty$, the assertion holds. Thus we may without loss of generality assume that $\eta_{1,\Phi} < \infty$.

Let us choose some DC-system[2] $\mathcal{A}$ on $\mathbb{R}^d$, denote $r_n := \sup\{\operatorname{diam} A\colon A \in \mathcal{A}_n\}$, and for a bounded Borel set A denote $g(A)$ a centre of a corresponding circumscribed sphere.

Further let $f(X) := p_2(X) - p_1(X)$ and $F_j := \{X \in \mathscr{X}_m\colon f(X) \in A_j\}$ for $A_j \in \mathcal{A}_n$. The sets F_j are measurable, translation invariant, since $f(X+z) = p_2(X) - p_1(X) = f(X)$ that is $X \in F_j$ iff $X + z \in F_j$, and $\mathbf{\Phi} = \sum \mathbf{\Phi}\llcorner F_j$ almost sure, where $\varphi\llcorner F_j$ denotes the restriction of the measure φ to the set F_j. Both follows from the definition of a location function.

Thus

$$\begin{aligned}
\mathrm{E}N_{2,\mathbf{\Phi}}(B) &= \mathrm{E}\int \mathbf{1}_B(p_1(X) + f(X))\, \mathbf{\Phi}(dX) \\
&= \mathrm{E}\sum \int \mathbf{1}_B(p_1(X) + f(X))\, \mathbf{\Phi}\llcorner F_j(dX) \\
&\le \mathrm{E}\sum \int \mathbf{1}_{B\oplus -A_j}(p_1(X))\, \mathbf{\Phi}\llcorner F_j(dX) \\
&= \mathrm{E}\sum \int \mathbf{1}_{t_{g(A_j)}(B\oplus -A_j)}(p_1(X))\, \mathbf{\Phi}\llcorner F_j(dX) \\
&\le \mathrm{E}\sum \int \mathbf{1}_{B\oplus \mathrm{B}_{r_n}}(p_1(X))\, \mathbf{\Phi}\llcorner F_j(dX) \\
&= \mathrm{E}N_{1,\mathbf{\Phi}}(B \oplus \mathrm{B}_{r_n}) \\
&= \eta_{1,\Phi}\mathscr{L}(B \oplus \mathrm{B}_{r_n}) \overset{n\to\infty}{\longrightarrow} \eta_{1,\Phi}\mathscr{L}(B),
\end{aligned}$$

[2]DC-system $\mathcal{A} = \{\mathcal{A}_n\}$ on X is a monotone sequence of countable disjoint decompositions of X satisfying the following conditions

1. for every n is $X = \bigcup A_j, A_j \in \mathcal{A}_n$ and $\mathcal{A}_n$ is a countable partition of X to Borel sets,
2. for every n and every $A \in \mathcal{A}_n$ there exist $A_1, \dots, A_k \in \mathcal{A}_{n+1}$ such that $A = A_1 \cup \cdots \cup A_2$ and
3. $\lim\limits_{n\to\infty} \sup\limits_{A\in\mathcal{A}_n} \operatorname{diam} A = 0.$

where the third equality follows from semi-stationarity of the process $\mathbf{\Phi}$ and the translation invariance of F_j.

Therefore $\eta_{2,\Phi} \leq \eta_{1,\Phi} < \infty$ and the above technique can be applied on $\eta_{1,\Phi}$ to get $\eta_{1,\Phi} \leq \eta_{2,\Phi}$ which implies $\eta_{1,\Phi} = \eta_{2,\Phi}$. □

Remark 11 *Let us now present an example of a semi-stationary, but nonstationary process.*

The definition of semi-stationarity property (1) is based on a first order characteristic. Thus it is sufficient to find a semi-stationary process, whose second order characteristics are not translation invariant.

In the following example a cluster process on $\mathbb{R}^2$ will be constructed in the way, that the intensity of the process of clusters — parent process — will decrease with an increasing distance from origin, but the expected number of points in a particular cluster will rise.

a) *Parent process $\mathbf{\Phi}_R$*
Let $\mathbf{\Phi}_R$ be a Poisson process on $\mathbb{R}^d$ with intensity function $\lambda_R(x) = \alpha f_R(x)$, $x \in \mathbb{R}^2$, where $\alpha > 0$, and f_R is an arbitrary positive measurable function. Thus

$$\Lambda_{\Phi_R}(A) = \int_A \lambda_R \, dx = \alpha \int_A f_R \, dx.$$

b) *Clusters $\boldsymbol{\xi}_x$*
Further let $\{\boldsymbol{\xi}_x\}$ be a collection of mutually independent point processes, independent from the parent process $\mathbf{\Phi}_R$, provided with intensity functions

$$\lambda_{\xi_x}(y) = f_S(x) \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{\|y-x\|^2}{2\sigma^2} \right\},$$

where $f_S = 1/f_R$.

c) *Cluster process $\mathbf{\Phi}$.*
Denote $\mathbf{\Phi} = \int \boldsymbol{\xi}_x \, \mathbf{\Phi}_R(dx)$, then the intensity measure Λ_Φ of the process $\mathbf{\Phi}$ satisfies

$$\begin{aligned}
\Lambda_\Phi(A) &= \mathrm{E} \int \mathrm{E}_{\mathbf{\Phi}_R} \boldsymbol{\xi}_x(A) \, \mathbf{\Phi}_R(dx) \\
&= \alpha \int f_R(x) \int \mathbf{1}_A(x) f_S(x) \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{\|y-x\|^2}{2\sigma^2} \right\} dy \, dx \\
&= \alpha \mathscr{L}(A).
\end{aligned}$$

Now we can define a location function of $\mathbf{\Phi}$ by $p(\{x\}) = x$. Then $N_{\mathbf{\Phi}} \equiv \mathbf{\Phi}$ and the point process $\mathbf{\Phi}$ is therefore semi-stationary, but nonstationary by the choice of an almost sure nonconstant function f_R.

Moreover if an arbitrary $\mathscr{H}^k$-process is constructed so that, to every location an independent random $\mathscr{H}^k$-set is placed, then the definition equation (1) of semi-stationarity restricts just the corresponding process of locations. By this technique further semi-stationary $\mathscr{H}^k$-processes, can be constructed.

A stationary process $\mathbf{\Phi}$ satisfies the semi-stationarity condition (1), whenever some location function exists. Here some examples of location functions are listed.

- If a $\mathscr{H}^k$-process $\mathbf{\Phi}$ fulfills

$$0 < \mathscr{H}^k(X) < \infty, \quad \left| \int_X \vec{x} \, \mathscr{H}^k(dx) \right| < \infty, \ \forall X \in \mathbf{\Phi} \text{ a.s.}$$

then $p(X)$ can be chosen as a gravity centre of a grain i.e.

$$p(X) = \int_X \vec{x} \, \mathscr{H}^k(dx).$$

The measurability of $p(X)$ follows from the measurability of the integral.

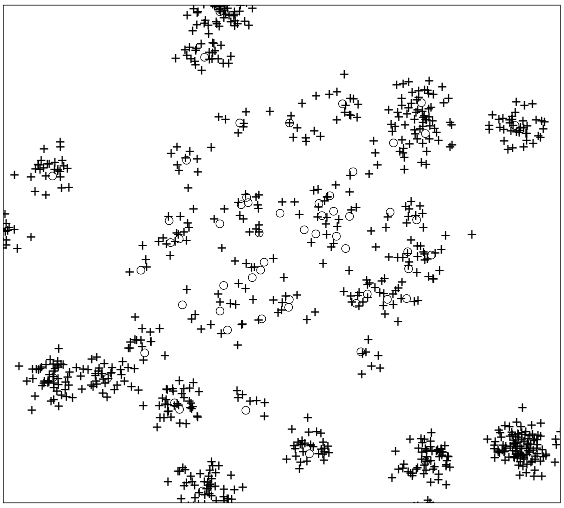

Figure 1: Cluster process $\mathbf{\Phi}$ with choice $f_R = e^{-\|x\|}$; $\{+ \in \mathbf{\Phi},\ \circ \in \mathbf{\Phi}_R\}$.

- If a process of compact sets is of interest, then a centre of a circumscribed sphere of a grain can be used as a convenient choice of a location function.
- For a proof of finiteness of η_Φ some significant point of grain can serve for a construction of a location function (e.g. the minimal point of compact set X under lexicographical order). Namely, the assumptions

$$\Lambda_\Phi(\mathscr{F}_B) < \infty; \forall B \in \mathfrak{B}, \mathscr{L}(B) < \infty \text{ and } p(X) \in X,\ X \in \mathbf{\Phi},$$

where $\mathscr{F}_B = \{F \colon F \in \mathscr{F},\ F \cap B \neq \emptyset\}$, imply $\eta_\Phi < \infty$, because $\eta_\Phi \mathscr{L}(B) = \mathrm{E} \int \mathbf{1}_B(p(X))\, \mathbf{\Phi}(dX) < \Lambda_\Phi(\mathscr{F}_B) < \infty$.

Definition 12 *We say that random point process $\mathbf{\Phi}$ on $\mathscr{X}_m$ satisfies* the decomposition condition*, if it is semi-stationary and the intensity η_Φ of its corresponding process of locations is finite.*

Compound Stationarity

Definition 13 *Two point processes $\mathbf{\Psi}$, $\mathbf{\Phi}$ on $\mathscr{X}_m$ and $\mathscr{X}_n$ respectively are called* compound stationary*, if their compound distribution is stationary, that is*

$$\Pr(\mathbf{\Psi} \in A, \mathbf{\Phi} \in B) = \Pr(t_z \mathbf{\Psi} \in A, t_z \mathbf{\Phi} \in B)$$

for every $A, B \in \mathfrak{F}$ and $z \in \mathbb{R}^d$.

Remark 14 *Clearly the compound stationarity implies stationarity of both processes, but the converse is not valid.*

Proposition 15 *Let $\mathbf{\Psi}$ and $\mathbf{\Phi}$ be compound stationary random point processes on $\mathscr{X}_m$ and $\mathscr{X}_n$ so that the process $\mathbf{\Psi}$ satisfies the decomposition condition. Then there exists a measure $\Lambda_0^{\Psi,\Phi}$ on $\mathscr{F} \times \mathscr{F}$ so that, every nonnegative measurable function $f \colon (\mathscr{F} \times \mathscr{F}, \mathfrak{F} \otimes \mathfrak{F}) \to (\mathbb{R}, \mathfrak{B})$ satisfies the equation*

$$\int f(X,Y)\, \Lambda_{\Psi,\Phi}(d(Y,X)) = \eta_\Psi \iint f(z+X, z+Y_0)\, \Lambda_0^{\Psi,\Phi}(d(Y_0,X))\, dz, \tag{3}$$

where $\Lambda_{\Psi,\Phi}$ is compound intensity of the processes $\mathbf{\Psi}$ and $\mathbf{\Phi}$, that is $\Lambda_{\Psi,\Phi}(\cdot) = \mathrm{E}(\mathbf{\Psi}, \mathbf{\Phi})(\cdot)$.

For the proof of the above proposition we will make a use of so called Palm distribution of random point process. If $(S, \mathfrak{S})$ and $(T, \mathfrak{T})$ are measurable spaces, we say that a mapping $\varrho\colon \mathfrak{S} \times T \to \langle 0, 1\rangle$ is a probability kernel from $(S, \mathfrak{S})$ to $(T, \mathfrak{T})$ if it satifies the following properties

1. the mapping $t \to \varrho(A, t)$ is measurable for every $A \in \mathfrak{S}$,
2. the mapping $A \to \varrho(A, t)$ is probability measure for every $t \in T$.

The existence of Palm distribution follows from the follwoing theorem.

Theorem 16 *Let Ψ be a random point process on $(X, \mathfrak{X})$ with locally finite intensity measure Λ_Ψ and let $Q_\Psi = \Pr \mathbf{\Psi}^{-1}$ be its distribution. The there exists a probability kernel $x \to P_x^\Psi$ from $(X, \mathfrak{X})$ to $(\mathcal{N}(X), \mathfrak{N}(\mathfrak{X})$ such that for every nonnegative mesurable function $f\colon X \times \mathcal{N}(X) \to \mathbb{R}$ the folowing equation holds*

$$\int_{\mathcal{N}(X)} \int_X f(x, \mu)\psi(dx) Q_\Psi(d\mu) = \int_X \int_{\mathcal{N}(X)} f(x, \mu) P_x^\Psi(d\mu)\Lambda_\Psi(dx). \tag{4}$$

Further if $P_x^\Psi, P_x'^\Psi$ are two probability kernels satisfying the above equation, then $P_x^\Psi(U) = P_x'^\Psi(U)$ Λ_Ψ-almost everywhere for every $U \in \mathfrak{N}(\mathfrak{X})$.

Definition 17 *Every probability kernel P_x^Ψ satifying equation (4) is called* Palm distribution *of the random point process $\mathbf{\Psi}$.*

Proof

$$\begin{aligned}
\mathrm{E}(\mathbf{\Psi}, \mathbf{\Phi})(f) &= \mathrm{E} \int \mathrm{E}[\textstyle\int f(X, Y)\mathbf{\Phi}(dX)|\mathbf{\Psi}]\, \mathbf{\Psi}(dY) \\
&= \mathrm{E} \iint \mathrm{E}[\textstyle\int f(X, Y)\mathbf{\Phi}(dX)|\mathbf{\Psi} = \mu]\, P_Y^\Psi(d\mu)\Lambda_\Psi(dY) \\
&= \eta_\Psi \iiint \mathrm{E}[\textstyle\int f(X, Y_0 + z)\mathbf{\Phi}(dX)|\mathbf{\Psi} = \mu]\, P_{Y_0+z}^\Psi(d\mu) Q_p^\Psi(dY_0)dz \\
&= \eta_\Psi \iiint \mathrm{E}[\textstyle\int f(X, Y_0 + z)\mathbf{\Phi}_z(dX)|\mathbf{\Psi}_z = t_z\nu]\, P_{Y_0}^\Psi(d\nu) Q_p^\Psi(dY_0)dz \\
&= \eta_\Psi \iiint \mathrm{E}[\textstyle\int f(X + z, Y_0 + z)\mathbf{\Phi}(dX)|\mathbf{\Psi} = \nu]\, P_{Y_0}^\Psi(d\nu) Q_p^\Psi(dY_0)dz,
\end{aligned}$$

where the third equality follows by Corollary 8. The fourth equality is valid because the compound stationarity of processes $\mathbf{\Phi}$, $\mathbf{\Psi}$ implies translation invariance of their mutual conditional distributions, that is that for every measurable function $g : (\mathcal{N}(X_m), \mathfrak{N}(\mathfrak{X}_m)) \to (\mathbb{R}, \mathfrak{B})$ the following holds

$$\mathrm{E}[g(\mathbf{\Phi})|\mathbf{\Psi} = \mu] = \mathrm{E}[g(\mathbf{\Phi}_z)|\mathbf{\Psi}_z = \mu] \text{ for } Q_\Psi\text{-almost all } \mu. \tag{5}$$

Thus

$$\Lambda_0^{\Psi,\Phi}(U) = \iint \mathrm{E}[\textstyle\int \mathbf{1}_U(X, Y)\mathbf{\Phi}(dX)|\mathbf{\Psi} = \nu]\, P_Y^\Psi(d\nu) Q_p^\Psi(dY).$$

$\square$

Remark 18 $\Lambda_0^{\Psi,\Phi}$ *is not a probability measure, but some kind of centring of an intensity measure of the process $\mathbf{\Phi}$ around a typical grain of the process $\mathbf{\Psi}$. One can see this using its form for independent processes,*

$$\Lambda_0^{\Psi,\Phi}(U) = \int \mathrm{E} \textstyle\int \mathbf{1}_U(X, Y)\, \mathbf{\Phi}(dX) Q_p^\Psi(dY).$$

Definition 19 *The measure $\Lambda_0^{\Psi,\Phi}$, from (3) will be called* intensity measure of a compound grain.

3 Cross Correlation Measure

The cross-correlation measure was introduced in [3] to describe a mutual relationship of two processes. It is suitable especially for tests of attractive or repulsive interactions.

Definition 20 *Let* $\mathbf{\Phi}$*,* $\mathbf{\Psi}$ *be a compound stationary random* $\mathscr{H}^k$ *(resp.* $\mathscr{H}^l$*)-processes and* λ_Φ^k*,* λ_Ψ^l *their intensities. Then the* cross-correlation measure *is defined by equation*

$$\mathcal{K}_{\Psi,\Phi}(B) = \frac{1}{\lambda_\Psi^k \lambda_\Phi^l \mathscr{L}(A)} \mathrm{E} \int_A \mu^k(\mathbf{\Phi}, B+x)\, \mu^l(\mathbf{\Psi}, dx),$$

where a set A *is arbitrary Borel set of positive Lebesque measure and* B *Borel set.*

For further computations let us denote its deterministic version by

$$\mathcal{C}_{Y,X}(B) = \int\limits_Y \int\limits_X \mathbf{1}_B(x-y)\, \mathscr{H}^k(dx)\mathscr{H}^l(dy),$$

where $X \in \mathcal{X}_k$, $Y \in \mathcal{X}_l$ with $\mathscr{H}^k(X), \mathscr{H}^l(Y) < \infty$. Notice that the definition of deterministic cross-correlation measure does not include any normalization.

Basic Properties

Proposition 21 *Let* $\mathbf{\Psi}$ *and* $\mathbf{\Phi}$ *be two compound stationary* $\mathscr{H}^l$*- and* $\mathscr{H}^k$*-processes and* A B *are arbitrary Borel sets. Then the following hold:*

1. *The cross-correlation measure can be expressed in the conditional form*

$$\mathcal{K}_{\Psi,\Phi}(B) = \frac{1}{\lambda_\Phi^k} \int \mathrm{E}[\mu^k(\mathbf{\Phi}, B) | \mu^l(\mathbf{\Psi}, \cdot) = \mu]\, P_0^\Psi(d\mu). \tag{6}$$

2. *The mixed second moment measure of* $\mathbf{\Psi}$ *and* $\mathbf{\Phi}$ *can be express by*

$$\mathrm{E}\mu^k(\mathbf{\Psi}, A)\mu^l(\mathbf{\Phi}, B) = \lambda_\Psi^l \lambda_\Phi^k \int_A \mathcal{K}_{\Psi,\Phi}(B-x)dx = \lambda_\Phi^k \lambda_\Psi^l \int_B \mathcal{K}_{\Phi,\Psi}(A-y)dy.$$

3. *The cross-correlation measure is symmetric in the sense*

$$\mathcal{K}_{\Psi,\Phi}(B-x) = \mathcal{K}_{\Phi,\Psi}(-B+x) \text{ for } \mathscr{L}\text{-almost all } x \in \mathbb{R}^d.$$

   If there exists a density $k_{\Psi,\Phi}$ *of the measure* $\mathcal{K}_{\Psi,\Phi}$*, then there exists also a density* $k_{\Phi,\Psi}$ *and these densities inherit the symmetry of the cross-correlation measure i.e.* $k_{\Psi,\Phi}(z) = k_{\Phi,\Psi}(-z)$ *for almost all* $z \in \mathbb{R}^d$*.*

4. *If the processes* $\mathbf{\Phi}$ *and* $\mathbf{\Psi}$ *are stochastically independent, then*

$$\mathcal{K}_{\Psi,\Phi}(B) = \mathscr{L}(B).$$

Proof

1. Using definition it holds

$$\begin{aligned}
\mathcal{K}_{\Psi,\Phi}(B) &= \frac{1}{\lambda_\Psi^l \lambda_\Phi^k \mathscr{L}(A)} \mathrm{E}\mathrm{E}_{\mathbf{\Psi}} \int_A \mu^k(\mathbf{\Phi}, B+x)\, \mu^l(\mathbf{\Psi}, dx) \\
&= \frac{1}{\lambda_\Psi^l \lambda_\Phi^k \mathscr{L}(A)} \iint \mathbf{1}_A \mathrm{E}[t_{-x}\mu^k(\mathbf{\Phi}, B) | \mu^l(\mathbf{\Psi}, \cdot) = \nu]\, P_x^\Psi(d\nu)\Lambda_\Psi(dx) \\
&= \frac{1}{\lambda_\Phi^k} \iint \frac{\mathbf{1}_A}{\mathscr{L}(A)} \mathrm{E}[t_{-x}\mu^k(\mathbf{\Phi}, B) | t_{-x}\mu^l(\mathbf{\Psi}, \cdot) = \mu]\, P_0^\Psi(d\mu)\mathscr{L}(dx) \\
&= \frac{1}{\lambda_\Phi^k} \int \mathrm{E}[\mu^k(\mathbf{\Phi}, B) | \mu^l(\mathbf{\Psi}, \cdot) = \mu]\, P_0^\Psi(d\mu),
\end{aligned}$$

   where the last equality follows by (5).

2. It holds

$$\begin{aligned}\mathrm{E}\mu^k(\mathbf{\Psi},A)\mu^l(\mathbf{\Phi},B) &= \mathrm{E}\mu^k(\mathbf{\Psi},A)\mathrm{E}[\mu^l(\mathbf{\Phi},B)|\mathbf{\Psi}]\\ &= \iint \mathbf{1}_A(x)\mathrm{E}[\mu^k(\mathbf{\Phi},B)|\mu^l(\mathbf{\Psi},\cdot)=\nu]P_x^{\Psi}(d\nu)\lambda_\Psi^l\mathscr{L}(dx)\\ &= \lambda_\Psi^l\lambda_\Phi^k\int_A \mathcal{K}_{\Psi,\Phi}(B-x)dx,\end{aligned}$$

where the last equality follows by (6) using the same technique as in the proof of 1. The statement then follows from symmetry.

3. The deterministic version of cross-correlation measure is symmetric by definition, that is $\mathcal{C}_{X,Y}(B)=\mathcal{C}_{Y,X}(-B)$. Therefore

$$\int_A \mathcal{K}_{\Psi,\Phi}(B-x)\,dx = \int_B \mathcal{K}_{\Phi,\Psi}(A-y)\,dy = \iint \mathbf{1}_B(y)\mathbf{1}_A(x+y)\,dy\,\mathcal{K}_{\Phi,\Psi}(dx)$$

$$= \iint \mathbf{1}_A(z)\mathbf{1}_B(z-x)\,dz\,\mathcal{K}_{\Phi,\Psi}(dx) = \int_A \mathcal{K}_{\Phi,\Psi}(-B+z)\,dz,$$

for an arbitrary Borel set A.

4. Using conditional form (6) one obtains

$$\mathcal{K}_{\Psi,\Phi}(B)=\frac{1}{\lambda_\Phi^k}\int \mathrm{E}[\mu^k(\mathbf{\Phi},B)|\mu^l(\mathbf{\Psi},\cdot)=\mu]P_0^{\Psi}(d\mu)=\frac{1}{\lambda_\Phi^k}\int \Lambda_\Phi^k(B)P_0^{\Psi}(d\mu)=\mathscr{L}(B).$$

$\square$

Existence of Density

The cross-correlation measure can have a density even in the case, where there is no density for its deterministic version. In many applications it is possible to reduce the problem of existence of a density to a question of existence of a density of measure $\mathrm{E}\,\mathcal{C}_{\boldsymbol{X},\boldsymbol{Y}}$, where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are suitable chosen random grains (cf. Theorem 25).

For $X,Y\subset\mathbb{R}^d$ denote $M_{X,Y}$ the set of all translations $z\in\mathbb{R}^d$ such that the intersection of X with $Y-z$ is nonempty, i.e.

$$M_{X,Y}=\{z\in\mathbb{R}^d\colon X\cap t_{-z}Y\neq\emptyset\}=Y-X=\{y-x\colon x\in X, y\in Y\}.$$

Theorem 22 *For $0\le k,l\le n$ let Λ be a complete measure on $(\mathscr{X}_k\times\mathscr{X}_l,\mathfrak{X}_k\otimes\mathfrak{X}_l)$ such that $\Lambda(\{(X,Y)\in\mathscr{X}_k\times\mathscr{X}_l\colon X\times Y$ is not $\mathscr{H}^{k+l}$-rectifiable$\})=0$. Then $M_{X,Y}$ is Λ almost everywhere $\mathscr{H}^{\min\{d,k+l\}}$-rectifiable. Moreover if*

$$\int \mathscr{H}^{\min\{d,k+l\}}(M_{X,Y}\cap\cdot)\,\Lambda(d(X,Y))\ll\mathscr{L}(\cdot)$$

on Borel sets, then the measure $\int\mathcal{C}_{X,Y}(\cdot)\,\Lambda(d(X,Y))$ has a density with respect to the Lebesgue measure.

In particular:

Corollary 23 *Let $(\Omega,\mathfrak{A},\Pr)$ be some probability space. Further for $0\le k,l\le n$ let $\boldsymbol{X}$, $\boldsymbol{Y}$ be random $\mathscr{H}^k$(resp. $\mathscr{H}^l$)-sets with $\mathscr{H}^k(\boldsymbol{X}),\mathscr{H}^l(\boldsymbol{Y})<\infty$ almost sure, such that $\boldsymbol{X}\times\boldsymbol{Y}$ is random $\mathscr{H}^{k+l}$-set. Then $M_{\boldsymbol{X},\boldsymbol{Y}}$ is random $\mathscr{H}^{\min\{d,k+l\}}$-set. Moreover if*

$$\Lambda^{\min\{d,k+l\}}_{M_{X,Y}}\ll\mathscr{L}$$

on Borel sets, then the measure $\mathrm{E}\mathcal{C}_{\boldsymbol{X},\boldsymbol{Y}}$ has a density with respect to the Lebesgue measure.

For arbitrary $\mathscr{H}^k$- and $\mathscr{H}^l$-rectifiable sets $X, Y \subset \mathbb{R}^d$ such that $X \times Y$ is a $\mathscr{H}^{k+l}$-rectifiable subset of $\mathbb{R}^{2d}$ denote $f_{X,Y}$ the restriction of the function $g\colon \mathbb{R}^{2d} \to \mathbb{R}^d$, defined by $g(x_1, \dots, x_{2d}) = (x_{d+1} - x_1, x_{d+2} - x_2, \dots, x_{2d} - x_d)$, to $X \times Y$. Then we set

$$J_{X,Y}(x,y) = (\frac{1}{\sqrt{2}})^{\max\{k+l-d,0\}} \mathrm{ap}J_{\min\{d,k+l\}} f_{X,Y}(x,y),$$

where $\mathrm{ap}J$ is the approximate Jacobian defined in [2, 3.2.22]. Measurability of the function $J_{X,Y}(x,y)$ with respect to product σ-algebra $\mathfrak{X}_k \otimes \mathfrak{X}_l \otimes \mathfrak{B}^d \otimes \mathfrak{B}^d$ on $\mathcal{X}_k \times \mathcal{X}_l \times \mathbb{R}^d \times \mathbb{R}^d$ was shown in [5].

For the proof of Theorem 22 we will need a kind of translative theorem. The following is slight generalisation of [5, 1.4.1].

Theorem 24 *For $0 \le k, l \le d$ let X be a $\mathscr{H}^k$-measurable, $\mathscr{H}^k$-rectifiable and Y a $\mathscr{H}^l$-measurable, $\mathscr{H}^l$-rectifiable subsets of $\mathbb{R}^d$ such that the set $X \times Y$ is $\mathscr{H}^{k+l}$-measurable and $\mathscr{H}^{k+l}$-rectifiable. Then for $\mathscr{H}^{d \wedge (k+l)}$-almost all $z \in \mathbb{R}^d$ the set $X \cap (Y - z)$ is $\mathscr{H}^{(k+l-d)\vee 0}$-rectifiable, $\mathscr{H}^{(k+l)\vee 0}$-measurable and an arbitrary $\mathscr{H}^{k+l}$-measurable function $h\colon X \times Y \to \mathbb{R}_+$ satisfies*

$$\int_{\mathbb{R}^d} \int_{X \cap (Y-z)} h(u, u+z)\, \mathscr{H}^{(k+l-d)\vee 0}(du) \mathscr{H}^{d \wedge (k+l)}(dz)$$
$$= \int_Y \int_X h(x,y) J_{X,Y}(x,y)\, \mathscr{H}^k(dx) \mathscr{H}^l(dy).$$

Proof The proof has to be done for cases $k + l \ge d$ and $k + l < d$ separately. The case of $k + l \ge d$ was treated in [5, 1.4.1]. To show the second one, $k + l < d$, denote

$$V_{X,Y} = \int_Y \int_X h(x,y) \mathrm{ap}J_{k+l} f_{X,Y}(x,y)\, \mathscr{H}^k(dx) \mathscr{H}^l(dy)$$
$$= \int_Y \int_{X_y} h(x') \mathrm{ap}J_{k+l} f_{X,Y}(x')\, \mathscr{H}^k(dx') \mathscr{H}^l(dy),$$

where $x' \in \mathbb{R}^{2d}$ and X_y is a slice of X, i.e. $X_y = \{(x,y)\colon x \in X\}$. Now apply co-area formula [2, 3.2.22] for the projection $\pi_2\colon X \times Y \to Y$. Thus $\pi_2^{-1}(y) = \{(x,y)\colon x \in X\}$, $\mathrm{ap}J_l \pi_2 = 1$ and

$$V_{X,Y} = \int_{X \times Y} h(v) \mathrm{ap}J_{k+l} f_{X,Y}(v) \mathrm{ap}J_l \pi_2(v)\, \mathscr{H}^{k+l}(dv)$$
$$= \int_{\mathbb{R}^d} \int_{f_{X,Y}^{-1}(z)} h(w)\, \mathscr{H}^0(dw) \mathscr{H}^{k+l}(dz)$$
$$= \int_{\mathbb{R}^d} \int_{X \cap (Y-z)} h(u, u+z)\, \mathscr{H}^0(du) \mathscr{H}^{k+l}(dz),$$

where the second equality follows by co-area formula [2, 3.2.22] applied to the function $f_{X,Y}$ and the third one from fact that the set $f_{X,Y}^{-1}(z)$ has the same amount of points as the set $X \cap (Y - z)$. □

Proof of Theorem 22 Choose $B \in \mathfrak{B}_n$ with $\mathscr{L}(B) = 0$. Theorem 24 for $(X, Y) \in \mathscr{X}_k \times \mathscr{X}_l$ implies

$$\mathcal{C}_{X,Y}(B) = \int_X \int_Y \mathbf{1}_B(x - y)\, \mathscr{H}^k(dx) \mathscr{H}^l(dy)$$
$$= \int_{M_{X,Y}} \int_{X \cap Y - z} \frac{\mathbf{1}_B(z)}{J_{X,Y-z}(u,u)}\, \mathscr{H}^{\max\{k+l-n,0\}}(du) \mathscr{H}^{\min\{d,k+l\}}(dz).$$

Set of all z such that $J_{X,Y-z}(u,u) = 0$ for some $u \in X \cap Y - z$ is a $\mathscr{H}^{\min\{d,k+l\}} \llcorner M_{X,Y}$-null set. The above equation is therefore well defined almost everywhere.

Denote the inner integral by $f(X,Y,z)$ and $f_i(X,Y,z) := \min\{f(X,Y,z), i\}$ for $i \in \mathbb{N}$. Then

$$\iint\limits_{M_{X,Y}} \mathbf{1}_B(z) f_i(X,Y,z)\, \mathscr{H}^{\min\{d,k+l\}}(dz) \Lambda(d(X,Y)) \leq$$

$$\leq \iint i \mathscr{H}^{\min\{d,k+l\}}(M_{X,Y} \cap B)\, \Lambda(d(X,Y)) = 0$$

by assumptions. Since $f_i \nearrow f$, Levi theorem implies

$$\int \mathcal{C}_{X,Y}(B)\, \Lambda(d(X,Y)) = 0.$$

Thus the assumptions of Radon-Nikodym theorem are satisfied and the measure

$$\int \mathcal{C}_{X,Y}(\cdot)\, \Lambda(d(X,Y))$$

has a density. $\square$

Sufficient conditions of existence of a density of a cross-correlation measure are presented in the following theorem.

Theorem 25 *Let $\mathbf{\Psi}$ and $\mathbf{\Phi}$ be compound stationary $\mathscr{H}^k$ (resp.$\mathscr{H}^l$)-processes such that $\mathbf{\Psi} \times \mathbf{\Phi}$ is a $\mathscr{H}^{k+l}$-process. If there exists an intensity measure of a compound grain $\Lambda_0^{\Psi,\Phi}$ such that the measure $\int \mathcal{C}_{X,Y}(\cdot) d\Lambda_0^{\Psi,\Phi}$ has a density, then the cross-correlation measure of processes $\mathbf{\Psi}$, $\mathbf{\Phi}$ has a density as well.*

Proof Recall Definition 19 of the intensity measure of compound grain and (3).

$$\mathcal{K}_{\Psi,\Phi}(B) = \frac{1}{\lambda_\Psi^k \lambda_\Phi^l \mathscr{L}(A)} \iiint \mathbf{1}_A(y) \mu^k(X, B+y) \mu^l(Y, dy) \Lambda_{\Psi,\Phi}(d(X,Y))$$

$$= \frac{\alpha}{\lambda_\Psi^k \lambda_\Phi^l \mathscr{L}(A)} \iiiint \mathbf{1}_A(y) \mu^k(t_z X, B+y) \mu^l(t_z Y, dy) \Lambda_0^{\Psi,\Phi}(d(X,Y)) \mathscr{L}(dz)$$

$$= \frac{\alpha}{\lambda_\Psi^k \lambda_\Phi^l \mathscr{L}(A)} \iiiint \mathbf{1}_A(y+z) \mathscr{L}(dz) \mu^k(X, B+y) \mu^l(Y, dy) \Lambda_0^{\Psi,\Phi}(d(X,Y))$$

$$= \frac{\alpha}{\lambda_\Psi^k \lambda_\Phi^l} \int \mathcal{C}_{X,Y}(B) \Lambda_0^{\Psi,\Phi}(d(X,Y)).$$

Finaly, the proof is finished by applying Theorem 22. $\square$

However, it is more convenient to use a conditional form of a cross-correlation measure for direct computation of a density.

Practical Examples

Here a few examples of computations of a density of the measure $\mathrm{E}\,\mathcal{C}_{\mathbf{X},\mathbf{Y}}$ shall be presented. A different treatment is needed for particular cases of dimension.

1. $k + l \geq d$
   This case is very easy to handle, since the measure $\mathcal{C}_{X,Y}$ already posseses a density. Thus it is possible to express directly a density of the cross-correlation measure.

$$\begin{aligned}
\mathcal{K}_{X,Y}(B) &= \mathrm{E} \iint \mathbf{1}_B(w-y) \mathbf{1}_{\mathbf{X}}(w) \mathbf{1}_{\mathbf{Y}}(y)\, \mathscr{H}^k(dw) \mathscr{H}^l(dy) \\
&= \mathrm{E} \iint \mathbf{1}_B(x) \frac{\mathbf{1}_{\mathbf{X}}(u) \mathbf{1}_{\mathbf{Y}}(u+x)}{J_{\mathbf{X},\mathbf{Y}}(u,u+x)}\, \mathscr{H}^{k+l-n}(du) \mathscr{L}(dx).
\end{aligned}$$

The last equality follows by Theorem 24 . Then the density is given by

$$k_{X,Y}(x) = \mathrm{E} \int_{\boldsymbol{X} \cap \boldsymbol{Y} - x} \frac{1}{J_{\boldsymbol{X},\boldsymbol{Y}-x}(u,u)} \mathscr{H}^{k+l-n}(du).$$

2. $0 < k + l < d$
This case is interesting by the fact that the probability contributes to the existence of the density as well as a deterministic space measure. The following settings can appear as an output of an intensity measure of a compound grain of some motion invariant Poisson process.
Choose a random $\mathscr{H}^k$-set $\boldsymbol{X}$ and $\mathscr{H}^l$-set $\boldsymbol{Y}$ in the way that $\boldsymbol{X} \times \boldsymbol{Y}$ is a random $\mathscr{H}^{k+l}$-set and $\mathscr{H}^{k+l}(\{z \in M_{\boldsymbol{X},\boldsymbol{Y}} \colon \mathrm{Tan}^{k+l}[\mathscr{H}^{k+l} \llcorner M_{X,Y}, z] \perp z\}) = 0$ a.s. Further let us denote the compound distribution of $\boldsymbol{X}$ and $\boldsymbol{Y}$ by $Q^{X,Y}$.
Then choose an uniformly distributed random variable $\boldsymbol{\varphi}$ on the space G_n of all rotations on $\mathbb{R}^d$. Thus the composition $(\boldsymbol{\varphi X}, \boldsymbol{\varphi Y})$ has a distribution $\overline{Q^{X,Y}}$, which satisfies for every $A \in \mathfrak{X}_k$ and $B \in \mathfrak{X}_l$ the equation

$$\overline{Q^{X,Y}}(A,B) = \int Q^{X,Y}(\varphi A, \varphi B) \vartheta_n(d\varphi),$$

where ϑ_n is the probability Haar measure on the space G_n.
Now, the explicit form of a density will be inferred.

$$\mathrm{E}\, \mathcal{C}_{\boldsymbol{\varphi X},\boldsymbol{\varphi Y}}(B) = \iint \mathcal{C}_{\varphi X,\varphi Y}(B)\, \vartheta_n(d\varphi) Q^{X,Y}(d(X,Y)). \tag{7}$$

Denote the inner integral by $V_{X,Y}$. Then according to Theorem 24 the following holds

$$\begin{aligned} V_{X,Y} &= \iiint \mathbf{1}_B(w-y) \mathbf{1}_{\varphi X}(w) \mathbf{1}_{\varphi Y}(y)\, \mathscr{H}^k(dw) \mathscr{H}^l(dy) \vartheta_n(d\varphi) \\ &= \iint\limits_{M_{\varphi X,\varphi Y}} \mathbf{1}_B(z) \int\limits_{\varphi X \cap \varphi Y - z} \frac{1}{J_{\varphi X, \varphi Y - z}(u,u)} \mathscr{H}^0(du)\ \mathscr{H}^{k+l}(dz) \vartheta_n(d\varphi). \end{aligned}$$

Moreover denote

$$f(X,Y,z) := \int\limits_{X \cap Y - z} \frac{1}{J_{X,Y-z}(u,u)} \mathscr{H}^0(du) \text{ pro } z \in M_{X,Y}.$$

The function f is rotation covariant, that is $f(X,Y,z) = f(\varphi X, \varphi Y, \varphi z)$, and the set $M_{X,Y}$ satisfies

$$M_{\varphi X, \varphi Y} = \varphi M_{X,Y}.$$

Therefore

$$\begin{aligned} V_{X,Y} &= \int \int\limits_{\varphi M_{X,Y}} \mathbf{1}_B(z) f(\varphi X, \varphi Y, z)\, \mathscr{H}^{k+l}(dz) \vartheta_n(d\varphi) \\ &= \int\limits_{M_{X,Y}} \int \mathbf{1}_B(\varphi z) \vartheta_n(d\varphi)\, f(X,Y,z)\, \mathscr{H}^{k+l}(dz), \end{aligned}$$

where the inner integral I expresses the probability that the point $\boldsymbol{\varphi} z$ hits the set B and by isotropy is a function only of $\|z\|$.

$$I(\|z\|) = \frac{1}{\mathscr{H}^{n-1}(\|z\|^{-1})} \int\limits_{\|z\|^{-1} \cap B} d\mathscr{H}^{n-1} = \frac{1}{\|z\|^{n-1} v_k} \int\limits_{\|z\|^{-1}} \mathbf{1}_B(u)\, \mathscr{H}^{n-1}(du),$$

where $\|z\|^{-1} = \{y : \|y\| = \|z\|\}$ and v_k is area of surface of a k-dimensional ball.

Then we may apply the co-area formula [2, 3.2.22] with the substitution $w = \|\cdot\|\llcorner M_{X,Y}$.

$$\begin{aligned} V_{X,Y} &= \int_{M_{x,y}} I(\|z\|) f(X,Y,z)\, \mathscr{H}^{k+l}(dz) \\ &= \int_{\|M_{X,Y}\|} \mathbf{1}_{\|B\|}(v) I(v) \int_{\|v\|^{-1}\cap M_{X,Y}} f(X,Y,z) \frac{1}{\mathrm{ap}J_1 w(z)}\, \mathscr{H}^{k+l-1}(dz)\mathscr{H}^1(dv), \end{aligned}$$

where $\mathrm{ap}J_1 w(z) = \frac{1}{\|z\|}|\Pi_V z|$, $V = \mathrm{Tan}^{k+l}[\mathscr{H}^{k+l}\llcorner M_{X,Y}, z]$. Obviously

$$\mathrm{ap}J_1 w(z) = 0 \Longleftrightarrow \mathrm{Tan}^{k+l}[\mathscr{H}^{k+l}\llcorner M_{X,Y}, z] \perp z,$$

and $\mathrm{ap}J_1 w(z) \neq 0$ $\mathscr{H}^{k+l}$-almost everywhere using assumptions.
The inner integral will by denoted by $h(v)$ and the co-area formula [2, 3.2.22] will be used backwards.

$$\begin{aligned} V_{X,Y} &= \int_{\|B\|} \int_{\|v\|^{-1}\cap B} \mathbf{1}_{\|M_{X,Y}\|}(v) \frac{1}{\|v\|^{n-1} v_k} h(v)\, \mathscr{H}^{n-1}(du)\mathscr{H}^1(dv) \\ &= \int_B \frac{1}{\|x\|^{n-1} v_k} h(\|x\|)\, \mathscr{L}(dx). \end{aligned}$$

Finaly let us just exchange the Lebesue measure with the distribution $Q^{X,Y}$ in equation (7). Then the density has a form

$$k_{\boldsymbol{\varphi X},\boldsymbol{\varphi Y}}(x) =$$

$$(\|x\|^{n-1} v_k)^{-1} \int_{\|x\|^{-1}\cap M_{X,Y}} \int_{X\cap Y-z} \frac{1}{J_{X,Y-z}(u,u)\mathrm{ap}J_1 w(z)}\, \mathscr{H}^0(du)\mathscr{H}^{k+l-1}(dz).$$

3. $k = l = 0$
   The case of point processes $\boldsymbol{X}$, $\boldsymbol{Y}$ is presented for completeness. Here the set $M_{\boldsymbol{X},\boldsymbol{Y}}$ is also a point process such that

$$M_{\boldsymbol{X},\boldsymbol{Y}} = \boldsymbol{Y} \oplus -\boldsymbol{X},$$

   and a deterministic version of a cross-correlation measure of realisations of $\boldsymbol{X}$ and $\boldsymbol{Y}$ is reduced to counting measure $\mu^0(M_{\boldsymbol{X},\boldsymbol{Y}},\cdot)$ i.e.

$$\mathrm{E}\mathcal{C}_{\boldsymbol{X},\boldsymbol{Y}} = \Lambda^0_{M_{\boldsymbol{X},\boldsymbol{Y}}}.$$

   Thus the problem of density of the measure $\mathrm{E}\mathcal{C}_{\boldsymbol{X},\boldsymbol{Y}}$ is reduced to the existence of an intensity function of the point process $M_{\boldsymbol{X},\boldsymbol{Y}}$.

References

[1] MATHERON G.: *Random sets and integral geometry*, John Wiley & Sons, New York-London-Sydney-Toronto, 1975

[2] FEDERER H.: *Geometric measure theory*, Springer-Verlag, Berling-Heidelberg-New York, 1969

[3] STOYAN D., OHSER J.: *Correlation between planar random structures, with ecological application*, Biom. J. vol.**24** no.7, 631-647, 1982

[4] R. SCHNEIDER, W. WEIL: *Stochastische Geometrie*, Teubner, Stuttgart-Leipzig 2000.

[5] ZÄHLE M. : *Random processes of Hausdorff rectifiable closed sets*, Math. Nachr. **108**, 49-72, 1982